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Algebraic Varieties

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Algebraic Varieties

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Preface

These are basically the notes that accompany my graduate course "Algebraic Geometry I" at Tsinghua. As with every course I teach, I revise the text until I cease teaching it and this is why until recently I resisted their publication. But as in the past few years the changes became marginal, the weight of such objections diminished, and I finally dropped them. Yet a warning is in order. One reason for these continuous revisions was that these notes were tailored to the "needs of the day". This of course changes with time and that makes me all the more aware of its deficiencies (by sometimes not giving a topic the treatment it deserves) and omissions (by skipping a nearby point of interest that would have merited discussion). On occasion I tried to make up for this by including some remarks in a smaller font.

As I hope this course will make clear, much of commutative algebra owes its existence to algebraic geometry and vice versa, and this is why there is no clear border between the two. This is why some familiarity with some commutative algebra is a prerequisite. But as a service to students lacking such background, I occasionally recall basic facts (all of it being standard fare in a first course on these subjects), also in a smaller font. Everything else we need has been included, so that these notes can be considered as essentially self-contained. (The only exception is the use of the Cohen-Macaulay property of a regular local ring (3.10.12), which we only use to prove a Bézout theorem.) In the end you will find that by learning algebraic geometry, you not only learn more commutative algebra, but also develop a geometrical way of thinking about it.

On `<www.staff.science.uu.nl/~looij101/>` I maintain a web page of this course, where among other things, I briefly explain what "Algebraic Geometry" is about and list some books for further reading. To repeat a recommendation that is made there, I encourage you to buy at least one other (preferably paper!) text book as a companion. Such a book may cover more or somewhat different ground, so that you may get a more balanced view of the subject.

The contents of these notes should be clear from a glance at the eponymous table. But I could characterize them also by what they lack: they deal with "Algebraic Geometry" over an algebraically closed field, and make no use of (co) homological methods. For example, in the last chapter, which among other things proves the Riemann-Roch theorem for curves, the classical notion of a repartition comes in place of sheaf cohomology. The two omissions are related, since sheaf cohomology in "Algebraic Geometry" is best developed in the setting of schemes (which is indeed the topic of a sequel to this course).

Eduard Looijenga
January 2019

Some conventions

Rings are always supposed to be commutative and to possess a unit and a ring homomorphism is required to take unit to unit. We allow that $1 = 0$, but in that case we get of course the zero ring $\{0\}$ and there cannot be any ring homomorphism going from this ring to a nonzero ring, as it must take unit to unit. Since a prime ideal of a ring is by definition not the whole ring, the zero ring has no prime ideals and hence also no maximal ideals. When R and R' are two rings, then $R \times R'$ is also one for componentwise addition and multiplication, the unit being $(1,1)$. The projections onto its factors are admitted as ring homomorphisms, but an inclusion obtained by putting one coordinate zero is not, as this is not unital, unless in that coordinate we have the zero ring (in other words, "$\times$" defines a categorical product but not a categorical sum).

We say that a ring is a *domain* ([1]) if its zero ideal is a prime ideal, in other words, if the ring is not the zero ring ($1 \neq 0$) and has no zero divisors.

Given a ring R, then an R-*algebra* is a ring A endowed with a ring homomorphism $\phi : R \to A$. When ϕ is understood, then for every $r \in R$ and $a \in A$, the product $\phi(r)a$ is often denoted by ra. In case R is a field, ϕ will be injective so that R may be regarded as a subring of A, but this need not be so in general. We say that A is *finitely generated as an* R-*algebra* if we can find $a_1, \ldots, a_n$ in A such that every element of A can be written as a polynomial in these elements with coefficients in R; in other words, if the R-algebra homomorphism $R[x_1, \ldots, x_n] \to A$ which sends the variable x_i to a_i is onto. This is not to be confused with the notion of finite generation of

[1] Since we assume all our rings to be commutative and with unit, this is the same notion as *integral domain*.

an R-module M which merely means the existence of a surjective homomorphism of R-modules $R^n \to M$ for some $n \geq 0$.

Similarly, a field L is said to be *finitely generated as a field* over a subfield K if there exist $b_1, \dots, b_n$ in L such that every element of L can be written as a fraction of two polynomials in these elements (the denominator being nonzero of course) with coefficients in K.

We denote the multiplicative group of the invertible elements (units) of a ring R by $R^\times$.

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