Advanced Lectures in Mathematics
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Uniformization, Riemann-Hilbert Correspondence, Calabi-Yau Manifolds & Picard-Fuchs Equations

edited by

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Uniformization, Riemann-Hilbert Correspondence, Calabi-Yau Manifolds, and Picard-Fuchs Equations

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ADVANCED LECTURES IN MATHEMATICS

Participants in the conference "Uniformization, Riemann-Hilbert Correspondence, Calabi-Yau manifolds, and Picard-Fuchs Equations"
Institute Mittag-Leffler, July 2015

Preface

The uniformization theorem for Riemann surfaces is one of the most important and beautiful theorems in mathematics. Its statement is clean, simple and natural, and it ties together several major areas of the main stream mathematics. The road from the initial conception by Klein and Poincaré to the final rigorous proofs by Koebe and Poincaré of the uniformization theorem was filled with inspiration, competition, excitement and disappointment.

Poincaré gave multiple proofs or attempts to establish different versions of the uniformization theorem. The work of Fuchs on differential equations with regular singularities inspired Poincaré's earlier work on differential equations, automorphic functions, and the uniformization theorem, which made him famous and also partially motivated Hilbert's twenty-first problem on the existence of linear differential equations with specified singular points and monodromy groups, which was in turn generalized to the Riemann-Hilbert correspondence.

After the Riemann mapping theorem for domains in the complex plane and the uniformization theorem for Riemann surfaces, many attempts were made to obtain higher dimensional generalizations. It seems that one of the most successful generalizations is the notion of Kähler-Einstein metrics for compact complex manifolds. Calabi-Yau manifolds form a special class of Kähler-Einstein manifolds and have many applications to subjects ranging from differential geometry, algebraic geometry, mathematical physics, topology, number theory, algebra, etc. They are higher dimensional generalizations of elliptic curves. Though many results are known about Calabi-Yau manifolds, much more is waiting to be explored. The periods of elliptic curves satisfy the Picard-Fuchs equation, which is a special differential equation with regular singularities. Periods of Calabi-Yau manifolds also satisfy Picard-Fuchs equations.

In view of the interconnection of these topics, a conference titled "Uniformization, Riemann-Hilbert Correspondence, Calabi-Yau manifolds, and Picard-Fuchs Equations" was held at Institute Mittag-Leffler in July 13–18, 2015. The purpose of the conference was to bring together many leading experts in these subjects to explore their historical development and interconnection between them.

To keep a permanent record of this conference and also to continue the fruitful interaction between the participants, we decided to edit a book which can serve as an overview of the many topics discussed. Almost all papers of this book were contributed by speakers of the conference. We hope and believe that they convey the lively atmosphere and accessible style of the conference.

All participants agreed that this conference was held at one of the most pleasant places for mathematicians: the great mathematics library, views of the surroundings, and fresh fruits during the tea break. We are grateful to Institute Mittag-Leffler for their hospitality, to many speakers for their excellent talks and

contributions to this book, and to referees for their help in reviewing and improving the papers in this book.

Lizhen Ji
Shing-Tung Yau

February 2017

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Hypergeometric Functions, from Riemann till Present

Frits Beukers

Abstract

Hypergeometric functions form a family of classical functions that occur frequently within many areas of mathematics and its applications. They were first introduced by Euler, who already discovered a great many surprising properties. Gauss continued this study by considering hypergeometric functions as solutions of a second order differential equation in the complex plane, including their multivaluedness. Riemann took up Gauss's study and made hypergeometric functions as prime example for his ideas on analytic continuation. It was also Riemann who named them Gauss hypergeometric functions. Although there exist many generalizations nowadays, we concentrate ourselves on these original functions. We briefly sketch Riemann's ideas and give an overview of developments around Gauss's hypergeometric function until recent times. There is an overlap of the first two sections with the author's summer school notes in [1].

1 Definition, first properties

The hypergeometric function of Gauss (although Euler-Gauss might be more appropriate) is a function of one complex variable z and three parameters a, b, c which we take to be in $\mathbb{R}$. Suppose that $a, b, c \in \mathbb{R}$ and $c \notin \mathbb{Z}_{\leqslant 0}$. Define *Gauss's hypergeometric function* by

$$ {}_2F_1(a,b,c|z) = \sum \frac{(a)_n (b)_n}{(c)_n n!} z^n. \tag{1.1}$$

The *Pochhammer symbol* $(x)_n$ is defined by $(x)_0 = 1$ and $(x)_n = x(x+1)\cdots(x+n-1)$. The radius of convergence of (1.1) is 1 unless a or b is a non-positive integer, in which cases we have a polynomial. Here are a couple of examples,