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Advanced Lectures in Mathematics
Volume 32

Handbook of Group Actions

Volume II

Companion to the volume
Handbook of Group Actions, Volume I

edited by

Lizhen Ji
Athanase Papadopoulos
Shing-Tung Yau

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Advanced Lectures in Mathematics, Volume 32
Handbook of Group Actions, Volume II

Companion to the volume
Handbook of Group Actions, Volume I

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ADVANCED LECTURES IN MATHEMATICS

Foreword to Volumes I and II

The decision of editing this Handbook came after an international conference we organized in Kunming (the capital of the Yunnan Province, China) on July 21–29, 2012, whose theme was "Group Actions and Applications in Geometry, Topology and Analysis".

Kunming is a wonderful place for meetings and for mathematical discussions, especially in the summer, when the weather is most favorable. The conference was a success, from the mathematical and the human point of view. The city is warm, and the landscape is beautiful. There is a big lake, and a mountain behind the lake. Mathematicians like beauty. Hermann Weyl said: "My work always tried to unite the truth with the beautiful, but when I had to choose one or the other, I usually chose the beautiful." (Quoted in *Hermann Weyl's Legacy*, Institute for Advanced Study.)

The first two volumes of this Handbook are a record of the Kunming conference, but above all, we want them to be a convenient source for people working on or studying group actions. In spite of the fact that there were 63 talks, we covered at Kunming only a small part of this broad subject. In fact, group actions are so important that it is surprising that there was no available handbook on that subject so far. It is certainly the ubiquity of group actions that makes such a project so vast and therefore difficult to attain, and our aim for the time being is to start it. The present two volumes are the first on this important subject, and more volumes in the same series will appear in the future. Other conferences on the same subject are also planned in the future; the next one will be in Sanya (Hainan Province).

This Handbook will serve as an introduction and a reference to both beginners, non-experts, experts and users of group actions.

The conference in Kunming would not have gone so smoothly without the generous and devoted help of the local organizers, namely, Provost Ailing Gong, Dean Xianzhi Hu, Party Secretary Fengzao Yang, Deputy Dean Yaping Zhang, Youwei Wen, and Jianqiang Zhang from the Kunming University of Science and Technology. We would like to thank them for their work and hospitality.

Many people have also helped with refereeing and reviewing the papers in this Handbook, and we would like to thank them all for their help.

L. Ji, A. Papadopoulos, and S.-T. Yau
Ann Arbor, Strasbourg, and Cambridge MA
November, 2014

Introduction

This is the second volume of the Handbook of Group Actions, which will contain several volumes.

The present volume is divided into five parts, where the chapters are organized according to the nature of the groups concerned or to the applications of the actions involved.

Part I: Geometric Topology

This part consists of 5 chapters.

Discrete groups such as fundamental groups of manifolds and their actions on topological spaces are essential in topology. In geometric topology, interactions between topology and the fundamental group (or its group ring) are particularly important, and this is reflected in various far-reaching conjectures and problems such as the Borel conjecture for aspherical manifolds, the Novikov conjecture, the Baum-Connes conjecture, the Farrell-Jones conjecture and the space form problem.

Chapter 1 by F. T. Farrell, A. Gogolev and P. Ontaneda discusses how exotic differentiable structures on higher-dimensional spheres and related unexpected nontrivial topology of the stable pseudo-isotopy space led to solutions of longstanding problems on the topology of the space of negatively curved metrics on Riemannian manifolds and the topology of the space of Anosov diffeomorphisms. The space of negatively curved metrics is a generalization of the Teichmüller space of hyperbolic surfaces. The authors discuss several results showing that in dimensions higher than ten, this space is disconnected and has nontrivial homotopy groups, in sharp contrast with the Teichmüller spaces of surfaces. They also describe an analogy between metrics of negative curvature and Anosov diffeomorphisms and they present several results on the existence of these diffeomorphisms, on the rigidity of manifolds admitting them and on the nontrivial topology of spaces of Anosov diffeomorphisms.

Chapter 2 by J. Guaschi and D. Juan-Pineda constitutes a comprehensive introduction to surface braid groups and a survey of results on the Farrell-Jones isomorphism conjecture and its variants, e.g. the fibered Farrell-Jones conjecture on the lower algebraic K-theory of group rings of surface braid groups. One purpose of these isomorphism conjectures is to compute the algebraic groups explicitly, though it is very difficult to carry them out in general. This paper also contains explicit computational results for surface braid groups. The combination of these two kinds of results makes surface braid groups special.

Groups acting on trees often have particular properties, generally related to decompositions of these groups, and results about the whole groups can be assembled from those of vertex and edge stabilizers. Chapter 3 by S. K. Roushon is a survey on the Farrell-Jones fibered isomorphism conjecture in both K- and L-theories and its relation with the vanishing of the Whitehead groups for groups acting on trees. The author presents both the classical material and the recent developments of the lower K-theory and the surgery L-theory of groups acting on trees. The paper ends with a list of open problems on the (fibered) isomorphism conjecture.

Motivated by Kazhdan's property T of groups acting on Hilbert spaces, some recent activity is concerned with actions of groups on Banach spaces. Affine actions of a group G on a Banach space E can be translated into properties of the first cohomology group of G with coefficients in the G-module formed by the Banach space E together with a representation of G. Fixed point properties of such group actions are equivalent to vanishing of associated cocycles. The existence of proper actions on Hilbert spaces has implications in the setting of the Baum-Connes conjecture for groups. Chapter 4 by P. W. Nowak is a survey on recent progress on group actions on Banach spaces and their fixed point properties, motivated by property T, but with a stress on actions on Banach spaces which are not Hilbert spaces. The author also thoroughly discusses the property known in the case of Hilbert spaces as a-T-menability, or the Haagerup property, that is, the existence of metrically proper affine isometric actions on Banach spaces. Applications to the dimension theory of boundaries of hyperbolic groups are mentioned.

One basic problem on spherical space forms asks which finite groups can act freely on spheres by homeomorphisms – or diffeomorphisms. In dimension three, this corresponds to the question of which finite groups can occur as fundamental groups of compact topological or smooth manifolds. In Riemannian geometry, the spherical space form problem amounts to the classification of compact Riemannian manifolds with constant sectional curvature, and it was solved quite satisfactorily. On the other hand, the topological spherical space form problem is more subtle, and there has been a lot of work on it. Chapter 5 by I. Hambleton concerns topological spherical space forms and it is an updated short survey on this problem, starting from the nineteenth century work, and ending with the work done recently based on Perelman's solution of the Poincaré conjecture. Some new directions of research in this field are also mentioned.

Part II: Representations and Deformations

This part consists of 5 chapters.

For a discrete subgroup Γ of a semisimple Lie group G, the most natural and obvious action is probably the action of Γ on the associated symmetric space G/K, where K is a maximal compact subgroup of G. Particular examples are the action of a Fuchsian group on the upper half-plane and the action of a Kleinian group on three-dimensional hyperbolic space. There are at least two ways of generalizing these examples. The first one is to consider representations of discrete groups into semisimple Lie groups and their actions on associated symmetric spaces, and the second one is to consider many actions of a given type, that is, families, or deformations of such actions, simultaneously.

Chapter 6 by R. D. Canary is a survey on the dynamics of the action of the outer automorphism group $\mathrm{Out}(\Gamma)$ of a word hyperbolic group Γ on the character variety of representations of Γ in a semisimple Lie group G. The stress is on two aspects of this theory: (1) the setting started by Labourie and developed later on by Guichard, Labourie and Wienhard on the proper discontinuity of the action on spaces of Anosov representations; (2) the works of Canary, Gelander, Lee, Magid, Minsky and Storm on the special case where Γ is the fundamental group of a compact-orientable 3-manifold with boundary and $G = \mathrm{PSL}(2, \mathbb{C})$.

In Chapter 7, J. Maubon surveys the Higgs bundle theory of Hitchin and Simpson and he shows how these bundles can be used in the representation theory of complex hyperbolic lattices into Lie groups of Hermitian type. He proves in particular the rigidity of the so-called *maximal* representations. The survey covers representations of surface groups but the main part of the survey concerns representations of higher-dimensional lattices. For this reason, the main Lie group that is involved is the Hermitian Lie group $\mathrm{SU}(p; q)$, since this is the only simple Lie group of Hermitian type into which maximal representations of higher-dimensional complex hyperbolic lattices are expected to exist.

Chapter 8 by K. Ohshika concerns deformation spaces of Kleinian groups, and limits of such groups. Deformation spaces are equipped with several different topologies (this was highlighted by Thurston and others), deformation spaces have several boundary structures, with complicated structure. Ohshika studies in particular the Bers boundaries for quasi-Fuchsian groups. He presents recent results that he obtained, which provide a complete classification of the geometric limits of quasi-Fuchsian groups. Combined with work of Kerckhoff and Thurston, these results explain why the action of the mapping class group on Teichmüller space does not extend continuously to a Bers boundary, and the author describes a quotient of this boundary, the so-called *reduced Bers boundary*, which admits a continuous action of the mapping class group.

Teichmüller spaces are parameter spaces which describe deformation of (equivalence classes of) complex structures on surfaces. Several chapters of the first volume of this handbook deal with Teichmüller spaces and their associated mapping class groups. In principle, complex structures are defined abstractly by charts. But there is another point of view, namely, to consider deformation of surfaces embedded in higher-dimensional manifolds, and in particular, four-dimensional manifolds. Diffeomorphism groups and hence mapping class groups of surfaces act on the space of embeddings of the surfaces. One question is how different these embeddings are when viewed from the point of view of the diffeomorphism group of the ambient space. Chapter 9 by S. Hirose is a survey of results on deformations of smooth surfaces embedded in four-dimensional manifolds with respect to these diffeomorphism groups and to the mapping class groups of the surfaces. Given an embedding e of the surface in the 4-manifold, the main question which is investigated is to what extent a diffeomorphism ϕ of an embedded surface extends to a diffeomorphism Φ of the ambient 4-manifold respecting the embedding e, that is, satisfying $\Phi \circ e = e \circ \phi$. The author surveys some flexibility and rigidity results on this question. Knotted and unknotted embeddings of closed surfaces in the 4-sphere and algebraic curves in $\mathbb{C}P^2$ are considered. Results on the Rokhlin quadratic form and on the Arf invariant are used.

One important and effective method to deform surfaces in four-dimensional symplectic manifolds is to consider Lefschetz fibrations. This subject was motivated by the study of Lefschetz fibrations in algebraic geometry, a basic tool in that area, and in particular in the study of algebraic surfaces. The works of Donaldson and Gompf showed that these fibrations give a fairly complete description of symplectic 4-manifolds. Motivated by the question of the geography of algebraic surfaces in terms of the ratio (or the slope) of two invariants of algebraic surfaces, in Chapter 10, N. Monden gives an introduction to Lefschetz fibrations with a particular stress on the relation with mapping class groups. The author presents several methods of construction of Lefschetz fibrations and he gives a summary of some results on the geography of symplectic 4-manifolds and, in particular, on the construction of some Lefschetz fibrations over the sphere S^2 which violate the so-called slope inequality.

Part III: Geometric Groups

This part consists of 3 chapters.

Among all infinite discrete groups, the modular group $\mathrm{SL}(2,\mathbb{Z})$ is special and it plays the role of a model for several classes of groups. One reason for its importance comes from its actions on very different spaces. Chapter 11 by A. M. Uludağ is a survey on several actions of $\mathrm{SL}(2,\mathbb{Z})$, starting from the very classical ones and ending with the recently discovered ones. The examples include the actions on Farey trees, on binary quadratic forms, on planar lattices, on the unit circle considered as the boundary of the upper half plane, and on dessins d'enfants, introduced by Grothendieck in the 1980s.

Simplicial sets are basic objects in algebraic topology and they are a generalization of simplicial complexes. They are related to the notion of simplicial group. Chapter 12 by J. Wu gives an introduction to simplicial groups and simplicial structures on geometric groups such as braid groups, link groups and mapping class groups.

By definition, a Lie group is a combination of a group and a compatible smooth manifold. If we ignore the underlying smooth structure, then we obtain a discrete group, or an abstract group. One basic question in Lie group theory is to what extent the smooth structure of a Lie group can be recovered from the group structure. For example, when is a group homomorphism a Lie group homomorphism? Similar questions can be raised for rational points of linear algebraic groups over general fields. Chapter 13 by I. A. Rapinchuk is a survey on known results on these questions and on their applications such as to character varieties of elementary subgroups of Chevalley groups over finitely generated commutative rings.

Part IV: Geometric Invariants and Growth of Discrete Groups

This part consists of 2 chapters.

One important question in the field of group actions in algebraic geometry concerns the actions of algebraic groups on algebraic varieties. The behavior of orbits of such actions is a basic question. This theory is one of the basic objects of geometric invariant theory and it is closely related to the problem of constructing moduli spaces in algebraic geometry.

In Chapter 14, D. P. Bǎć and N. Q. Thǎńg give a summary of classical geometric invariant theory over algebraically closed fields as well as a survey of geometric invariant theory over non-algebraically closed fields, i.e., actions of algebraic groups on affine varieties over non-algebraically closed fields. The authors discuss questions related to the notion of stability, of various types and closures that are akin to this algebraic geometry setting and of other topological properties of orbits.

One basic point of view in geometric group theory is to consider a finitely generated group as a metric space. One way to do this is to endow the group with a word metric (which depends on the choice of a generating set). One geometric object associated to such a group (or a word metric) is the growth series, whose coefficients are the number of group elements at fixed distance from the identity element. This gives rise to the important notion of spherical growth series. Chapter 15 by M. Fujii gives an introduction to a method of computing spherical growth series of finitely generated groups via finite-state automata. The theory is illustrated by several explicit examples of growth series related to the pure Artin group of dihedral type. The paper also contains background material on Artin groups as well as a survey on the known groups for which the growth series is rational or irrational, for some or for an arbitrary generating set.

Part V: Music and Group Actions

Chapter 16 by A. Papadopoulos, concerns applications of groups and group actions in arts, and more precisely in music theory and music composition. Several examples are presented, especially from the compositions and the theoretical work of the French composer Olivier Messiaen. Since most readers of this handbook are not familiar with the details on the relation between mathematics and music — although most of them know that such a relation exists — the chapter also contains an introduction, with a historical overview, on this subject.

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