Representation Theory, Automorphic Forms & Complex Geometry

Wilfried Schmid

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A Tribute to Wilfried Schmid

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Preface

This volume represents contributions from two conferences held in 2013. The first was a workshop at the Tsinghua Sanya International Mathematics Forum (Hainan, China) centered around three topics close to Wilfried Schmid's recent research interests: mixed Hodge modules, the unitary dual of real reductive Lie groups, and Voronoi summation formulas for automorphic forms. The second conference, "Representation Theory, Automorphic Forms, and Complex Geometry", was hosted by Harvard University from May 20–23 in honor of Prof. Schmid's 70th birthday.

I wish to thank Prof. Shing-Tung Yau for his generosity in initiating both conferences, and for his irreplaceable leadership role in obtaining funding for them. I am also grateful to him and to Lizhen Ji for suggesting the publication of this volume, as well as for their help in putting it together. I hope it will be a fitting tribute to the strong research and deep ideas that have marked Schmid's work over the last 50+ years. Finally, I very much appreciate the assistance of Eduardo Cattani and Dragan Milicic for explaining details of Schmid's earlier work, as well as to my other coorganizers of those conferences (Bill Casselman, Dorian Goldfeld, Benedict Gross, Mark Kisin, and Peter Trapa).

I am particularly grateful to Barry Mazur for organizing a session on open problems at the Harvard conference. Seven open problem submissions appear in this volume, and others can be found at this website (which he established): `http://www.math.harvard.edu/conferences/schmid_2013/problems/index.html`.

A brief mathematical biography of Wilfried Schmid

Though born in Hamburg, Wilfried Schmid grew up in West Germany's capital city Bonn. This was a fortuitous location, for through his parents' acquaintance he received early encouragement to go into mathematics from Friedrich Hirzebruch, the founder of the Max Planck Institute for Mathematics and the leader of post-war German mathematics. At age 17 he came to Princeton, NJ in 1960 with his parents for a year. He audited a few classes at Princeton University, where he enrolled formally as a sophomore in 1961. His academic strength was not limited solely to mathematics: he received the highest possible marks in all but one course in college, and was named

Valedictorian of the Class of 1964. His graduation speech about Princeton's role in the international realm is one of four valedictory addresses specifically noted on the Princeton website.

From 1964–1967 Schmid was a Ph.D. student of Philip Griffiths at U.C. Berkeley. At this time the representation theory of real reductive groups had just seen a major landscape shift due to the work of Harish-Chandra and Robert Langlands. Schmid was part of a group of influential Lie theorists (including Nolan Wallach and Roger Howe) who were either students or young postdocs at Berkeley at the time. After his striking thesis, he briefly stayed on the Berkeley faculty before moving to Columbia University, where at age 27 he became the youngest Full Professor in Columbia's history. In 1978 he moved to Harvard University.

Schmid's work spread from representation theory, to algebraic geometry, and then to analytic number theory, all while maintaining its core connection to geometric analysis. Below we summarize some of the main contributions of Schmid has made to these areas. Of course even more could be said about his contributions to the profession (e.g., as a founding editor of two prestigious journals, *Journal of the A.M.S.* and *The Cambridge Journal of Mathematics*) and the impact of his work in K-12 mathematics education.

Representation theory. The discrete series representations of a real reductive Lie group form the basic building blocks of Harish-Chandra's monumental treatment of representation theory. Harish-Chandra proved a famous character formula for discrete series representations, but a full construction and understanding remained a difficult problem (as it remains today in the p-adic setting).

Schmid's thesis and work shortly following it proved a conjecture of Langlands, who posited that discrete series representations could be constructed from certain actions on cohomology groups. Thus when combined with the work of Harish-Chandra, this gives a construction of all irreducible representations of real reductive groups. In later work with his student Henryk Hecht, Schmid proved Blattner's conjecture, which describes the restriction of a discrete series representation to a maximal compact subgroup (see also Vogan's contribution the Problem Session). Finally, Atiyah and Schmid together gave a different construction of discrete series using the Index Theorem, which streamlined the use of heavy machinery of Harish-Chandra on which previous work relied.

In the 1980s, Schmid's work was influenced heavily by the algebraic-geometric constructions of representations discovered by Beilinson and Bernstein. Together with Hecht, Milicic, and Wolf, he gave a foundationally-important treatment of Harish-Chandra modules through the theory of $\mathcal{D}$-modules. This allows for technical simplifications (for example, avoiding analytic difficulties in earlier approaches), and yields the simplest-known construction and exhaustion of the discrete series.

Schmid and Hecht also jointly proved the Osborne conjecture on $\mathfrak{n}$-homology and the Langlands classification. A collaboration of Schmid and Kari Vilonen in the late 1990s later settled the Barbasch-Vogan conjecture, which gives a precise description of the support of the Fourier transform of the (distributional) character of an irreducible admissible representation of a linear reductive group.

Algebraic geometry. Shortly after receiving his Ph.D. Schmid collaborated with Griffiths on the topic of variations of Hodge structure, which had been introduced by Griffiths a few years earlier. Their 1969 joint paper studies the type of homogeneous spaces that occur in Griffiths' work, particularly Lie-theoretic aspects which were fundamental to later work (e.g., by Griffiths and Carayol, where Griffiths-Schmid varieties reappear in the guise of Mumford-Tate domains). Their 1973 Tata survey gives a widely-cited exposition of these topics, including Deligne's theory of mixed Hodge structures.

In 1973 Schmid published a 109-page tour de force in *Inventiones Mathematicae* on variations of Hodge structure. Far and away his most-cited paper, it includes his famous $SL(2)$ and nilpotent orbit theorems, and establishes the existence of a limiting Hodge structure for the punctured disc (thereby proving a conjecture of Deligne). The techniques are ingenious and involve a serious application of Lie theory and complex analysis. Carlos Simpson's contribution to the Problem Session contains further discussion concerning Hodge structure.

After a break from the topic for many years, Schmid returned to it in joint work with Cattani and Kaplan. They extended the $SL(2)$ orbit theory to variation in several variables. As opposed to his 1973 *Inventiones* paper, which used $SL(2)$ orbits to prove the existence of the limiting mixed Hodge structure, here that becomes the starting point and the $SL(2)$ orbits now correspond to real splittings of the mixed Hodge structure. Their work is again marked by an application of Lie theory (along with a dose of several complex variables) to prove deep results in algebraic geometry. The three also resolved a famous conjecture of Deligne equating intersection cohomology with L^2 cohomology.

Automorphic distributions. In the mid-1990's Wilfried Schmid became interested in applying analytic concepts from representation theory to automorphic forms via their boundary value distributions. The latter control the embedding of a model of a unitary representation into spaces of automorphic functions, and were earlier studied by Ehrenpreis-Mautner and others (but without new applications). Having studied the intricacies of various completions of Harish-Chandra modules, Schmid was in a prime position to introduce new techniques to analytic number theory. Indeed, his first paper on the topic gave a very sharp and conceptual estimate on the regularity of these distributions, which refines earlier results on sums of the form $\sum_{n\leq X} a_n e^{2\pi i n\alpha}$, where $\alpha \in \mathbb{R}$ and the a_n are the Hecke eigenvalues of an automorphic form for $SL(2,\mathbb{R})$.

I was very fortunate to meet Professor Schmid while visiting Harvard at the time he was completing this paper, and to then later collaborate with him on a series of ten papers on automorphic distributions on higher rank groups. These papers mainly concern two additional topics: a Voronoi-style summation formula for Hecke eigenvalues on $GL(n)$, and a method to handle the archimedean theory of integral representations of L-functions. More information on these topics can be found in Templier's article and in my submission to the Problem Session, respectively. Brandon Bate's article performs a detailed analysis of metaplectic Eisenstein series through the lens of automorphic distributions. Diana Shelstad's article and James Arthur's Problem Session contributions both concern aspects of representation theory motivated by automorphic forms.

The future: Hodge theory and the unitary dual. Wilfried Schmid is a rare mathematician who both began his career bursting out the gate with success, and then methodically built upon his experience and skills to approach new problems in different areas later in his career. More than half a century after his thesis on geometric descriptions of discrete series representations, he and Kari Vilonen are pursuing a program to understand unitarity via Hodge Theory. This involves new work on Saito's theory of mixed Hodge modules, including Schmid's recent generalization of his orbit theorems. The joint paper by Dan Barbasch and Dan Ciubotaru in this volume is in part motivated by this approach, while Christian Schnell's article contains a modern survey of mixed Hodge modules.

As always, Schmid's research is noted for its originality, its potential reach, and its technical strength.

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