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Compressible Flow and Euler's Equations

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Introduction

The equations describing the motion of a perfect fluid were first formulated by Euler in 1752 (see [9], [10]), based, in part, on the earlier work of D. Bernoulli [1]. These equations were among the first partial differential equations to be written down, preceded, it seems, only by D'Alembert's 1749 formulation [8] of the 1-dimensional wave equation describing the motion of a vibrating string in the linear approximation. In contrast to D'Alembert's equation, however, we are still, after the lapse of two and a half centuries, far from having achieved an adequate understanding of the observed phenomena which are supposed to lie within the domain of validity of Euler's equations.

The phenomena displayed in the interior of a fluid fall into two broad classes, the phenomena of sound, the linear theory of which is acoustics, and the phenomena of vortex motion. The sound phenomena depend on the compressibility of a fluid, while the vortex phenomena occur even in a regime where the fluid may be considered to be incompressible. The formation of shocks, the subject of the present monograph, belongs to the class of sound phenomena, but lies in nonlinear regime, beyond the range covered by the linear acoustics.

Let us make a short review of the history of the study of the sound phenomena in fluids, in particular the phenomena of the formation of shocks in the nonlinear regime. At the time when the equations of the fluid motion were formulated, thermodynamics was in its infancy, however, it was already clear that the local state of a fluid as seen by a comoving observer is determined by two thermodynamic variables, say pressure and temperature. Of these, only pressure entered the equations of motion, while the equations involve also the density of the fluid. Density was already known to be a function of pressure and temperature for a given type of fluid. However, in the absence of an additional equation, the system of equations at the time of Euler, which consisted of the momentum equations and the equation of continuity, was underdetermined, except in the incompressible limit. The additional equation was supplied by Laplace in 1816 [13] in the form of what was later to be called adiabatic condition, and allowed him to make the first correct calculation of the sound speed.

The first work on the formation of shocks was done by Riemann in 1858 [17]. Riemann considered the case of isentropic flow with plane symmetry, where the equations of fluid mechanics reduces to a system of conservation laws for two unknowns and with two independent variables, a single space coordinate and time. He introduced for such systems the so-called Riemann invariants, and with the help of these showed that solutions which arise from smooth initial conditions develop infinite gradients in finite time.

In 1865 the concept of entropy was introduced into theoretical physics by Clausius [7], and the adiabatic condition was understood to be the requirement that the entropy of each fluid element remains constant during its evolution.

The first general result on the formation of singularity in 3-dimensional fluids was obtained by Sideris in 1985 [18]. Sideris considered the compressible Euler equations in the case of a classical ideal gas with adiabatic index $\gamma > 1$ and with initial data which coincide with those of a constant state outside a ball. The assumptions of his theorem on the initial data were that there is an annular region bounded by the sphere outside which the constant state holds, and a concentric sphere in its interior, such that a certain integral in this annular region of $\rho - \rho_0$, the departure of the density ρ from its value ρ_0 in the constant state, is positive, while another integral in the same region of ρv^r, the radial momentum density, is non-negative. These integrals involve kernels which are functions of the distance from the center. It is also assumed that everywhere in the annular region the specific entropy s is not less than its value s_0 in the constant state. The conclusion of the theorem is that the maximal time interval of existence of a smooth solution is finite. The chief drawback of this theorem is that it tells us nothing about the nature of the breakdown. Also the method relies the strict convexity of the pressure as a function of density displayed by the equation of state of an ideal gas, and does not extend to more general equation of state.

The most recent and complete results on the formation of shocks in three dimensional fluids were obtained by Christodoulou in 2007 [5]. Christodoulou considered the relativistic Euler equations in three space dimensions for a perfect fluid with an arbitrary equation of state. He considered the regular initial data on a spacelike hyperplane Σ_0 in Minkowski spacetime which outside a sphere coincide with the data corresponding to a constant state. He considered the restriction of the initial data to the exterior of a concentric sphere in Σ_0 and the maximal classical development of this data. Under suitable restriction on the size of the departure of the initial data from those of the constant state, he proved certain theorems which give a complete description of the maximal classical development. In particular, theorems give a detailed description of the geometry of the boundary of the domain of the maximal classical solution and a detailed analysis of the behavior of the solution at this boundary.

The aim of the present monograph is to derive analogous results for the classical, non-relativistic, compressible Euler's equations taking the data to be irrotational and isentropic, and to give a proof of these results which is considerably simpler and completely self-contained. The present monograph in fact not only gives simpler proofs but also sharpens some of the results. In addition the present monograph explains in depth the ideas on which the approach is based. Finally certain geometric aspects which pertain only to the non-relativistic theory are discussed.

We shall presently explain the basis of the approach of the present monograph (also of the previous one dealing with the relativistic case). This basic idea can be thought as an extension of the method of Riemann invariants combined with the method of the partial hodograph transformation, to the case of more than one space dimension. We first recall some basic facts about Riemann invariants. In the case of one space dimension, the isentropic Euler system reads

$$\partial_t \rho + \partial_x(\rho v) = 0$$
$$\partial_t v + v\partial_x v = -\frac{1}{\rho}\partial_x p$$

and it can be written as a single equation of the velocity potential ϕ:

$$(g^{-1})^{\mu\nu}\partial_\mu\partial_\nu\phi = 0$$

or, setting $\psi_\mu = \partial_\mu\phi$,

$$(g^{-1})^{\mu\nu}\partial_\mu\psi_\nu = 0$$

where g is the following Lorentzian metric on the spacetime manifold $\mathcal{M}$:

$$g = -\eta^2 dt^2 + (dx - vdt)^2.$$

Here

$$v = -\psi_x$$

is the fluid velocity and

$$h = \psi_t - \frac{1}{2}\psi_x^2$$

is the enthalpy. The pressure p is a given function of h and

$$\rho = dp/dh$$

while the sound speed η is given by

$$\eta^2 = dp/d\rho$$

Riemann invariants are the functions R_1, R_2 defined on the cotangent space $T_p^*\mathcal{M}$, which are the two functionally independent solutions of the following eikonal equation:

$$g_{\mu\nu}\frac{\partial R}{\partial\psi_\mu}\frac{\partial R}{\partial\psi_\nu} = 0$$

i.e.

$$-\eta^2(\frac{\partial R}{\partial\psi_t})^2 + (\frac{\partial R}{\partial\psi_x} + \psi_x\frac{\partial R}{\partial\psi_t})^2 = 0$$

We define the vectorfields N_1, N_2 on $\mathcal{M}$ by

$$N_1 := \frac{\partial R_1}{\partial \psi_\mu} \frac{\partial}{\partial x^\mu}, \quad N_2 := \frac{\partial R_2}{\partial \psi_\mu} \frac{\partial}{\partial x^\mu}$$

These are null vectorfields with respect to g. We choose R_1 and R_2 so that the integral curves of N_1 and N_2 are the incoming and outgoing null curves respectively. Then R_1, R_2 as functions on $\mathcal{M}$ satisfy

$$N_2 R_1 = 0, \quad N_1 R_2 = 0 \tag{1}$$

Let us introduce the acoustical coordinates (t, u) so that u is constant along the outgoing null curves. Then the vectorfields

$$L = \frac{\partial}{\partial t}, \quad \underline{L} = \eta^{-1} \kappa L + 2T$$

where

$$T = \frac{\partial}{\partial u} \quad \text{and} \quad \kappa = -\frac{\partial x}{\partial u} \tag{2}$$

are null vectorfields with respect to g, and the integral curves of L and $\underline{L}$ are outgoing and incoming null curves respectively. Therefore L and $\underline{L}$ are colinear to N_2 and N_1 respectively. Therefore Equations (1) are equivalent to

$$L R_1 = 0, \quad \underline{L} R_2 = 0 \tag{3}$$

To write down explicit expressions for R_1, R_2 we use, instead of (ψ_t, ψ_x), the following variables in $T_p^* \mathcal{M}$:

$$h = \psi_t - \frac{1}{2} \psi_x^2, \quad v = -\psi_x$$

Then for any function $f = f(h, v)$ defined on $T_p^* \mathcal{M}$, we have

$$\frac{\partial f}{\partial \psi_x} + \psi_x \frac{\partial f}{\partial \psi_t} = -\frac{\partial f}{\partial v}$$

Let us introduce a function $r = r(h)$ by

$$\frac{dr}{dh} = \frac{1}{\eta}, \quad r(0) = 0$$

Then

$$R_1 = r + v, \quad R_2 = r - v$$

are the two functionally independent solutions of the eikonal equation, therefore the two Riemann invariants. From the first equation of (3), we know that

$$R_1 = R_1(u)$$

is determined by initial data, while to obtain R_2, we consider the second equation of (3), namely the equation

$$\eta^{-1}\kappa\frac{\partial R_2}{\partial t}+2\frac{\partial R_2}{\partial u}=0$$

Here κ enters, which is defined by (2). To obtain an equation for κ we consider the following equations:

$$\frac{\partial x}{\partial t}=c_+,\quad c_+=v+\eta$$

We shall derive an equation for κ. From (3) we have

$$\frac{\partial\kappa}{\partial t}=-\frac{\partial c_+}{\partial u}=-\frac{1}{2}\frac{\partial R_1}{\partial u}+\frac{1}{2}\frac{\partial R_2}{\partial u}-\frac{d\eta}{dh}\frac{\partial h}{\partial u}$$

while

$$\frac{\partial h}{\partial u}=\frac{dh}{dr}\frac{\partial r}{\partial u}=\frac{1}{2}\eta(\frac{\partial R_1}{\partial u}+\frac{\partial R_2}{\partial u})$$

Substituting the above we obtain

$$\frac{\partial\kappa}{\partial t}=\frac{1}{2}(-1-\eta\frac{d\eta}{dh})\frac{\partial R_1}{\partial u}+\frac{1}{2}(1-\eta\frac{d\eta}{dh})\frac{\partial R_2}{\partial u}$$

Since

$$\frac{\partial R_1}{\partial u}=\frac{2}{\eta}\frac{\partial h}{\partial u}-\frac{\partial R_2}{\partial u}$$

we have

$$\frac{\partial\kappa}{\partial t}=\frac{1}{2\eta}(-2-2\eta\frac{d\eta}{dh})\frac{\partial h}{\partial u}+\frac{\partial R_2}{\partial u}$$

Let us define

$$H=-2h-\eta^2$$

Noting that by the second equation of (3)

$$\frac{\partial R_2}{\partial u}=-\frac{\kappa}{2\eta}\frac{\partial R_2}{\partial t}$$

We conclude that κ satisfies the following equation:

$$\frac{\partial\kappa}{\partial t}=m'+\kappa e'$$

with

$$m'=\frac{1}{2\eta}\frac{dH}{dh}\frac{\partial h}{\partial u},\quad e'=-\frac{1}{2\eta}\frac{\partial R_2}{\partial t}$$

The main idea in the 1-dimensional case is that R_1, R_2 as well as the rectangular coordinate x are smooth functions of (t, u). The partial hodograph transformation is the transformation

$$(t, u) \longmapsto (t, x)$$

from acoustical to rectangular coordinates. The Jacobian is

$$\frac{\partial(t,x)}{\partial(t,u)} = \begin{vmatrix} 1 & 0 \\ v+\eta & -\kappa \end{vmatrix} = -\kappa$$

and vanishes when κ vanishes. This means R_1, R_2 are not smooth in (t, x) when shocks form.

In the case of more than one space dimension, in particular, the case of three space dimensions, we do not have Riemann invariants. We work instead with the first order variations, which are defined through the *variation fields*:

$$\partial_\mu, \quad \mathring{R}_i = \epsilon_{ijk} x^j \frac{\partial}{\partial x^k}, \quad x^\mu \partial_\mu - I; \quad \mu = 0,1,2,3, \quad 1 \leqslant i,j,k \leqslant 3$$

Here I is the multiplication operator by 1. These fields are the generators of the subgroup of the scale-extended Galilean group, the invariance group of the compressible Euler system, which leaves the constant state invariant. These first order variations satisfy the linear wave equation

$$\Box_{\tilde{g}} \psi = 0 \tag{4}$$

where $\tilde{g}$ is the conformal acoustical metric:

$$\tilde{g} = \Omega g, \quad \Omega = \frac{\rho}{\eta}, \quad g = -\eta^2 dt^2 + \sum_i (dx^i - v^i dt)^2$$

Like the equations satisfied by R_1, R_2 in the case of one space dimension, Equation (4) does not depend on the Galilean structure, but depends only on the properties of $(\mathcal{M}, g)$ as a Lorentzian manifold. Actually, it depends more sensitively on the conformal class of $(\mathcal{M}, g)$. This is similar to the fact that in the case of one space dimension, null curves depend only on the conformal class of the acoustical metric. Also like in the case of one space dimension, we shall work in the *acoustical coordinates* (t, u, ϑ). Here u is the *acoustical function* in $\mathcal{M}$, whose level sets C_u are outgoing null hypersurfaces. The level curves of $\vartheta \in \mathbb{S}^2$ on each C_u are the generators of C_u. Like in the case of one space dimension, we denote by L the tangent vectorfield of the generator of C_u parametrized by t, and $\underline{L}$ the incoming null normals of $S_{t,u} := C_u \bigcap \Sigma_t$, whose definitions are formally the same as one space dimensional case. We also denote by T the tangent vectorfield of the inward normal curves to the $S_{t,u}$ in Σ_t parametrized by u. Here Σ_t is the level set of the function t, which is isometric to the Euclidean space.

To obtain a fundamental energy estimate for this linear equation in (t, u, ϑ), we need two *multiplier vectorfields* K_0, K_1. These are non-spacelike and future-directed with respect to g (a requirement which actually depends only on the conformal class of g). They are linear combinations of L and $\underline{L}$, with coefficients which are smooth in (t, u, ϑ). The concept of multiplier vectorfields originates from Noether's theorem [14] on conserved currents. A modern more general treatment of *compatible currents* is found in [4]. In order to obtain higher order energy estimates, we consider the nth order variations by applying a string of *commutation vectorfields* of length $n-1$ to the first order variations:

$$T, \quad Q := (1+t)L, \quad R_i = \Pi \mathring{R}_i$$

Here Π is the orthogonal projection from $T_p\Sigma_t$ to $T_pS_{t,u}$. Then we obtain an inhomogeneous wave equation for the nth order variation ψ_n:

$$\Box_{\tilde{g}}\psi_n = \rho_n \tag{5}$$

where ρ_n is determined by the deformation tensors of the commutation vectorfields. Actually ρ_n depends on up to the $(n-1)$th order derivatives of the deformation tensors, and $\rho_1 = 0$. The use of commutation fields originates in [12]. Multiplier fields and commutation fields on general curved spacetimes have first been used in [6].

After we solve (5) in the acoustical coordinates, we need to go back to the original rectangular coordinates. Again we must consider the inverse of the transformation:

$$(t, u, \vartheta) \longmapsto (t, x)$$

where $\vartheta \in \mathbb{S}^2$ and $x \in \mathbb{R}^3$. This is what replaces the partial hodograph transformation in higher dimensions. The Jacobian of this transformation is

$$\kappa\sqrt{\det \not{g}} \tag{6}$$

Here $\not{g}$ is the induced acoustical metric on $S_{t,u}$. So we consider the system satisfied by the rectangular coordinates on each C_u:

$$\frac{\partial x^i}{\partial t} = L^i = -\eta\hat{T}^i - \psi_i$$

which is a fully nonlinear system for x^i. Here $\hat{T}$ is the inward unit normal of $S_{t,u}$ in Σ_t, whose expression is the ratio of a homogeneous quadratic polynomial in $\frac{\partial x^i}{\partial \vartheta^A}$ to the square root of a homogeneous quartic polynomial in $\frac{\partial x^i}{\partial \vartheta^A}$. The estimates of the derivatives of x^i reduce to the estimates of the derivatives of χ and μ. These are defined as follows:

$$2\chi = \not{\mathcal{L}}_L \not{g}, \quad \mu = \eta\kappa$$

where κ is the magnitude of T. Thus χ is the second fundamental form of $S_{t,u}$ in C_u. Finally the way we estimate χ, μ is to study the geometric structure equations of the foliation of $\mathcal{M}$ by surfaces $S_{t,u}$.

To summarize, in the case of n space dimensions, the role of Riemann invariants is played by the first order variations ψ, which shall be proved to be smooth functions of (t, u, ϑ). Moreover, we shall show that the x^i are also smooth functions of (t, u, ϑ). This shall be done through estimates on χ and μ based on the geometric structure equations. One of these equations is the same as in the one space dimensional case:

$$L\mu = m + \mu e$$

where

$$m = \frac{1}{2}\frac{dH}{dh}Th, \quad H = -2h - \eta^2$$

and

$$e = \frac{1}{2\eta^2}(\frac{\rho}{\rho'})'Lh + \frac{1}{\eta}\hat{T}^i(L\psi_i)$$

As a consequence of these facts, the boundary of the maximal classical development contains a singular part $\mathcal{H}$, where the Jacobian (6) vanishes. Since $\sqrt{\det \not{g}}$ is, by virtue of the estimates for χ, bounded from below by a positive constant, κ, equivalently μ, vanishes on $\mathcal{H}$. Thus, the inverse of the transformation

$$(t, u, \vartheta) \longmapsto (t, x)$$

is not differentiable at $\mathcal{H}$. Therefore the ψ are not differentiable with respect to the rectangular coordinates at $\mathcal{H}$. Nevertheless, $\mathcal{H}$, the zero-level set of μ, a smooth function of (t, u, ϑ), is a smooth hypersurface in $\mathcal{M}$ relative to the differential structure induced by the acoustical coordinates. This is because we can show that $L\mu$ is bounded from above by a strictly negative function at $\mathcal{H}$, therefore $\mathcal{H}$ is a non-critical level set of μ.

The first main result in the present monograph can be thought as an existence theorem (**Theorem 17.1**) for the nonlinear wave equation of the velocity potential

$$(g^{-1})^{\mu\nu}\partial_\mu\partial_\nu\phi = 0$$

with small initial data:

The solution of the above nonlinear wave equation can be extended smoothly to the boundary of the maximal solution in acoustical coordinates (t, u, ϑ), and the solution is also smooth in rectangular coordinates before μ becomes 0, since the differential structures induced by acoustical coordinates and rectangular coordinates are equivalent to each other when $\mu > 0$.

Theorem 17.1 also gives a lower bound for the time when the shock forms (i.e. $\mu = 0$), and the energy estimates for solution as well as various geometric quantities associated to the acoustical spacetime $(\mathcal{M}, g)$.

Based on this existence theorem, we find some conditions on initial data which guarantee the formation of shocks in finite time (see **Theorem 18.1**). The conditions are imposed on a Σ_t-integral of the following function:

$$(1 - u + t)\underline{L}\psi_0 - \psi_0 \tag{7}$$

where

$$\psi_0 := \partial_t \phi$$

is one of the first order variations. The principal part of function (7), namely, $(1-u+t)\underline{L}\psi_0$ determines the properties of function m, which, in turn, determines the formation of shocks. Moreover, the spherical mean of function (7) on $S_{t,u}$ satisfies an ordinary differential inequality in the parameter t. Then we can connect the properties of m near the point where the shocks form and the properties of m on the initial hypersurface Σ_0 by using this ordinary differential inequality. Then the necessary properties of m then follow from its properties on the initial hypersurface Σ_0.

Also based on the existence theorem, we can give a geometric description of the boundary of the maximal classical solution in acoustical differential structure (**Proposition 19.1**):

The boundary contains a regular part $\underline{C}$*, which is an incoming null hypersurface in* $(\mathcal{M}, g)$*, and a singular part* $\mathcal{H}$*, on which the function* μ *vanishes.* $\mathcal{H}$ *is a spacelike hypersurface in* $(\mathcal{M}, g)$*, and it has the common past boundary with* $\underline{C}$*, denoted by* $\partial_{-}\mathcal{H}$*, which is a 2-dimensional spacelike surface in* $(\mathcal{M}, g)$*. However, the singular boundary, from the intrinsic point of view, is a null hypersurface in* $(\mathcal{M}, g)$*, on which the acoustical metric* g *degenerates in acoustical coordinates.*

The corresponding description of the singular boundary in the standard differential structure (that is, in rectangular coordinates) is given in **Proposition 19.3**.

Moreover, we establish a trichotomy theorem (**Theorem 19.1**) describing the behavior of the past-directed null geodesics initiated at the singular boundary:

For each point q *of the singular boundary, the intersection of the past null geodesic conoid of* q *with any* Σ_t *in the past of* q *splits into three parts, the parts corresponding to the outgoing and to the incoming sets of null geodesics ending at* q *being embedded discs with a common boundary, an embedded circle, which corresponds to the set of the remaining null geodesics ending at* q*. All outgoing null geodesics ending at* q *have the same tangent vector at* q*.*

Finally, considering the transformation from one acoustical function to another, we show that the foliations corresponding to different families of outgoing null hypersurfaces have equivalent geometric properties and degenerate in precisely the same way on the same singular boundary (See **Proposition 19.2**).

Let us now give an outline of the present monograph. The first four chapters concern the geometric set up. Then in Chapter 5, we obtain energy estimates for the linear wave equation associated to the conformal acoustical metric. Chapter 6 deals with the preliminary estimates for the deformation tensor of the commutation vectorfields, the precise estimates of which are given in Chapters 10 and 11. We also introduce the basic bootstrap assumptions on variations as well as on χ and μ in Chapters 5 and 6. Chapters 8 and 9 are crucial in the whole work, because it is here that estimates for χ and μ are derived which do not lose derivatives, thus allowing us to close the bootstrap argument. Chapter 8 concerns the estimates for the top order spatial derivatives of χ. In fact only the top order angular derivatives are involved. While Chapter 9 concerns the estimates for the top order spatial derivatives of μ. In Chapter 12, based on the bootstrap assumptions on variations, we recover the bootstrap assumptions on χ and μ, except **C1**, **C2** and **C3**, which are recovered in Chapter 13. In Chapter 14, based on a crucial lemma (**Lemma 8.11**) established in Chapter 8, we estimate the *borderline* contribution from the top order spatial derivatives of χ and μ. Then in Chapter 15, we obtain the energy estimates for the top order variations. These are allowed to blow up as shocks begin to form. We then revisit the lower order energy estimates and show that the estimates of each preceding order blow up successively more slowly until we finally reach energy estimates of a certain order which do not blow up at all. These bounded energy estimates allow us to close the bootstrap argument (See Chapters 16 and 17).

In regard to the notational conventions, Latin indices take the values $1, 2, 3$, while Greek indices take the values $0, 1, 2, 3$. Repeated indices are meant to be summed, unless otherwise specified.

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