Advanced Lectures in Mathematics (ALM)

Advanced Lectures in Mathematics
Volume 31

Handbook of Group Actions

Volume I

Companion to the volume
Handbook of Group Actions, Volume II

edited by

Lizhen Ji
Athanase Papadopoulos
Shing-Tung Yau

International Press
www.intlpress.com

HIGHER EDUCATION PRESS

Advanced Lectures in Mathematics, Volume 31
Handbook of Group Actions, Volume I

Companion to the volume
Handbook of Group Actions, Volume II

Volume Editors:
Lizhen Ji (University of Michigan, Ann Arbor)
Athanase Papadopoulos (Université de Strasbourg, France)
Shing-Tung Yau (Harvard University)

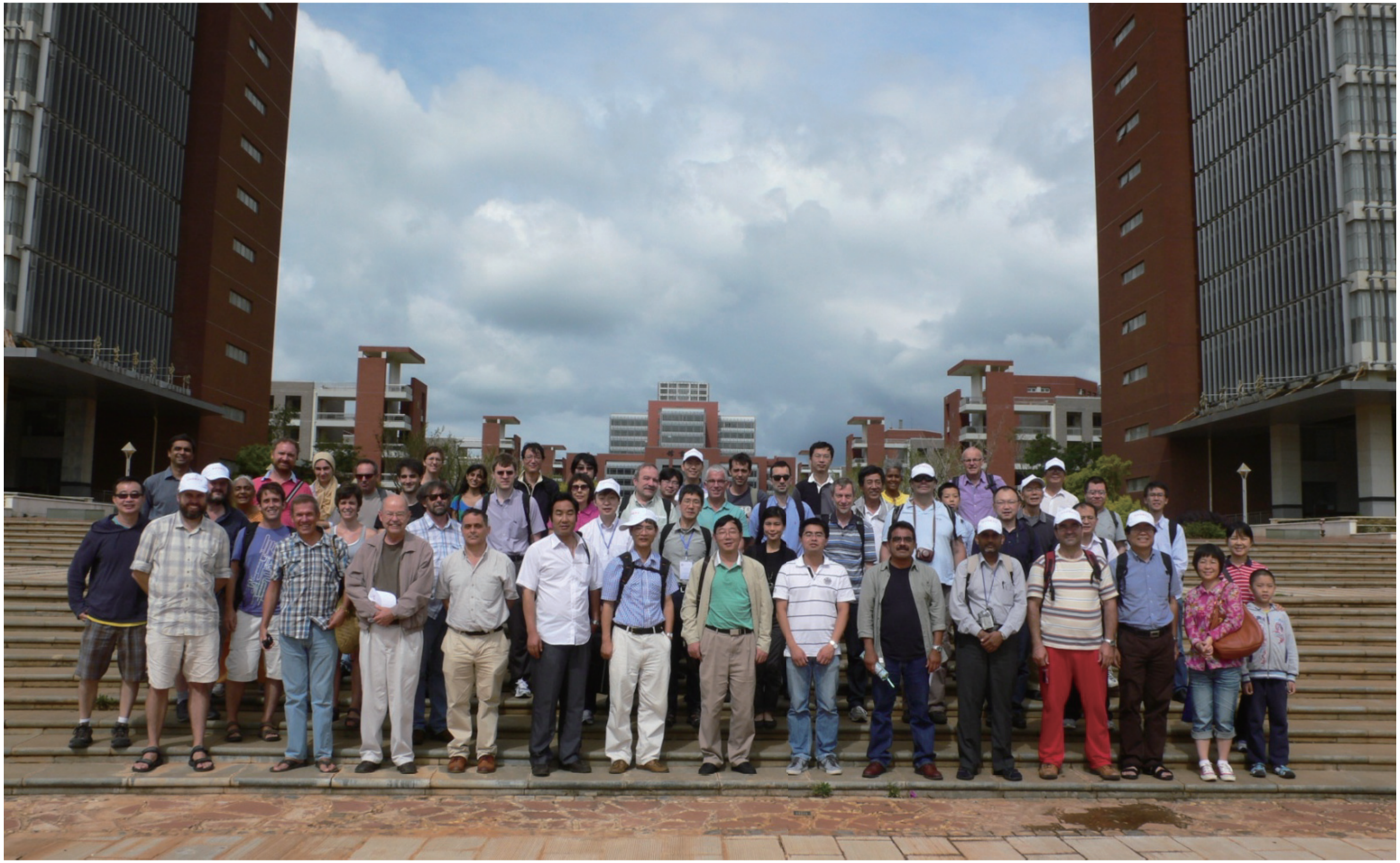
ISBN: 978-1-57146-300-5

Printed in the United States of America.

19 18 17 16 15 1 2 3 4 5 6 7 8 9

ADVANCED LECTURES IN MATHEMATICS

A group picture in front of the library of the original campus of Kunming University of Science and Technology



A group picture on the new campus of Kunming University of Science and Technology

Foreword to Volumes I and II

The decision of editing this Handbook came after an international conference we organized in Kunming (the capital of the Yunnan Province, China) on July 21–29, 2012, whose theme was "Group Actions and Applications in Geometry, Topology and Analysis".

Kunming is a wonderful place for meetings and for mathematical discussions, especially in the summer, when the weather is most favorable. The conference was a success, from the mathematical and the human point of view. The city is warm, and the landscape is beautiful. There is a big lake, and a mountain behind the lake. Mathematicians like beauty. Hermann Weyl said: "My work always tried to unite the truth with the beautiful, but when I had to choose one or the other, I usually chose the beautiful." (Quoted in *Hermann Weyl's Legacy*, Institute for Advanced Study.)

The first two volumes of this Handbook are a record of the Kunming conference, but above all, we want them to be a convenient source for people working on or studying group actions. In spite of the fact that there were 63 talks, we covered at Kunming only a small part of this broad subject. In fact, group actions are so important that it is surprising that there was no available handbook on that subject so far. It is certainly the ubiquity of group actions that makes such a project so vast and therefore difficult to attain, and our aim for the time being is to start it. The present two volumes are the first on this important subject, and more volumes in the same series will appear in the future. Other conferences on the same subject are also planned in the future; the next one will be in Sanya (Hainan Province).

This Handbook will serve as an introduction and a reference to both beginners, non-experts, experts and users of group actions.

The conference in Kunming would not have gone so smoothly without the generous and devoted help of the local organizers, namely, Provost Ailing Gong, Dean Xianzhi Hu, Party Secretary Fengzao Yang, Deputy Dean Yaping Zhang, Youwei Wen, and Jianqiang Zhang from the Kunming University of Science and Technology. We would like to thank them for their work and hospitality.

Many people have also helped with refereeing and reviewing the papers in this Handbook, and we would like to thank them all for their help.

L. Ji, A. Papadopoulos, and S.-T. Yau
Ann Arbor, Strasbourg, and Cambridge MA
November, 2014

Introduction

The subject of this Handbook is groups and group actions. Although groups are omnipresent in mathematics, the notion of group was singled out relatively recently. We recall that the first definition of a (finite) group was formulated by Cayley around the middle of the nineteenth century. But the concept itself is inherent in the work of Galois (as a group of permutations of solutions of polynomial equations), and it is also contained — at least implicitly — in works of Ruffini, Lagrange and Gauss.

On the other hand, the idea of group is closely related to that of symmetry, or rather, to the mathematics behind symmetry, and the use of groups, seen as symmetries, can be traced back to antiquity. In fact, the notion of symmetry reflects a group action, not only in mathematics, but also in other sciences, including chemistry, biological physics, and the humanities. Symmetry is also one of the most fundamental concepts in art.

In mathematics, the notion of abstract group is at the heart of the formulation of many problems. Still, it is usually the concept of transformation group, or of a group acting on a space, rather than that of group alone, which is of fundamental importance. A group action brings in an additional notion to the group at hand, coming from the space on which the group acts. It is also the group action which makes groups interesting, useful and understandable. The precise identification of a group with a group of symmetries of a space is made through the action of the group on that space. But as the same group can act on different spaces, this group can be realized in several different ways as a group of symmetries.

The notion of transformation group was inherent in eighteenth-century geometry, in particular in projective geometry. But it was Klein, in his *Erlangen program* manifesto, and mathematicians like Lie, Poincaré and others who worked in the spirit of this program (some of them without being aware of the program) who highlighted the importance of a transformation group as a basic concept associated to a geometry, with the view that a geometry is characterized (and, in a certain way, it is defined) by a transformation group rather than by a space.

Central to contemporary research is the study of discrete group actions on homogeneous spaces, in particular on manifolds of constant curvature and locally symmetric spaces of finite volume. The most famous of such groups are probably the Fuchsian groups, the Kleinian groups and the arithmetic subgroups of semisimple Lie groups, where not only the groups are studied individually, but their deformation theory is also very rich.

Other interesting classes of examples of infinite groups are, on the one hand, the automorphism groups $\mathrm{Aut}(F_n)$ and the outer automorphism groups $\mathrm{Out}(F_n)$

of a free group F_n on n generators ($n \geq 2$), and on the other hand, the automorphism groups $\mathrm{Aut}(\pi_1(S_g))$ and the outer automorphism groups $\mathrm{Out}(\pi_1(S_g))$ of fundamental group of (say, closed) surfaces S_g of genus g ($g \geq 2$), i.e., the mapping class groups of S_g. One can also mention the Coxeter groups.

It can easily be argued that the free group is much simpler to apprehend than a surface group; for instance, one can easily visualize the Cayley graph of the free group F_n, a regular tree with vertices of order $2n$, and hence, one can have a good picture of the geometry and combinatorics of that group, whereas the Cayley graph of the fundamental group $\pi_1(S_g)$ is more complex. However, it turns out that the theory of automorphism and of outer automorphisms of the free group F_n is much less understood than that of the automorphism (and the outer automorphism) group of the surface group S_g. The reason is that many actions of $\mathrm{Aut}(\pi_1(S_g))$ and $\mathrm{Out}(\pi_1(S_g))$ arising naturally from the geometry and the topology of surfaces have been studied, whereas for the groups $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$, there are not as many actions on geometric or topological spaces. The Coxeter groups are understood via to their action on Coxeter complexes.

The reader can refer to the beginning of the article by L. Ji in this volume, where many group actions are listed.

The present volume of the *Handbook of Group Actions* is more especially concerned with discrete group actions. It consists of 12 chapters, and it is divided into four parts. Each part emphasizes special discrete groups and their actions.

Part I: Geometries and General Group Actions

This part contains 2 chapters.

Chapter 1 is by S.-T. Yau. It is a record of the talk that the author gave at the Kunming conference, whose main theme was a view of a generalized geometry based on the notion of operators rather than on that of space. The relation with physics is also discussed. The concept of group is essential here, as a group of operators and as a gauge group. Several constructions of Riemannian geometry can be done in this setting, including the definitions of the Dirac and the Laplace operators, the differential topology of operator geometry, Hodge theory, Yang-Mills theory and conformal field theory. There is also a version of that theory for discrete spaces.

Chapter 2 is by L. Ji, and it is a summary of group actions that arise in mathematics. It attempts to cover all the major fields where group actions play an important role and to convey a sense of how broad group actions are in mathematics and other sciences. Hopefully it will give some content to the statement that group actions and symmetry, which are the same thing, are everywhere.

Part II: Mapping Class Groups and Teichmüller Spaces

This part concerns mapping class groups and Teichmüller spaces. The two topics are related, because the action of the mapping class group of a surface on the Teichmüller space of that surface constitutes one of the most interesting (and may be the most interesting) action of that group, in terms of the richness and the developments of the underlying theory, and also in terms of applications. Furthermore, Teichmüller spaces equipped with actions of mapping class groups are the primary source of holomorphic group actions in high dimensions, including infinite dimensions. Teichmüller spaces are also related in other ways to the subject of group actions; for instance, an element of a Teichmüller space can be seen as a Fuchsian group acting on hyperbolic 2-space. This makes a relation between the present section and the section in Volume II of this handbook which deals with representations and deformations of subgroups of Lie groups.

In Chapter 3, A. Papadopoulos surveys some actions of mapping class groups. The latter admit actions which are of very different natures on spaces associated to surface: group-theoretic, holomorphic, combinatorial, topological, metric, piecewise-linear, etc. The author reviews in more detail actions on spaces of foliations and laminations, namely, measured foliations, unmeasured foliations, general geodesic laminations and the reduced Bers boundary. The chapter also contains a section on perspectives and open questions on actions of mapping class groups.

In Chapter 4, W. Su surveys two horofunction compactifications of Teichmüller space which are also spaces on which the mapping class group naturally acts. The horofunction boundary of a space is defined relatively to a certain metric. The two horofunction spaces that are studied in this chapter are associated to the Teichmüller metric and to the Thurston metric. The relation between these compactifications with Thurston and Gardiner-Masur's compactifications is reviewed (results of Walsh and of Lui and Su), and the isometry groups of Teichmüller space equipped with the two metrics are considered.

In Chapter 5, F. Herrlich studies Teichmüller disks, that is, embeddings of the hyperbolic disk in Teichmüller space that are holomorphic and isometric. More precisely, the author studies the stabilizers of these discs in the Schottky space $\mathcal{S}_g$ of a closed Riemann surface of genus g, a quotient of the Teichmüller space $\mathcal{T}_g$ by a certain (non-normal) torsion-free subgroup of the mapping class group. The Schottky space is an infinite orbifold covering of Riemann's moduli space. The stabilizer of a Teichmüller disk is sometimes a lattice in $\mathrm{PSL}(2,\mathbb{R})$. Schottky space is, like Teichmüller space, a complex manifold. The author studies in particular the stabilizers in Schottky space of the Teichmüller disks and more generally the behavior of these disks under the covering map $\mathcal{S}_g \to \mathcal{T}_g$.

Chapters 6 and 7 concern infinite-dimensional Teichmüller spaces.

Chapter 6 by E. Fujikawa concerns actions of mapping class groups of surfaces of infinite type. There are various groups which play the role of a mapping class groups, and various spaces which play the role of Teichmüller spaces, in this infinite-dimensional setting, and the author considers some of them. In particular, she considers the action of the so-called asymptotically trivial mapping class group on the asymptotic Teichmüller space, a space which was first introduced by Sullivan. The main result she describes in this context is that for surfaces satisfying a condition of bounded geometry (a quasi-isometry invariant condition which involves lower and upper bounds on certain classes of geodesics, when the Riemann surface is equipped with a hyperbolic metric), the asymptotically trivial mapping class group coincides with that of the so-called stable quasiconformal mapping class group, that is, the subgroup of conformal mapping classes which have representatives which are the identity outside a compact subset. She then introduces another Teichmüller space, which is called the intermediate Teichmüller space, which is the quotient of the classical (quasiconformal) Teichmüller space by the asymptotically trivial mapping class group. Under the same bounded geometry condition, this space inherits a complex structure from that of the quasiconformal Teichmüller space. In general, the asymptotically trivial Teichmüller modular group is a proper subgroup of the group of holomorphic automorphisms of the asymptotic Teichmüller space. The author then studies the dynamics of the various actions that arise, and conditions under which such group actions are properly discontinuous. She also gives an asymptotic version of the Nielsen realization problem.

Chapter 7 by K. Matsuzaki is a survey of the complex analytic theory of the universal Teichmüller space and of some of its subspaces. Roughly speaking, the universal Teichmüller space is the space of equivalence classes of hyperbolic metrics on the unit disc, where two structures are considered equivalent if they differ by an isotopy which induces the identity on the boundary S^1 of the disc. This space can also be defined as a certain quotient of the group of diffeomorphisms of the unit circle. It is termed universal because it contains naturally the Teichmüller spaces of all hyperbolic surfaces. In this theory, the representation of the elements of a Teichmüller space by Fuchsian groups is useful if not essential. One of the important concepts that are studied in detail in this chapter is a natural subset of the universal Teichmüller space which is not the Teichmüller space of a surface, namely, a space of equivalence classes of diffeomorphisms of the circle with Hölder continuous derivatives. The author shows that this space is equipped with a complex structure modeled on a complex Banach space. This complex structure is described through a careful study of the Bers embedding of the space in the space of Schwarzian derivatives. The diffeomorphisms of the circle with Hölder continuous derivatives are characterized by certain properties of their quasiconformal extensions to the unit disc, and the theory bears relations with the space of asymptotically conformal maps studied by Carleson.

Chapter 8 by T. Satoh concerns mapping class groups of surfaces, and it has a more algebraic nature. It is a survey of the Johnson homomorphisms associated to mapping class groups. These are homomorphisms associated to graded quotients of a certain descending filtration of these groups. Johnson defined in the 1980s the first homomorphism in the sequence, as a tool to study the Torelli group. A similar theory for automorphisms of free groups was developed before, by Andreadakis, in his thesis in the 1960s. The Johnson homomorphism for surface mapping class groups was generalized later on to the so-called Johnson homomorphisms of higher degrees, and several people did extensive work on them, including Morita, Hain, Satoh and others, and there are recent results on the same subject by Kawazumi-Kuno and by Massuyeau-Turaev.

Part III: Hyperbolic Manifolds and Locally Symmetric Spaces

Chapter 9 by G. J. Martin is a survey on the various aspects of the geometry and arithmetic of Kleinian groups. The author examines the geometry of Kleinian groups and he gives geometric conditions on isometry groups of hyperbolic 3-space in order to be discrete. He studies in detail the two-generator groups, giving several generalizations of Jørgensen's inequality for discreteness, and he discusses the classification of arithmetic generalized triangle groups. One motivation for this study is a problem which Siegel posed in 1943, namely, to identify the minimal covolume lattices of isometries of hyperbolic n-space, and more generally of rank-one symmetric spaces.

Chapter 10 by G. Prasad and A. S. Rapinchuk is a survey on several results related to the basic question: *Can you hear the shape of a drum*? They concern locally symmetric spaces of finite volume. The problem asks whether two Riemannian manifolds having the same spectrum, i.e., the same set of eigenvalues, are isometric. A closely related question is the so-called iso-length spectrum problem for locally symmetric spaces: if two Riemannian manifolds have the same length spectrum, i.e., the same set of lengths of closed geodesics, are they isometric or at least commensurable? The major portion of this paper deals with this latter question and with related problems on algebraic groups and their maximal algebraic tori and the authors give a fairly complete and detailed survey of results in this direction.

Part IV: Knot Groups

This part contains two chapters on representations of knot groups and twisted Alexander polynomials. The twisted Alexander polynomial is defined as a pair consisting of a group and a representation of that group. It generalizes the classical Alexander polynomial. The twisted Alexander polynomial is naturally defined for links in S^3 and more generally for finitely presentable groups. In some instances it can easily be calculated.

Chapter 11 by T. Morifuji is a survey on representations of knot groups and twisted Alexander polynomials, with a special focus on the twisted Alexander polynomial for finitely presentable groups introduced by Wada. This polynomial is associated to a representation into $\mathrm{SL}(2,\mathbb{C})$. There are applications to fibering and genus detecting problems of knots in S^3. The twisted Alexander polynomial of a knot is seen as a $\mathbb{C}$-valued rational function on the character variety of the knot group, and it is also expressed in terms of Reidemeister torsion. The chapter also contains a comprehensive introduction to the classical Alexander polynomials and to the algebraic theory which is behind it (presentations of knot groups, Wirtinger presentations, Tietze transformations and Fox derivatives), as well as on representations of knot groups into $\mathrm{SL}(2,\mathbb{C})$, their character varieties and their deformations. The author focuses on the deformation of an abelian representation to a nonabelian one and of a reducible representation to an nonreducible one.

Chapter 12 by M. Suzuki is devoted to the study of the existence of epimorphisms between knot groups. The author indicates by some examples how to detect the existence of a meridional epimorphism (that is, an epimorphism that preserves meridians) between knot groups and he gives explicit descriptions of some non-meridional epimorphisms. He shows that the existence of an epimorphism between finitely presentable groups implies that their twisted Alexander polynomials are divisible. He makes connections with other works on the subject, and in particular with the so-called Simon conjecture (a problem in Kirby's list) whose general case was settled recently by Agol and Liu. The result says that every knot group admits an epimorphism onto at most finitely many knot groups.

Contents

Part I: Geometries and General Group Actions

Part II: Mapping Class Groups and Teichmüller Spaces

Part III: Hyperbolic Manifolds and Locally Symmetric Spaces