Nonlinear Analysis in Geometry and Applied Mathematics, Part 2

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Nonlinear Analysis in Geometry and Applied Mathematics, Part 2

editors

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Nonlinear Analysis in Geometry and Applied Mathematics, Part2

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Preface

Partial differential equations (PDEs) play a central role in mathematics and physics. PDEs can be used to describe the fundamental laws of nature, to probe the geometry and topology of manifolds, to construct moduli spaces, and to study the algebraic structure of complex manifolds and projective varieties. At its most basic level, the study of PDEs amounts to two questions: Firstly, given an equation, do solutions exist? And secondly, if a solutions exists, how much regularity can be guaranteed? For a randomly chosen equation the answers to these questions are almost always "no", and "none" respectively. However, PDEs with origins in geometry or physics often have hidden structure which make their study more tractable. It is of fundamental importance to identify and exploit these hidden structures.

The present volume details some of the activities of the 2015–2016 thematic program on PDEs held at the Harvard University Center of Mathematical Science and Applications (CMSA). The articles contained in this book represent a broad swath of current PDE research, ranging from works on minimal surfaces, to the existence and regularity theory of elliptic equations, to fluid dynamics, and conservation laws. Conspicuously absent is any material on mathematical general relativity, which appeared in the first volume of this series edited by Lydia Bieri, Piotr Chruściel, and Shing-Tung Yau. We hope that this volume will provide a window to the exciting developments of the CMSA thematic program on PDEs.

The editors would like to thank the Harvard CMSA for their hospitality and support, and the NSF and Evergrande Group for financial support. The editors would also like to express their gratitude to all the authors who contributed to this book.

The Editors:
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Lagrangian representations between linear transport and scalar conservation laws

Stefano Bianchini, Paolo Bonicatto, and Elio Marconi

Abstract. In this note we present a unifying approach for two classes of first order partial differential equations: we introduce the notion of Lagrangian representation in the settings of continuity equation and scalar conservation laws. This yields, on the one hand, the uniqueness of weak solutions to transport equation driven by a two dimensional BV nearly incompressible vector field. On the other hand, it is proved that the entropy dissipation measure for scalar conservation laws in one space dimension is concentrated on countably many Lipschitz curves.

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1. Introduction

This paper deals with the *continuity* and *transport* equations

$$\partial_t u + \operatorname{div}(u\mathbf{b}) = 0, \tag{1.1a}$$

$$\partial_t u + \mathbf{b} \cdot \nabla u = 0, \tag{1.1b}$$

where $u\colon I \times \mathbb{R}^d \to \mathbb{R}$ is an unknown scalar function and $\mathbf{b}\colon I \times \mathbb{R}^d \to \mathbb{R}^d$ is a given vector field (we write $I = (0, T)$, for some $T > 0$).

Strictly related to these equations, we will also consider the following scalar conservation law

$$\partial_t u + \operatorname{div}(\mathbf{f}(u)) = 0, \tag{1.2}$$

where $u\colon I \times \mathbb{R}^d \to \mathbb{R}$ is an unknown scalar function and the *flux* $\mathbf{f}\colon \mathbb{R} \to \mathbb{R}^d$ is a given smooth map.

These two classes of first order partial differential equations arise naturally in Mathematical Physics: both (1.1a) and (1.2) express the *conservation* of a physical quantity. The key difference between the models is that in the continuity equation (1.1a) the quantity is driven by a given vector field $\mathbf{b}$, while in the conservation law (1.2) the velocity field depends on the quantity u itself in a non-linear way.

The common feature is that both equations can be solved (at least formally) by the *method of characteristics*. Indeed, considering for simplicity

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Allard's interior regularity Theorem: an invitation to stationary varifolds

Camillo De Lellis

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Introduction

This is a small set of notes, taken from the last lectures of a course given in Spring 2012 at the University of Zürich. The aim is to give a short, reader-friendly but nonetheless detailed introduction to Allard's interior regularity theory for stationary integral varifolds. Allard's results, which are 40 years old (see [**2**]), form a pillar of the theory of minimal surfaces, which has been used a number of times in the literature, sometimes to reach really spectacular geometric applications. On the other hand I know only one textbook which reports them, Simon's *Lecture notes on geometric measure theory* (see [**13**]). Because of lack of time, I was not able to cover the material exposed in [**13**] in my course and I therefore looked for a suitable reduction which would anyway allow me to prove the key results. These lecture notes assume that the reader is familiar with some more advanced measure theory (Hausdorff measures, covering arguments and density theorems, see Chapters 2, 4 and 6 of [**9**]), has a coarse knowledge of rectifiable sets (definition, area formula and approximate tangents, see Chapter 4 of [**5**]) and knows a little about harmonic functions (see e.g. [**7**]).

There is, however, a price to pay. At a first glance an obvious shortcoming is that we restrict the theory to varifolds with bounded generalized mean curvature, whereas a suitable integrability assumption is usually sufficient. This is not a major point, since we cover stationary varifolds in smooth Riemannian manifolds (cf. Exercise 1.6). A second drawback is that hypothesis (H2) in Allard's ε-regularity Theorem 3.2 is redundant. Still, the statement given here suffices to draw the two major conclusions of Allard's theory. A third disadvantage is that a few estimates coming into the proof of Theorem 3.2 are stated in a fairly suboptimal form (the reader might compare, for instance, the crude estimates of Proposition 5.1 to those of Theorem 20.2 of [**13**]). In spite of these drawbacks, I still hope that these notes will give to the reader not only a quick access to the most relevant ideas, but also to

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The Weyl and Minkowski problems, revisited

Pengfei Guan

The Weyl and Minkowski problems are two inspiring sources for the theory of Monge-Ampère equation and fully nonlinear equations in general. The seminal works of Nirenberg [**23**], Pogorelov [**25, 28**] and Cheng-Yau [**6**] played important role in the development of geometric fully nonlinear PDEs. Though these two problems were solved longtime ago, there are many important geometric problems of current interest can be traced back to them. We discuss some recent work which are closely related to these two classical problems:

a. The intermediate Christoffel-Minkowski problem;
b. Isometric embedding of surfaces to 3-dimensional Riemannian manifolds.

The emphasis here is on issues of *regularity and convexity estimates* for solutions of nonlinear PDEs.

1. The Minkowski problem

The classical Minkowski problem was considered by Minkowski in [**22**]. Suppose M is a closed strongly convex hypersurface in the Euclidean space $\mathbb{R}^{n+1}$, the Gauss map $\nu : M \to \mathbb{S}^n$ is a diffeomorphism, where at any point $p \in M$, $\nu(p)$ is the unit outer normal at p. Let us denote $\kappa = (\kappa_1, \cdots, \kappa_n)$ to be the principal curvatures and $K = \kappa_1 \cdots \kappa_n$ the Gauss curvature of M respectively.

The Minkowski problem: *given a positive function φ on $\mathbb{S}^n$, find a closed strongly convex hypersurface whose Gauss curvature is $K = \frac{1}{\varphi}$ as a function on its outer normals.*

By the Divergence Theorem, φ has to satisfy equation

$$\int_{\mathbb{S}^n} x_i \varphi = \int_{\mathbb{S}^n} \frac{x_i}{K(x)} = \int_M \nu \cdot \vec{E}_i = 0, i = 1, \ldots, n+1, \tag{1.1}$$

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A survey of non-uniqueness results for the anisotropic Calderón problem with disjoint data

Thierry Daudé, Niky Kamran, and Francois Nicoleau

ABSTRACT. After giving a general introduction to the main known results on the anisotropic Calderón problem on n-dimensional compact Riemannian manifolds with boundary, we give a motivated review of some recent non-uniqueness results obtained in [**5, 6**] for the anisotropic Calderón problem at fixed frequency, in dimension $n \geq 3$, when the Dirichlet and Neumann data are measured on disjoint subsets of the boundary. These non-uniqueness results are of the following nature: given a smooth compact connected Riemannian manifold with boundary (M, g) of dimension $n \geq 3$, we first show that there exist in the conformal class of g an infinite number of Riemannian metrics $\tilde{g}$ such that their corresponding Dirichlet-to-Neumann maps at a fixed frequency coincide when the Dirichlet data Γ_D and Neumann data Γ_N are measured on disjoint sets and satisfy $\overline{\Gamma_D \cup \Gamma_N} \neq \partial M$. The corresponding conformal factors satisfy a nonlinear elliptic PDE of Yamabe type on (M, g) and arise from a natural but subtle gauge invariance of the Calderón when the data are given on disjoint sets. We then present counterexamples to uniqueness in dimension $n \geq 3$ to the anisotropic Calderón problem at fixed frequency with data on disjoint sets, which do not arise from this gauge invariance. They are given by cylindrical Riemannian manifolds with boundary having two ends, equipped with a suitably chosen warped product metric. This survey concludes with some remarks on the case of manifolds with corners.

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On the existence of $W_p^{1,2}$ solutions for fully nonlinear parabolic equations under either relaxed or no convexity assumptions

N.V. Krylov

Abstract. We establish the existence of solutions of fully nonlinear parabolic second-order equations like $\partial_t u + H(v, Dv, D^2v, t, x) = 0$ in smooth cylinders without requiring H to be convex or concave with respect to the second-order derivatives. Apart from ellipticity nothing is required of H at points at which $|D^2v| \le K$, where K is any fixed constant. For large $|D^2v|$ some kind of relaxed convexity assumption with respect to D^2v mixed with a VMO condition with respect to t, x are still imposed. The solutions are sought in Sobolev classes. We also establish the solvability without almost any conditions on H, apart from ellipticity, but of a "cut-off" version of the equation $\partial_t u + H(v, Dv, D^2v, t, x) = 0$.

1. Introduction and main results

In this paper we consider parabolic equations

(1.1)
$$\partial_t v(t,x) + H[v](t,x) = 0,$$

where

$$H[v](t,x) = H(v(t,x), Dv(t,x), D^2v(t,x), t, x),$$

in subdomains of

$$\mathbb{R}^{d+1} = \{(t,x) : t \in \mathbb{R}, x \in \mathbb{R}^d\}.$$

Let $\Omega \in C^{1,1}$ be an open bounded subset of $\mathbb{R}^d$. Fix $T \in (0,\infty)$ and set

$$\Pi = [0,T) \times \Omega$$

(if the t-axis is directed vertically, $[0,T) \times \Omega$ is indeed looking like a pie). Fix

$$p > d \quad \text{and a measurable function} \quad \bar{G} \ge 0 \quad \text{on} \quad \mathbb{R}^{d+1}.$$

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Viscosity solutions and the minimal surface system

O. Savin

ABSTRACT. We give a definition of viscosity solution for the minimal surface system and prove a version of Allard regularity theorem in this setting.

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1. Introduction

There are two main approaches to the theory of nonlinear elliptic scalar equations. One of them is variational (the L^2 approach), and it is based on energy estimates. This applies to equations with divergence structure. The second one, which regards more general nonlinear equations, is the viscosity solution approach (or L^∞ approach), and it is based solely on the maximum principle.

For the general theory of nonlinear elliptic systems only the variational approach seems to be successful. The reason is that the maximum principle does not extend to graphs when the codimension is higher than one.

In this short note we show that purely nonvariational techniques can be employed in the special situation of the minimal surface system.

Minimal submanifolds are usually studied from the geometric measure theory point of view. There are not many available results concerning the minimal surface system. This is due in part to the examples of Lawson and Osserman [**LO**] which show a quite different situation with respect to the minimal surface equation. Uniqueness does not hold, and the existence of classical (Lipschitz) solutions to the Dirichlet problem with smooth data may fail as well. They also gave an example of nontrivial global Lipschitz solution to the Bernstein problem $u : \mathbb{R}^4 \to \mathbb{R}^3$ obtained as a suitable scaling of the Hopf map $\eta : S^3 \to S^2$,

$$u(x) = \frac{\sqrt{5}}{2}|x| \; \eta\left(\frac{x}{|x|}\right), \qquad \eta(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2z_1\bar{z}_2).$$

However, the existence of classical solutions and the Bernstein theorem hold under specific bound assumptions involving the principal values of Du, see [**F, JX, W**].

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The Aleksandrov-Bakelman-Pucci estimate and the Calabi-Yau equation

Valentino Tosatti and Ben Weinkove

Abstract. We give two applications of the Aleksandrov-Bakelman-Pucci estimate to the Calabi-Yau equation on symplectic four-manifolds. The first is solvability of the equation on the Kodaira-Thurston manifold for certain almost-Kähler structures assuming S^1-invariance, extending a result of Buzano-Fino-Vezzoni. The second is to reduce the general case of Donaldson's conjecture to a bound on the measure of a superlevel set of a scalar function.

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1. Introduction

Yau's Theorem [**28**] states that one can prescribe the volume form of a Kähler metric on a compact Kähler manifold M^n within a given cohomology class. The proof reduces, via a continuity method, to obtaining uniform C^∞ *a priori* estimates on a potential function u solving the complex Monge-Ampère equation

$$(\omega + \sqrt{-1}\partial\overline{\partial}u)^n = e^F\omega^n, \quad \omega + \sqrt{-1}\partial\overline{\partial}u > 0, \quad \sup_M u = 0, \tag{1.1}$$

for a given smooth function F. A key step in Yau's paper [**28**] was the acclaimed L^∞ estimate of u, which he obtained using a Moser iteration argument.

There are now alternative proofs of the L^∞ estimate, which have been used to extend Yau's Theorem to different settings [**14, 3, 20, 21, 4, 9, 24**]. In particular, Cheng and Yau (see [**2**, p. 75]) used the Aleksandrov-Bakelman-Pucci (ABP) estimate to prove an L^2 stability result for the complex Monge-Ampère equation, and later Błocki [**3**] used this idea to give a new proof of the L^∞ estimate. Recall that the ABP estimate states, roughly

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Self-similar solutions to the Navier-Stokes equations: a survey of recent results

Zachary Bradshaw and Tai-Peng Tsai

Abstract. We survey the various constructions of forward self-similar solutions (and generalizations of self-similar solutions) to the Navier-Stokes equations. We also include and prove an extension of a recent result from [**7**].

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1. Introduction

The Navier-Stokes equations are a system of partial differential equations that describe the evolution of a viscous incompressible fluid's velocity field v and associated pressure π. In three dimensional space they are

$$\begin{aligned}\partial_t v - \Delta v + v\cdot\nabla v + \nabla\pi &= 0\\ \nabla\cdot v &= 0\end{aligned}\qquad \text{in } \mathbb{R}^3\times[0,\infty), \tag{1.1}$$

and are supplemented with some initial data v_0. If the nonlinearity $v\cdot\nabla v$ is omitted, this becomes Stokes system.

Leray proved in [**32**] that, if $v_0\in L^2$, then a global in time weak solution v to (1.1) exists. Hopf later generalized this to bounded domains (where the problem is supplemented with an appropriate boundary condition) in [**20**]. Leray's construction is based on the a priori bound

$$\operatorname*{ess\,sup}_{0<t'<t}\int |v(x,t')|^2dx + \int_0^t\int 2|\nabla v(x,t')|^2dx\,dt' \le \int |v_0(x)|^2dx. \tag{1.2}$$

Formally this is a result of testing (1.1) against a solution v and noting that the nonlinear term vanishes due to incompressibility. This energy inequality is identical to that satisfied by solutions to the Stokes system. Neither Leray nor Hopf were able to say much more about these weak solutions; this remains true of researchers today. In particular, we still do not know if Leray's weak solutions are unique or if they are smooth, even with smooth, compactly supported v_0 (partial and conditional results are available, but the general questions remain open).

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