Advanced Lectures in Mathematics (ALM)

ALM 1: Superstring Theory
ALM 2: Asymptotic Theory in Probability and Statistics with Applications
ALM 3: Computational Conformal Geometry
ALM 4: Variational Principles for Discrete Surfaces
ALM 6: Geometry, Analysis and Topology of Discrete Groups
ALM 7: Handbook of Geometric Analysis, No. 1
ALM 8: Recent Developments in Algebra and Related Areas
ALM 9: Automorphic Forms and the Langlands Program
ALM 10: Trends in Partial Differential Equations
ALM 11: Recent Advances in Geometric Analysis
ALM 12: Cohomology of Groups and Algebraic K-theory
ALM 13: Handbook of Geometric Analysis, No. 2
ALM 14: Handbook of Geometric Analysis, No. 3
ALM 15: An Introduction to Groups and Lattices: Finite Groups and Positive Definite Rational Lattices
ALM 16: Transformation Groups and Moduli Spaces of Curves
ALM 17: Geometry and Analysis, No. 1
ALM 18: Geometry and Analysis, No. 2
ALM 19: Arithmetic Geometry and Automorphic Forms
ALM 20: Surveys in Geometric Analysis and Relativity
ALM 21: Advances in Geometric Analysis
ALM 22: Differential Geometry: Under the Influence of S.-S. Chern
ALM 23: Recent Developments in Geometry and Analysis
ALM 24: Handbook of Moduli, Volume I
ALM 25: Handbook of Moduli, Volume II
ALM 26: Handbook of Moduli, Volume III
ALM 27: Number Theory and Related Areas
ALM 28: Selected Expository Works of Shing-Tung Yau with Commentary, Volume I
ALM 29: Selected Expository Works of Shing-Tung Yau with Commentary, Volume II

Advanced Lectures in Mathematics
Volume 28

Selected Expository Works of Shing-Tung Yau with Commentary

Volume I

Companion to the volume
Selected Expository Works of Shing-Tung Yau with Commentary, Volume II

edited by

Lizhen Ji · Peter Li
Kefeng Liu · Richard Schoen

International Press
www.intlpress.com

高等教育出版社
HIGHER EDUCATION PRESS

Advanced Lectures in Mathematics, Volume 28
Selected Expository Works of Shing-Tung Yau with Commentary, Volume I

Companion to the volume
Selected Expository Works of Shing-Tung Yau with Commentary, Volume II

Volume Editors:
Lizhen Ji (University of Michigan)
Peter Li (University of California at Irvine)
Kefeng Liu (University of California at Los Angeles)
Richard Schoen (Stanford University)

Consulting Editor:
Steve Nadis

ISBN: 978-1-57146-293-0

Printed in the United States of America.

18 17 16 15 14 1 2 3 4 5 6 7 8 9

Contents

Preface

In the early spring of 2013, Lizhen Ji asked me to write comments about my collected or selected works. I was too busy at the time to take on such a task. At one point, however, I gave in to his request and decided to write comments about my survey articles. Upon tallying them up, I was surprised to see that I had written far more survey articles than I had remembered.

Since I was a child, I have always been interested in history. Hence when I started to write these commentaries, I tried to stick to the facts to the best of my memory. I also consulted friends who participated in these events and looked at letters and emails that I had kept over the past forty years.

This does not mean that there are absolutely no mistakes in the statements. Nevertheless, I believe that these accounts can be interesting—and maybe even important—for students who'd like to know something about how the various papers were written and what my friends and I thought about the approaches we took.

In the course of putting together this collection, I received strong support from Lizhen Ji, Hao Xu, Kefeng Liu, Shiu-Yuen Cheng, and Hung-Hsi Wu. I am also very grateful to the publishers led by Liping Wang, Yushan Deng, and others. My friend Steve Nadis agreed to be the consulting editor for this project. I am extremely thankful for all of their help, without which this project likely would not have materialized.

Shing-Tung Yau
June 30, 2014

Shing-Tung Yau: His Mathematics and Writings

Lizhen Ji

Why *selected* works?

There has been a long tradition of publishing collected or selected works of distinguished mathematicians. There are several good reasons for doing this, and it has served many purposes. Probably the most obvious one is that collected and selected works provide an easy access to papers that are scattered in different journals, some of which are not easily accessible to many people. Otherwise, few people, if any, will take the time and trouble to dig up all the papers of their admired mathematicians–especially not those papers that are far away from their interests, of their admired mathematicians and read them. On the other hand, reading papers of a master dealing with different subjects or areas conveys the underlying unity and hence a big picture of mathematics, and it also allows one to gain a historical perspective (or to enter the history). In other words, collected and selected works are more than the simple sum of individual papers.

Indeed, as Abel said famously, we learn "by studying the masters, not their pupils." Even though the world is becoming smaller, few people have many chances to interact with masters who are alive. Of course, the next best way to learn from masters is to read and study their collected works.

Naturally, publishing collected or selected works is also an honor to the authors of these papers. It should be mentioned that collected works of some people can bring honor of the genre of collected works.

Now, with the wide and easy use of e-papers and e-books, most papers in journals can be obtained easily online, and a mere reprinting of papers is probably not as valuable as before. Of course, the value of selected works still stands. For example, holding and reading a beautifully printed book is definitely different from viewing papers online or on e-book readers. But they should also provide something else. Several additional things seem to be reasonable: descriptions of how ideas in the paper were formed and time and place the papers were written, relations between papers with the advantage of hindsight, and developments of subjects after the papers were published, and visions for the future. In other words, they should explain the circumstances of the birth of papers and proper, impacts of the papers, and fitting these papers in the grand scheme of mathematics.

These additional things are especially important to beginners, non-experts and even some experts. Most people often concentrate on the best known theorems and most important papers of great mathematicians, but even masters struggled and stumbled sometimes on their mathematical trips. How they found good problems and their ways in their careers, made progress and reached peaks is best described by their own papers, recollections and commentaries, but not textbooks where everything is polished and presented in a streamlined matter, without mentioning that textbooks and research books might not cover some gems in the original papers that are not directly related to the themes of the books. But many people, especially younger ones, often prefer to read polished textbooks.

Of course, reading mathematics papers can be difficult (more difficult than reading textbooks), and proper arrangement of related papers and additional guides from the masters are certainly valuable and helpful. Such collected or selected works of distinguished mathematicians often tell good stories of the authors and their mathematics, and browsing through them can be enjoyable and beneficial to people who are not interested in some specific results in the papers.

In these works of expository writings of Shing-Tung Yau, all these things are printed together with his survey papers and papers on open problems. One reason for restricting these volumes to expository papers of Yau is practical. Yau has been very creative and prolific. The collected works including all his papers (both research and expository papers) up to now will occupy too many volumes. Besides, he is also still very active and productive, and the time for collected works may not be ripe yet.

Why expository writing?

Probably some explanation is needed for publishing these volumes of expository writings of Yau now. Briefly, it is the right time for Yau to share his perspectives and his vision on the broad area of geometric analysis, and his expository writings provide a unique means to this end. They will render a valuable service to the mathematics community.

Colloquium talks have been a common means of communication between mathematicians from different subjects, after they were made successful and popular by Klein and Hilbert in Göttingen about 100 years ago. More expository talks such as "Basic notion seminars" and "What is...?" have also sprung up in many places. They provide effective ways for people to learn and enjoy some beautiful pieces of mathematics, which are outside their fields of specialty. Though there are many books and papers dealing with all kinds of subjects in mathematics, one difficulty is that there are too many of them. It is difficult for people to find the right books and papers, and people may lack the motivation to read mathematics outside their specialties, especially when they involve difficult and technical material. Many people choose to study mathematics not for fame or fortune, but for the beauty and enjoyment of the discipline. To really appreciate the beauty and power of mathematics, one has to roll up one's sleeves and do the work. But not many people can work in many different subjects in mathematics. In the history of mathematics, only a few people have been universal mathematicians. Some obvious names in the recent times include Gauss, Riemann, Poincaré, Hilbert, Weyl, and Hadamard. In spite of the difficulties, one can still enjoy and appreciate many facets of the rich world of mathematics by listening to expository talks and talking to experts. In the abstract world of mathematics, direct interaction and communication is still vital, and the virtual internet is no replacement yet.

Next to listening to talks, one can try to read expository writings and informal comments and notes on technical papers. The former is like colloquium talks, and the latter is like conversations at the colloquium tea or dinner. Expository writings include books, survey papers and descriptions of open problems. It often happens that expository writings are less valuable in a short period than highly technical and original papers, which can give people priorities and more credit. But in the long run, books and expository papers might be read by more people and have a bigger and longer impact. Think of Euler. How many of his papers are still read by people now? But his two elementary books on analysis and algebras are still printed and read by many people. What about Hilbert? His paper on the open problems and his report (or survey) on algebraic number theory are probably most read among his papers. Among contemporary mathematicians, we can think of the expository writings of people such as Atiyah, Milnor and Serre, which have had a huge impact on the modern mathematics. Though not everyone likes every

Photographs

Curriculum Vitae

Shing-Tung Yau (丘成桐)

Last updated on June 12, 2014

ADDRESS:	William Caspar Graustein Professor of Mathematics Department of Mathematics Harvard University Cambridge, MA 02138, U.S.A.
BORN:	Shantou, Guangdong, China, April 4, 1949
EDUCATION:	Ph.D., Mathematics, University of California, Berkeley, 1971

HONORARY DEGREE:

May 11, 2014	Doctor of Humane Letters, University of Illinois at Chicago
Jun 17, 2011	Doctor of Mathematics, University of Waterloo
Jun 05, 2010	Doctor of Science, National Cheng Kung University
May 18, 2009	Doctor of Science, Lehigh University
Nov 2005	Doctor of Science, National Taiwan University
May 22, 2005	Doctor of Science, Polytechnic University in Brooklyn
Nov 2004	Doctor of Science, Hong Kong University of Science and Technology
Jul 2004	Doctor of Science, National Central University
Mar 2003	Doctor of Science, Zhejiang University
Oct 16, 2002	Doctor of Science, Macau University
Jun 24, 2000	Doctor of Science, National Tsing Hua University, Taiwan
Jun 21, 1997	Doctor of Science, National Chiao Tung University, Taiwan
Sept 1987	Master of Arts, Harvard University
Dec 1980	Doctor of Science, The Chinese University of Hong Kong

RESEARCH INTERESTS:

Differential geometry, differential equations, and general relativity

POSITIONS:

2014–Present	Director of the Center for Mathematical Sciences and Applications, Harvard University
2013–Present	Professor of Physics, Harvard University
2008–2012	Chair of Department of Mathematics, Harvard University

Commentary on

Métriques de Kähler-Einstein Sur les Variétés Ouvertes

In 1978 I was on my way from Stanford to attend the International Congress of Mathematicians in Helsinki, where I was to give a plenary talk. I was invited and spent a month at IHES to communicate with our colleagues in France.

I could not speak French at all. A Stanford graduate student, Petric Ames, picked me up after I arrived at the airport and accompanied me to the apartment at IHES.

His parents came from the French part of North Africa. I did not realize that his French carries what the Parisians considered to be an accent. We went to many places in Paris for sightseeing. We met Bernard Saint-Donat on the streets of Paris, when Ames and I decided to watch a movie called "Hitler." B. Saint-Donat wanted to take us to see the great Paris and watch some opera. But P. Ames insisted on watching the movie. It ended up that B. Saint-Donat got angry at the two guys from America who weren't cultured enough to appreciate the great Paris. Well, later we did go to see many great museums in Paris of which I was highly impressed. I never cease to admire French culture every time I visit Paris.

In Paris, I met many mathematicians from America and from other countries. This includes Robert Langlands, Blaine Lawson, Kenneth Ribet, Yoichi Miyaoka, and many mathematicians in France including Jean-Pierre Serre, Pierre Deligne, Jean-Pierre Bourguignon, Marcel Berger, and others.

Nicolaas Kuiper was the director of IHES at that time. He gave me a very warm welcome, inviting me to his home for lunch. Robert Connelly of Cornell University had just found a non-convex polyhedron that was not rigid and flexible in three spaces. N. Kuiper was extremely impressed by this result. He took us to meet an artist in Paris whose artwork included such flexible polyhedrons. Apparently the artist knew how to build such things long ago without knowing that mathematicians were interested in them.

Y. Miyaoka had just finished his proof of the Miyaoka-Yau inequality for algebraic surfaces, based on the ideas of Fedor Bogomolov. I told him what Kähler-Einstein metrics could do, mentioning that the argument also worked in higher dimensions. The log version of the inequality could also be derived. I gave a talk on this at the Polytechnic University, after being invited to do so by my friend J.-P. Bourguignon. He made notes of my talk and recorded my work with S. Y. Cheng on the existence of a Kähler-Einstein metric on the complement of a divisor of normal crossing as long as $K + D$ is positive. (The log Chern number inequality follows from this rather straightforwardly.)

At that time, I had just finished my work with Richard Schoen on the structure of three manifolds with positive scalar curvature, which is related to the positive mass conjecture. I gave a talk on this, and B. Lawson showed great interest in it. R. Schoen and I were also in the process of writing up a proof, which demonstrated that in the category of manifolds with positive scalar curvature, it is possible to perform surgery with codimension not less than three that preserved the property of admitting such metrics. This paper appeared in *Manuscripta Math.* in 1979. I mentioned our results to B. Lawson before it was published. A year later, B. Lawson wrote a paper that appeared in the *Annals of Math* 1980 with Mikhael Gromov on a different way to perform such a surgery. (It was later discovered that there were some mistakes in the formula they used, although they may have been corrected by now.) R. Schoen and I realized its importance for studying the topology of manifolds with positive scalar curvature, but we did not know the right experts on spin cobordism to transform the surgery result to provide effective classification for simply connected manifolds with positive scalar curvature.

Shing-Tung Yau

Selected Expository Works of
Shing-Tung Yau with Commentary
ALM 28, pp. 3–6

Métriques de Kähler-Einstein Sur les Variétés Ouvertes*†

Shing-Tung Yau‡

1. On va s'intéresser à trouver des métriques de Kähler-Einstein sur des variétés ouvertes (par exemple des domaines bornés de $\mathbb{C}^n$). Calabi a ramené la recherche des telles métriques pour certains domaines à l'étude de l'équation de Monge-Ampère réellè sur $\mathbb{R}^m$

$$\det \frac{\partial^2 u}{\partial x^i \partial x^j} = 1. \tag{1}$$

L'étude de cette équation est le cadre de la géométrie dite “affine” puisque l'opérateur est invariant sous le groupe $\mathrm{sl}_m(\mathbb{R})$.

Théorème 2: *Si u est une fonction convexe définie sur* $\mathbb{R}^m$, *la seule solution de* (1) *est un polynôme quadratique.*

Le cas $m = 2$ est dû à Jörgens, le cas $m \leqslant 5$ à Calabi (cf.[1]) et les généralisations à Pogorelov. Après Pogorelov, Cheng et Yau (cf.[2]) ont donné une preuve analytique différente de celle de Pogorelov.

3. En considérant le graphe de u, on définit une métrique affine invariante par $\mathrm{sl}_{m+1}(\mathbb{R})$ par $\sum u_{ij} dx^i \otimes dx^j$ (on note $u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$); la formule est plus compliquée si $\det u_{ij} \neq 1$.

On veut montrer que u est un polynôme du second degré, donc que $\sum\limits_{i,j,k=1}^{m} u_{ijk}^2 = 0$. La meilleure méthode consiste à considérer u_{ijk} comme 3-tenseur sur une variété et à travailler dans la métrique définie par u. On considère donc

$$S = \sum u^{ir} u^{js} u^{kt} u_{ijk} u_{rst}.$$

(pour le cas complexe on prendra

$$S = \sum u^{i\bar{r}} u^{\bar{j}s} u^{k\bar{t}} u_{i\bar{j}k} u_{\bar{r}s\bar{t}}).$$

4. Dans le cas complexe, on considère l'équation sur $\mathbb{C}^n$

$$\det \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} = 1.$$

*Notes de J. P. Bourguignon

†S. T. Yau, Métriques de Kähler-Einstein sur les varietes ouvertes, *Asterisque*, **58** (1978), 163–167.

‡Department of Mathematics, Stanford University et I. H. E. S.

Commentary on

The Classical Plateau Problem and the Topology of 3-manifolds

In the fall of 1977, I was moving from my visiting position at UCLA to another visiting position at Berkeley at the invitation of Chern, who wanted me to try out an offer from Berkeley. I rented a two-bedroom apartment in north Berkeley, where my mother came to stay with me. It was a productive year, working with Richard Schoen and S.-Y. Cheng. I taught a course on the differential geometry of isometric embeddings. (My lecture notes were later given to J. Hong at Fudan University. He wrote a nice book with Han, based partially on my notes.)

Before I started the semester, Chern asked me to go with him to a joint Japan-United States conference on minimal submanifolds and geodesics that he, Osserman, Otsuki, and Obata had arranged. I took the advantage of this trip to visit Hong Kong. I passed Tokyo, flying on United Airlines to Hong Kong, When the airline collected my ticket for the first leg, they mistakenly took my return ticket too. That caused me huge problems when I tried to go from Hong Kong back to Tokyo. But I was still glad that I went, as it was my first time back in Hong Kong since I left in 1969. I was very pleased to see my brother, my sisters, and my friends, although I found out that housing situation in Hong Kong was much worse than in America. This was also my first visit to Japan. The city of Tokyo was as crowded as Hong Kong, so I felt right at home there.

Although I was reasonably famous at that time, as I had just finished the proof of the Calabi conjecture, the Japanese hosts considered age to be important, as do the Chinese. Hence I was not invited to their important banquets nor to the Japanese opera. However, Jim Simons and Blaine Lawson happened to be in Tokyo too, so we went to some bars there, even though I did not drink alcohol. The Vietnam War had not ended long before, and the Japanese were still very much influenced by the American presence in Vietnam. We saw Japanese actors performing cowboy dances and music, which didn't look too different from the kind of things you'd see on American TV. Jim Simons had many interesting comments about them. I met and had nice discussions with a few Japanese mathematicians, including Mrs. Miyaoka (before she married).

I had just finished my work with Bill Meeks on the embedding of the Douglas solution of the Plateau problem. So I talked about that at the conference. But I also announced an interesting conjecture that caught some of my Japanese colleague's interest. I proposed that the first eigenvalue of an embedded closed minimal surface in the unit three sphere must be equal to two. When I told this to E. Calabi in 1979 at the IAS, he was quite excited about the idea. Calabi told my student that this conjecture could offer some good insights into minimal surfaces in a sphere. The conjecture is still not solved. But Choi and Wang took some steps towards a solution ten years later, making a substantial contribution to the theory of minimal surfaces in a sphere.

Shing-Tung Yau

Selected Expository Works of
Shing-Tung Yau with Commentary
ALM 28, pp. 9–10

The Classical Plateau Problem and the Topology of 3-manifolds*

William H. Meeks III†, Shing-Tung Yau‡

It was an old problem whether there is a disk with least area bounding a given closed curve in $\boldsymbol{R}^3$. This was solved about forty years ago by Douglas and Rado in a somewhat generalized sense. Only in 1969 Osserman proved that the solution of Douglas has no branch points in the interior. Later Gulliver even proved that there is no false branch point. Hence, the Douglas solution is an immersion of the disk.

It was a general question whether the solution is actually embedded when γ is a Jordan curve on the boundary of a convex region. In this note, we announce a solution of this problem in the more general case when γ lies on the convex boundary of a three-dimensional manifold. This provides an interesting proof of Dehn's lemma and the sphere theorem in 3-manifold theory. Some other new results in 3-manifold theory can also be proved with our approach.

For simplicity, all curves are C^3 and all manifolds are smooth. If γ is a Jordan curve in a Riemannian 3-manifold M^3 then we will call a conformal mapping $f : D^2 \to M^3$ a Douglas-Morrey solution on Plateau's problem if f has least energy with respect to all piecewise smooth mappings of the disk D^2 into M^3 such that $f|\partial D^2$ is a monotonic parameterization of γ.

Theorem 1. *If M^3 is a compact Riemannian 3-manifold with convex boundary and γ is a Jordan curve on the boundary which contracts to a point in M^3, then there exists a Douglas-Morrey solution to Plateau's problem for γ.*

Remark 1. *The compactness of M^3 can be replaced by a Morrey type condition that M^3 is homogeneous regular. The above theorem is a consequence of Morrey's solution of Plateau's problem.*

Theorem 2. *If M^3 is a Riemannian 3-manifold with convex boundary, γ is a Jordan curve on the boundary. and $f : D^2 \to M^3$ is a Douglas-Morrey solution to Plateau's problem then f is an embedding.*

It should be noted that the fact that the above f is an immersion was proved by R. Ossermann [3] for curves in R^3 and by R. Gulliver [1] for curves in general

*William H. Meeks III, Shing-Tung Yau, Minimal Submanifolds and Geodesics, Kaigai PUblications, Tokyo, 1978, 101–102.

†This research was supported in part by NSF Grant No. Mcs76-07147.
‡This author was supported by the Sloan Fellowship.

Commentary on

Geometric Bounds on the Low Eigenvalues of a Compact Surface

I gave a presentation with Rick Schoen and Scott Wolpert regarding work we did at Stanford in the 1970s, when both Rick and Scott were graduate students. In 1974, Rick and I were chatting in my office about eigenvalues. Scott wandered into the office with a cup of coffee in his hand, asking what we were doing. So we joked that we could work on any problem on Riemann surfaces, including eigenvalues of the Laplacian for Riemann surfaces. Rick and I then told him how that problem can be related to finding isoperimetric constants for the Poincaré metric. The interesting result is that when the number is greater than $4g - 1$, the eigenvalue is always greater than or equal to 1/4. I'd known that result for quite a while. (It had also been found, independently, by P. Buser.) But I wanted to understand the behavior of eigenvalues lower than $4g - 1$. So the three of us looked into that, finding that between $2g - 2$ to $4g - 1$, the eigenvalues have a positive lower bound depending only on genus. The lower eigenvalues can go to zero when the Riemann surface moves to the boundary of the moduli space of the Riemann surfaces. I thought that was a nice result. But we still did not know how to find the above-mentioned best constant that depends on g only. So I talked about this result at the Hawaii conference, but Scott complained that the writing in this paper was poor, which did not please us. It was, in any case, a result of interest. Yet I wonder what happens in higher dimensional Kähler-Einstein manifolds: Can one replace the length of geodesics by the volume of special Lagrangians? That is a question for future study.

Shing-Tung Yau

Selected Expository Works of
Shing-Tung Yau with Commentary
ALM 28, pp. 13–19

Geometric Bounds on the Low Eigenvalues of a Compact Surface*

R. Schoen[†], S. Wolpert, S. T. Yau

In this note we announce upper and lower bounds for the low eigenvalues of a negatively curved compact surface in terms of the lengths of certain families of geodesics (Main Theorem). Let M be a compact oriented surface of genus $g > 1$ endowed with a metric of Gauss curvature K satisfying $-1 \leqq K \leqq -k$ for some constant $k > 0$. Let $0 = \lambda_0 < \lambda_1 \leqq \lambda_2 \leqq \cdots$ be the eigenvalues of the Laplace operator on M acting on functions. Our main result says that for $1 \leqq n \leqq 2g-3, \lambda_n$ is bounded above and below by positive constants (depending on g and k) times the length of the shortest subdivision of M into $n+1$ pieces by simple closed geodesics. Also, it follows that λ_{2g-2} has positive upper and lower bounds depending on g, k. Basically, our results say that λ_n can be small only for a surface which is nearly divided into $n+1$ pieces, each piece having negative Euler characteristic. Since one can construct surfaces having specified lengths for the disjoint simple closed Poincaré geodesics, it follows that for any n with $1 \leqq n \leqq 2g-3$, there exists a sequence of surfaces of genus g and $K \equiv -1$ so that λ_n tends to zero and λ_{n+1} has a positive lower bound. We only sketch a proof of our result in this note. Full details will appear elsewhere.

For $1 \leqq n \leqq 2g-3$, we consider the family of curves which consist of a disjoint union of simple closed geodesics dividing M into $n+1$ components. We let c_n denote the class of all such curves. Define a number ℓ_n by

$$\ell_n = \min\{L(C) : C \in c_n\}$$

where $L(C)$ is the length of C. We now state our main result.

Main Theorem. Let M be a compact oriented surface of genus $g > 1$ with a metric of Gauss curvature K. Suppose for some constant $k > 0$ we have $-1 \leqq K \leqq -k$. There exist positive constants α_1, α_2 depending only on g such that for $1 \leqq n \leqq 2g-3$, we have $\alpha_1 k^{3/2} \ell_n \leqq \lambda_n \leqq \alpha_2 \ell_n$ and $\alpha_1 k \leqq \lambda_{2g-2} \leqq \alpha_2$.

We outline the main steps in the proof of the theorem.

1. We use a variant of the Ahlfors-Schwarz lemma to reduce the case of variable curvature to that of curvature identically equal to -1. If our given metric is $\sigma = \sigma(z)|dz|^2$, and $\mu = \mu(z)|dz|^2$ is the Poincaré metric, we use the curvature

*R. Schoen, S. Wolpert, S. T. Yau, Geometric Bounds on the Low Eigenvalues of a Compact Surface, Proceedings of Symposia in Pure Mathematics, Volume 36, 1980.
[†]1980 Mathematics Subject Classification 53-XX.