A Course in Hodge Theory

With Emphasis on Multiple Integrals

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by Hossein Movasati

Instituto de Matemática Pura e Aplicada
Rio de Janeiro, Brazil

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This book is dedicated to the memory of my parents Rogayeh and Ali, and to my family Sara and Omid

گندمی را زیر خاک انداختند

پس ز خاکش خوشه‌ها بر ساختند

بار دیگر کوفتندش ز آسیا

قیمتش افزود و نان شد جان‌فزا

باز نان را زیر دندان کوفتند

گشت عقل و جان و فهم هوشمند

باز آن جان چونک محو عشق گشت

یعجب الزراع آمد بعد کشت

If seeds are planted firmly in the ground,
Wheat will eventually grow all around;
Then in the mill they grind it to make bread-
Its value soars now with it men are fed;
Next by men’s teeth the bread is ground again,
Life, wisdom, and intelligence they gain,
And when in love that life becomes effaced
Farmers rejoice the seed’s not gone to waste!

Jalal al-Din Muhammad Balkhi, Masnavi, Book 1, Section 148. Translation: Jawid Mojaddedi, Oxford World’s Classic. Calligraphy: `nastaliqonline.ir`.

Preface

The main objective of the present book is to give an introduction to Hodge theory and its main conjecture, the so-called Hodge conjecture. We aim to explore the origins of Hodge theory much before the introduction of Hodge decomposition of the de Rham cohomology of smooth projective varieties. This is namely the study of elliptic, abelian and multiple integrals originated from the works of Cauchy, Abel, Jacobi, Riemann, Poincaré, Picard and Lefschetz, among many others. Therefore, the reader is warned that he or she will find in this book a partial presentation of the modern Hodge theory. The present book is intended to be an incomplete resuscitation of Picard and Simart's treatise *Théorie des fonctions algébriques de deux variables indépendantes* after almost a century, keeping in mind that the main object of study is the multiple integral itself and not other by-products. A complete analysis of this treatise and other contributions need a historian in mathematics, and is beyond the scope of this book. Another main emphasis of this book is on the computational aspects of the theory such as computing homologies by means of vanishing cycles, de Rham cohomologies, Gauss–Manin connections, Hodge cycles, etc. The development of Hodge theory during the last decades has put it far from its origin and the introduction of mirror symmetry by string theorists and the period manipulations of the B-model Calabi–Yau varieties, have risen the need for a text in Hodge theory with more emphasis on periods and multiple integrals. We aim to present materials which are not covered in J. Lewis's book *A survey of the Hodge conjecture,* nor in C. Voisin's books *Hodge theory and complex algebraic geometry, I and II*. Therefore, the reader will not find in this book some of the fundamental theorems in modern Hodge theory. We have tried to keep the text self-sufficient; however, a basic undergraduate knowledge of Complex Analysis, Differential Equations, Algebraic Topology and Algebraic Geometry will make the reading of the text smoother. The text is mainly written for two primary target audiences: graduate students who want to learn Hodge theory and get a flavor of why the Hodge conjecture is hard to deal with, and mathematicians who use periods and multiple integrals in their research and would like to put them in a Hodge theoretic framework. We hope that our text, together with those mentioned above, makes Hodge theory more accessible to a broader public.

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Rio de Janeiro, RJ, Brazil
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Hossein Movasati

Frequently used notations

$\mathsf{k}, \bar{\mathsf{k}}$	A field of characteristic zero and its algebraic closure.
$\bar{\mathbb{Q}}$	The field of algebraic numbers.
R	A finitely generated ring over the field k.
$\mathsf{T} := \mathrm{Spec}(\mathsf{R})$	A parameter space.
Θ_{T}	The set of vector fields in T.
$t \in \mathsf{T}$	A point in the parameter space.
$\mathrm{Mat}(n \times m, \mathsf{R})$	The set of $n \times m$ matrices with entries in R.
$\mathrm{Mat}(n, \mathsf{R})$	The set of $n \times n$ matrices with entries in R.
$V^{\vee}$	The dual of an R-module V. We always write a basis of a free R-module of rank r as a $r \times 1$ matrix. For a basis δ of V and α of $V^{\vee}$ we denote by $$[\delta, \alpha^{\mathsf{tr}}] := [\alpha_j(\delta_i)]_{i,j}$$ the corresponding $r \times r$ matrix.
M^{tr}	The transpose of a matrix M. We also write $M = [M_{ij}]$, where M_{ij} is the (i, j) entry of M. The indices i and j always count the rows and columns, respectively.
d	The differential operator or a natural number which is the degree of a tame polynomial.
$\mathbb{P}^{n+1}$	The projective space.
$(x_0, x_1, \ldots, x_{n+1})$	Homogeneous coordinates of $\mathbb{P}^{n+1}$
$x = (x_1, x_2, \ldots, x_{n+1})$	Affine coordinates of $\mathbb{C}^{n+1}$, for $n = 1, 2$ we use the classical notations (x, y) and (x, y, z), respectively.
f	A tame polynomial in $\mathsf{R}[x_1, x_2, \ldots, x_{n+1}]$.
s	The parameter in the tame polynomial $f - s$.
x^{β}, x^{α}	Monomials.
$\nu_1, \nu_2, \ldots$	The weights of the variables $x_1, x_2, \ldots$.
U	An open set in the usual topology or an affine variety.
L_t	A fiber of a tame polynomial.
X	A smooth hypersurface in $\mathbb{P}^{n+1}$.

ρ, ρ_0	The Picard number of a surface X and $\rho_0 := b_2 - \rho$, where b_2 is the second Betti number of X.
$h^{i,j}$	Hodge numbers of the projective variety X.
r	Dimension of the moduli space of hypersurfaces.
Y	A subvariety of X of codimension 1. It is usually the hyperplane section of X.
Z, Z_i	Algebraic subvarieties/cycles of X.
Z_∞	The algebraic cycle obtained by intersection of X with a linear $\mathbb{P}^{\frac{n}{2}+1}$.
$\check{Z}$	A primitive algebraic cycle, that is, $\check{Z} \cdot Z_\infty = 0$.
n, m	The dimension of X and any number between 0 and $2n$, respectively.
$H^m_{\mathrm{dR}}(X)$, $H^m_{\mathrm{dR}}(U)$	Algebraic de Rham cohomology
$\omega, \eta, \omega_\beta, \eta_\beta$	Differential forms in U or elements of $H^m_{\mathrm{dR}}(X)$ or $H^m_{\mathrm{dR}}(U)$ etc..
$\mathit{\Delta}$	The discriminant in R of a tame polynomial or a simplex.
$\mathit{\Delta}^n$	The n-dimensional simplex.
$\hat{\mathit{\Delta}}$	A divisible element in R in order to get tameness for f.
$\check{\mathit{\Delta}}$	The double discriminant which is an element in R.
$H_m(X,\mathbb{Z}), H_m(U,\mathbb{Z})$	The singular homology with coefficients in $\mathbb{Z}$.
$H_m(X,\mathbb{Z})_0, H^m_{\mathrm{dR}}(X)_0$	The primitive (co)homology.
$H_m(U,\mathbb{Z})_\infty$, $H_m(U,\mathbb{Z})_{\check{\infty}}$	The $\mathbb{Z}$-module of cycles at infinity.
u	Polarization which is an element in $H^2_{\mathrm{dR}}(X)$ obtained by $X \subset \mathbb{P}^{n+1}$.
∂	The boundary map.
δ	A homology class.
λ	A path in a topological space.
σ, τ	Maps derived from the Leray-Thom-Gysin isomorphism.
$\delta = \{\delta_t\}_{t \in U}$	A continuous family of cycles.
$G_{\mathrm{tors}}, G_{\mathrm{free}}$	The torsion and free subgroups of an abelian group G, respectively.
$\cup$	Cup product in singular or de Rham cohomology.
$\cap$	Cap product in singular (co)homologies.
$\mathbb{S}^n$	The n-dimensional sphere.
$\mathbb{B}^n$	The n-dimensional ball.
$a \cdot b$, $\langle a, b \rangle$	Intersection of topological or algebraic cycles a and b.
$\mathrm{Hodge}_n(X,\mathbb{Z})$	The $\mathbb{Z}$-module of Hodge cycles.
A_β	Rational numbers which are responsible for distinguishing between differential forms.
$\mathit{\Gamma}(a)$, $B(a_1, a_2, \cdots)$	The $\mathit{\Gamma}$ and B-function, respectively.
B_β	B-factors of the periods of the Fermat variety.
C	The set of critical values of a map.
I, x^I	A basis of the Milnor module.
X^d_n	The Fermat variety.
$\mathsf{H}, \mathsf{H}', \mathsf{H}''$	Brieskorn modules.

F^i	Hodge filtration.
W	Weight filtration.
$\frac{\omega}{df}$	Gelfand-Leray form.
M	Gauss-Manin system of the tame polynomial f.
$\mathrm{jacob}(f)$	Jacobian ideal of the polynomial f.
μ	The Milnor number of the tame polynomial f.
μ_d	The group of d-th roots of unity.
∇	Gauss-Manin connection.
∇_v	Connection along a vector field v.
$\mathbf{P}$	Period matrix, period map.
ξ	An invariant of Hodge cycles.
V_δ	Hodge locus corresponding to the Hodge cycle δ.
$(x)_y$	Pochhammer symbol.
$\langle x\rangle_y := (x-y+1)_{y-1}$	A modified Pochhammer symbol.
$\{x\}$	Fractional part of x.

Contents