Contents

Preface **xi**

1 Introduction **1**
1.1 The problem . . . 2
1.2 The eikonal equation . . . 2
1.3 The transport equations . . . 8
1.4 Semi-classical differential operators . . . 18
1.5 The Schrödinger operator on a Riemannian manifold . . . 23
1.6 The plan . . . 24

2 Symplectic Geometry **25**
2.1 Symplectic vector spaces . . . 25
2.2 Lagrangian complements . . . 27
2.3 Equivariant symplectic vector spaces . . . 30
2.4 Symplectic manifolds . . . 32
2.5 Darboux style theorems . . . 33
2.6 The space of Lagrangian subspaces of a symplectic vector space . 38
2.7 Lagrangian subspaces transverse to a pair of Lagrangian subspaces . . . 40
2.8 The Maslov line bundle . . . 42
2.9 A look ahead: A simple example of Hamilton's idea . . . 43

3 The Language of Category Theory **47**
3.1 Categories . . . 47
3.2 Functors and morphisms . . . 48
3.3 Example: Sets, maps and relations . . . 50
3.4 The linear symplectic category . . . 55

4 The Symplectic "Category" **63**
4.1 Clean intersection . . . 64
4.2 Composable canonical relations . . . 65
4.3 Transverse composition . . . 66
4.4 Lagrangian submanifolds as canonical relations . . . 67
4.5 The involutive structure on $\mathcal{S}$ . . . 68
4.6 Reductions in the symplectic "category" . . . 68
4.7 Canonical relations between cotangent bundles . . . 69

4.8 The canonical relation associated to a map 70
4.9 Pushforward of Lagrangian submanifolds of the cotangent bundle 72
4.10 Pullback of Lagrangian submanifolds of the cotangent bundle . . 75
4.11 The moment map 76
4.12 Double fibrations 86
4.13 Exact symplectic manifolds and exact canonical relations 91

5 Generating Functions **97**
5.1 Fibrations 97
5.2 The generating function in local coordinates 100
5.3 Example: A generating function for a conormal bundle 101
5.4 Example: The generating function of a geodesic flow 102
5.5 The generating function for the transpose 105
5.6 The generating function for a transverse composition 106
5.7 Generating functions for clean composition 108
5.8 Reducing the number of fiber variables 109
5.9 The existence of generating functions 113
5.10 The Legendre transformation 116
5.11 The Hörmander-Morse lemma 117
5.12 Changing the generating function 125
5.13 The Maslov bundle 125
5.14 Identifying the two definitions of the Maslov bundle 129
5.15 More examples of generating functions 131

6 The Calculus of $\frac{1}{2}$-Densities **137**
6.1 The linear algebra of densities 137
6.2 Densities on manifolds 145
6.3 Pull-back of a density under a diffeomorphism 147
6.4 Densities of order 1 148
6.5 The principal series representations of $\mathrm{Diff}(X)$ 149
6.6 The push-forward of a density of order one by a fibration 149

7 The Enhanced Symplectic "Category" **151**
7.1 The underlying linear algebra 151
7.2 Half densities and clean canonical compositions 154
7.3 Rewriting the composition law 155
7.4 Enhancing the category of smooth manifolds and maps 155
7.5 Enhancing a map enhances the corresponding canonical relation . 158
7.6 The involutive structure of the enhanced symplectic "category" . 158
7.7 The symbolic distributional trace 161
7.8 The Maslov enhanced symplectic "category" 165

8 Oscillatory $\frac{1}{2}$-Densities **167**
8.1 Definition of $I^k(X, \Lambda)$ in terms of a generating function 168
8.2 Semi-classical Fourier integral operators 171
8.3 The symbol of an element of $I^k(X, \Lambda)$ 173
8.4 Symbols of semi-classical Fourier integral operators 175

8.5 The Keller-Maslov-Arnold description of the line bundle $\mathbb{L}$ 180
8.6 Microlocality 183
8.7 Semi-classical pseudo-differential operators 189
8.8 The local theory 195
8.9 The semi-classical Fourier transform 198
8.10 Semi-classical differential and pseudo-differential operators 202
8.11 Description of the space $I^k(X, \Lambda)$ 206
8.12 The clean version of the symbol formula 208
8.13 Clean composition of Fourier integral operators 209
8.14 An abstract version of stationary phase 212

9 Pseudodifferential Operators 217
9.1 Semi-classical pseudo-differential operators with compact microsupport 217
9.2 Classical ΨDO's with polyhomogeneous symbols 219
9.3 Semi-classical pseudo-differential operators 224
9.4 The symbol calculus 229
9.5 The formal theory of symbols 235
9.6 The Weyl calculus 238
9.7 The structure of $I(X, \Lambda)$ as a module over the ring of operators . 239
9.8 Lagrangian distributions 240
9.9 Conormal distributions 243
9.10 The semi-classical wavefront set 246

10 Trace Invariants 253
10.1 Functions of pseudo-differential operators 253
10.2 The wave operator for semi-classical pseudo-differential operators 256
10.3 The functional calculus modulo $O(\hbar^\infty)$ 257
10.4 The trace formula 259
10.5 Spectral invariants for the Schrödinger operator 260
10.6 An inverse spectral result: Recovering the potential well 263
10.7 Semi-classical spectral invariants for Schrödinger operators 265
10.8 An inverse result for the Schrödinger operator with a magnetic field 267
10.9 Counterexamples 268
10.10 The functional calculus on manifolds 270

11 Fourier Integral Operators 273
11.1 Semi-classical Fourier integral operators 273
11.2 The lemma of stationary phase 275
11.3 The trace of a semi-classical Fourier integral operator 276
11.4 The mapping torus of a symplectic mapping 282
11.5 The Gutzwiller formula 286
11.6 The Donnelly theorem 292

12 Integrality in Semi-Classical Analysis **295**
12.1 Introduction . . . 295
12.2 Line bundles and connections . . . 297
12.3 Integrality in DeRham theory . . . 302
12.4 Integrality in symplectic geometry . . . 304
12.5 Symplectic reduction and the moment map . . . 307
12.6 Coadjoint orbits . . . 312
12.7 Integrality in semi-classical analysis . . . 315
12.8 The Weyl character formula . . . 316
12.9 The Kirillov character formula . . . 322
12.10 The GKRS character formula . . . 325
12.11 The pseudodifferential operators on line bundles . . . 326
12.12 Spectral properties of the operators, $A_\hbar$ . . . 330
12.13 Equivariant spectral problems in semi-classical analysis . . . 331

13 Spectral Theory and Stone's Theorem **337**
13.1 Unbounded operators, domains, spectra and resolvents . . . 338
13.2 Self-adjoint operators on a Hilbert space . . . 344
13.3 Stone's theorem . . . 349
13.4 The spectral theorem . . . 360
13.5 The Calderon-Vallaincourt theorem . . . 365
13.6 The functional calculus for Weyl operators . . . 369
13.7 Kantorovitz's non-commutative Taylor's formula . . . 371
13.8 Appendix: The existence of almost holomorphic extensions . . . 376

14 Differential Calculus, Cartan's Identity, Moser Trick **379**
14.1 Superalgebras . . . 379
14.2 Differential forms . . . 380
14.3 The d operator . . . 380
14.4 Derivations . . . 381
14.5 Pullback . . . 382
14.6 Chain rule . . . 383
14.7 Lie derivative . . . 383
14.8 Cartan's formula . . . 384
14.9 Integration . . . 386
14.10 Stokes theorem . . . 386
14.11 Lie derivatives of vector fields . . . 386
14.12 Jacobi's identity . . . 388
14.13 A general version of Cartan's formula . . . 388
14.14 The Moser trick . . . 391

15 The Method of Stationary Phase **395**
15.1 Gaussian integrals . . . 395
15.2 The integral $\int e^{-\lambda x^2/2} h(x) dx$ . . . 397
15.3 Gaussian integrals in n dimensions . . . 398
15.4 Using the multiplication formula for the Fourier transform . . . 398
15.5 A local version of stationary phase . . . 399
15.6 The formula of stationary phase . . . 401

15.7 Group velocity 404
15.8 The Fourier inversion formula 406
15.9 Fresnel's version of Huygen's principle 407
15.10 The lattice point problem 410
15.11 Van der Corput's theorem 415

16 The Weyl Transform **419**
16.1 The Weyl transform in the physics literature 420
16.2 Definition of the semi-classical Weyl transform 421
16.3 Group algebras and representations 421
16.4 The Heisenberg algebra and group 422
16.5 The Stone–von-Neumann theorem 424
16.6 Constructing $\rho_\hbar$ 424
16.7 The "twisted convolution" 426
16.8 The group theoretical Weyl transform 427
16.9 Two two by two matrices 428
16.10 Schrödinger representations 428
16.11 The Weyl transform 429
16.12 Weyl transforms with symbols in $L^2(\mathbb{R}^{2n})$ 431
16.13 Weyl transforms associated to linear symbols 431
16.14 The one parameter group generated by L 433
16.15 Composition 434
16.16 Hilbert-Schmidt Operators 435
16.17 Proof of the irreducibility of $\rho_{\ell,\hbar}$ 437
16.18 Completion of the proof 438

Bibliography **441**

Index **443**