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Advanced Lectures in Mathematics
Volume 28

Selected Expository Works of Shing-Tung Yau with Commentary

Volume I

Companion to the volume
Selected Expository Works of Shing-Tung Yau with Commentary, Volume II

edited by

Lizhen Ji · Peter Li
Kefeng Liu · Richard Schoen

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ADVANCED LECTURES IN MATHEMATICS

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Preface

In the early spring of 2013, Lizhen Ji asked me to write comments about my collected or selected works. I was too busy at the time to take on such a task. At one point, however, I gave in to his request and decided to write comments about my survey articles. Upon tallying them up, I was surprised to see that I had written far more survey articles than I had remembered.

Since I was a child, I have always been interested in history. Hence when I started to write these commentaries, I tried to stick to the facts to the best of my memory. I also consulted friends who participated in these events and looked at letters and emails that I had kept over the past forty years.

This does not mean that there are absolutely no mistakes in the statements. Nevertheless, I believe that these accounts can be interesting—and maybe even important—for students who'd like to know something about how the various papers were written and what my friends and I thought about the approaches we took.

In the course of putting together this collection, I received strong support from Lizhen Ji, Hao Xu, Kefeng Liu, Shiu-Yuen Cheng, and Hung-Hsi Wu. I am also very grateful to the publishers led by Liping Wang, Yushan Deng, and others. My friend Steve Nadis agreed to be the consulting editor for this project. I am extremely thankful for all of their help, without which this project likely would not have materialized.

Shing-Tung Yau
June 30, 2014

Shing-Tung Yau: His Mathematics and Writings

Lizhen Ji

Why *selected* works?

There has been a long tradition of publishing collected or selected works of distinguished mathematicians. There are several good reasons for doing this, and it has served many purposes. Probably the most obvious one is that collected and selected works provide an easy access to papers that are scattered in different journals, some of which are not easily accessible to many people. Otherwise, few people, if any, will take the time and trouble to dig up all the papers of their admired mathematicians–especially not those papers that are far away from their interests, of their admired mathematicians and read them. On the other hand, reading papers of a master dealing with different subjects or areas conveys the underlying unity and hence a big picture of mathematics, and it also allows one to gain a historical perspective (or to enter the history). In other words, collected and selected works are more than the simple sum of individual papers.

Indeed, as Abel said famously, we learn "by studying the masters, not their pupils." Even though the world is becoming smaller, few people have many chances to interact with masters who are alive. Of course, the next best way to learn from masters is to read and study their collected works.

Naturally, publishing collected or selected works is also an honor to the authors of these papers. It should be mentioned that collected works of some people can bring honor of the genre of collected works.

Now, with the wide and easy use of e-papers and e-books, most papers in journals can be obtained easily online, and a mere reprinting of papers is probably not as valuable as before. Of course, the value of selected works still stands. For example, holding and reading a beautifully printed book is definitely different from viewing papers online or on e-book readers. But they should also provide something else. Several additional things seem to be reasonable: descriptions of how ideas in the paper were formed and time and place the papers were written, relations between papers with the advantage of hindsight, and developments of subjects after the papers were published, and visions for the future. In other words, they should explain the circumstances of the birth of papers and proper, impacts of the papers, and fitting these papers in the grand scheme of mathematics.

These additional things are especially important to beginners, non-experts and even some experts. Most people often concentrate on the best known theorems and most important papers of great mathematicians, but even masters struggled and stumbled sometimes on their mathematical trips. How they found good problems and their ways in their careers, made progress and reached peaks is best described by their own papers, recollections and commentaries, but not textbooks where everything is polished and presented in a streamlined matter, without mentioning that textbooks and research books might not cover some gems in the original papers that are not directly related to the themes of the books. But many people, especially younger ones, often prefer to read polished textbooks.

Of course, reading mathematics papers can be difficult (more difficult than reading textbooks), and proper arrangement of related papers and additional guides from the masters are certainly valuable and helpful. Such collected or selected works of distinguished mathematicians often tell good stories of the authors and their mathematics, and browsing through them can be enjoyable and beneficial to people who are not interested in some specific results in the papers.

In these works of expository writings of Shing-Tung Yau, all these things are printed together with his survey papers and papers on open problems. One reason for restricting these volumes to expository papers of Yau is practical. Yau has been very creative and prolific. The collected works including all his papers (both research and expository papers) up to now will occupy too many volumes. Besides, he is also still very active and productive, and the time for collected works may not be ripe yet.

Why expository writing?

Probably some explanation is needed for publishing these volumes of expository writings of Yau now. Briefly, it is the right time for Yau to share his perspectives and his vision on the broad area of geometric analysis, and his expository writings provide a unique means to this end. They will render a valuable service to the mathematics community.

Colloquium talks have been a common means of communication between mathematicians from different subjects, after they were made successful and popular by Klein and Hilbert in Göttingen about 100 years ago. More expository talks such as "Basic notion seminars" and "What is...?" have also sprung up in many places. They provide effective ways for people to learn and enjoy some beautiful pieces of mathematics, which are outside their fields of specialty. Though there are many books and papers dealing with all kinds of subjects in mathematics, one difficulty is that there are too many of them. It is difficult for people to find the right books and papers, and people may lack the motivation to read mathematics outside their specialties, especially when they involve difficult and technical material. Many people choose to study mathematics not for fame or fortune, but for the beauty and enjoyment of the discipline. To really appreciate the beauty and power of mathematics, one has to roll up one's sleeves and do the work. But not many people can work in many different subjects in mathematics. In the history of mathematics, only a few people have been universal mathematicians. Some obvious names in the recent times include Gauss, Riemann, Poincaré, Hilbert, Weyl, and Hadamard. In spite of the difficulties, one can still enjoy and appreciate many facets of the rich world of mathematics by listening to expository talks and talking to experts. In the abstract world of mathematics, direct interaction and communication is still vital, and the virtual internet is no replacement yet.

Next to listening to talks, one can try to read expository writings and informal comments and notes on technical papers. The former is like colloquium talks, and the latter is like conversations at the colloquium tea or dinner. Expository writings include books, survey papers and descriptions of open problems. It often happens that expository writings are less valuable in a short period than highly technical and original papers, which can give people priorities and more credit. But in the long run, books and expository papers might be read by more people and have a bigger and longer impact. Think of Euler. How many of his papers are still read by people now? But his two elementary books on analysis and algebras are still printed and read by many people. What about Hilbert? His paper on the open problems and his report (or survey) on algebraic number theory are probably most read among his papers. Among contemporary mathematicians, we can think of the expository writings of people such as Atiyah, Milnor and Serre, which have had a huge impact on the modern mathematics. Though not everyone likes every book written by Serge Lang, many people will probably agree that they certainly have had nontrivial impact on mathematics and have been enjoyed by mathematicians from different areas of mathematics.

There are many selected and collected works of famous mathematicians. Usually they consist of their research papers plus some expository writings. But there does not seem to be entire volumes of selected papers that consist of only expository papers by a single mathematician up to now. Writing expository papers takes different skills, and writing it well takes time and broad knowledge, hence it is not easy at all. In this sense, Yau is special among contemporary mathematicians. Besides his many highly original and deep contributions to mathematics, he has also written many expository papers covering the broad areas of geometry and analysis.

Writing versus person

There is a Chinese saying that a person's writing resembles the person, i.e., it reflects his or her character. This implies that a mathematician's work and his writing also resemble the person. Instead of commenting on the style of Yau's mathematics and his expository writings, it might be helpful to comment on him as a person.

If one thinks of Yau, the following features stand out. He is powerful and persistent (or stubborn). Once he sets up a goal, he will overcome all obstacles to carry through and achieve it. He strongly and confidently believes in what he is doing. He is full of energy and is always active. It is difficult to imagine him without working on multiple projects or organizing something at any time, while keeping others busy at the same time. For example, he travels extensively and frequently, but the concept of jet lag simply does not exist for him.

He has broad interests, and fine divisions between various things also do not exist for him. For example, the usual gap between pure mathematics and applied mathematics is not visible to him, and the more obvious gap between mathematics and humanities does not matter to him either. He is brave and also delicate (or subtle, as Bacon said that mathematics makes one subtle). He has strong opinions about many things and often express them clearly, both verbally and in writing. He has a curious mind and is full of all kinds of ideas, some of which are rather unusual. His approach to everything is global and massive (the quote below from Armand Borel confirms this).

Yau is highly motivated, or driven, by the ideal to do good for all of mathematic and the mathematics community. By all accounts, he has left his share of permanent marks on the field.

A glimpse of the mathematics of Yau

Based on the above brief summary of Yau as a person, we will briefly take a look at his mathematics and his achievements in expository writing.

Shing-Tung Yau is a mathematician of great originality, vision and technical power. His work, together with his dedication to mathematics and to its effective dissemination, has permanently affected many subjects ranging from differential geometry, differential equations, topology, and algebraic geometry to mathematical physics. He is truly a global mathematician and teacher.

Originality and technical power

Yau has proposed and emphasized the philosophy that the subject of nonlinear partial differential equations should be combined with geometry (including differential geometry, complex geometry and algebraic geometry). Many deep results by him and his co-authors have shown that only through geometry, can one thoroughly understand nonlinear differential equations such as the Monge-Ampère equation, and that geometric structures can be constructed effectively by nonlinear partial differential equations coupled with the linear theory of Index theorems, which are still the only way to build geometric structures.

Probably the most striking example is the construction of Calabi-Yau metrics, which lead to immediate solution of several longstanding open problems in algebraic geometry and has contributed to the extensive development of mathematical physics, in particular string theory. Other examples of geometric structures constructed in this way include the geometric structures on three manifolds by Hamilton's Ricci flow, the gluing arguments by Taubes and Uhlenbeck for the existence of anti self-dual connections and anti self-dual metrics on four manifolds, and the existence of Hermitian Yang-Mills connections on higher dimensional complex manifolds by Uhlenbeck-Yau.

Before Yau's work in geometric analysis in the 1970s, theories of differential equations were largely separately from differential geometry, and applications of nonlinear differential equations to complex geometry and algebraic geometry could not be imagined. It is fair to say that Yau and his friends such as Rick Schoen, Leon Simon, Shiu-Yuen Cheng, Karen Uhlenbeck, Peter Li, and Richard Hamilton founded the subject of geometric analysis in the 1970s, which was followed by Cliff Taubes, Simon Donaldson and others in the 1980s. Geometric analysis has become the most powerful development in the past 40 years in geometry and topology.

It might not be well-known that the existence of Hermitian Yang-Mills connections is absolutely essential for the construction of supersymmetric heterotic strings and has had a huge impact in algebraic geometry and complex geometry.

For example, the proof of Uhlenback and Yau is still the only proof that can be used for applications such as the Higgs field and variations of Hodge structures. Yau's work on linear differential equations is also very original and influential. For example, the gradient estimate for harmonic functions has become a standard technique in geometric analysis, and the Li-Yau estimate for the heat equation is fundamental to all parabolic equations and has motivated the major estimates of Ricci flow, which was used to prove the Poincaré conjecture.

Another example of the great originality and technique of Yau is the application of harmonic maps and minimal surfaces. Meeks and Yau proved the famous classical problem of embedding of the Douglas solution of the minimal surface equation, which was turned around to solve the equivariant Dehn's Lemma, which was a crucial step in the solution of the famous Smith conjecture for group actions on the three sphere. Schoen and Yau used the theory of harmonic maps to prove the famous Einstein conjecture on the positivity of mass in general relativity—a proof that constitutes one of the most important justifications of relativity theory. (If the mass were negative, the whole system would be unstable and fall apart!) Many important consequences of the solution of positive mass in geometry have been developed, and this led to Yau's recent work with Mu-Tao Wang on the correct definition of quasi-local mass in general relativity.

Vision and open problems

Yau has formulated conjectures and proposed many open problems that have had far reaching consequences. For example, the concept of mirror symmetry in string theory has had many unexpected applications in algebraic geometry, and understanding the mirror symmetry conjecture has been a major challenge in theoretical physics and mathematics. In 1996, Strominger-Yau-Zaslow proposed a new construction of mirror manifolds, called the SYZ conjecture. This conjecture has been the guiding principle for a whole generation of people working on mirror symmetry.

In 1982, Yau proposed a list of 120 open problems when he organized a special year on differential geometry at IAS, Princeton. They covered a broad range of topics. Some of them have been very influential and have changed the subject. For example, Problem 65 called for a notion of rank of nonpositively curved manifolds, which extends the one for locally symmetric spaces, and asked whether higher rank metrics are locally symmetric metrics.

This problem was completely and independently solved by Ballmann, and Burns-Spatzier. The efforts to solve this problem have completely changed the subject of manifolds of nonpositive curvature and attracted many people to work on understanding geometry, topology and the analysis of manifolds of nonpositive curvature. A currently active topic deals with CAT(0)-geodesic metric spaces and their quotients, which are not necessarily smooth manifolds and a natural generalization of nonpositively curved Riemannian manifolds.

After the success of this list of open problems, Yau proposed another 100 open problems in 1992. This list has also had a huge influence on topics covered. For example, one problem can be stated as follows: *Prove that a compact Kähler manifold with positive first Chern class admits a Kähler metric if and only if the manifold is stable in the sense of geometric invariant theory, meaning that the tangent bundle is stable as a bundle and the automorphism group is reductive.*

This problem has been studied intensively by many top experts around the world since the 1980s, and is the focus of current research in algebraic geometry and complex geometry.

Yau also raised many problems in other papers and contexts. All these problems show the global perspective of Yau towards mathematics. More importantly, they teach generations of geometers (in the broadest sense) the importance of understanding and appreciating the connections between different parts of mathematics.

Expository writing of Yau and related activities

Besides several hundred research papers, Yau has written many survey papers, as these two volumes show. We will only highlight several surveys here. The paper "Survey on partial differential equations in differential geometry," in the volume *Seminar on Differential Geometry*[1], edited by Yau, was the first major survey of geometric analysis. In some sense, it is the record of the firsthand experience of this emerging subject by one of the most important practitioners.

His well-known list of open problems, "Problem section, in *Seminar on Differential Geometry,*[2] was written at the same time for a special year (1979–1980) at the Institute for Advanced Study, Princeton.

This special year was mainly organized by Yau and had a lasting impact on both geometric analysis and also on the Institute for Advanced Study (IAS). It is perhaps best to quote from an article written on the history of the IAS by the distinguished mathematician Armand Borel, who was a longtime permanent member at IAS. Borel wrote: [3]

> *In the sixties, considerable progress was made in... Algebraic groups, arithmetic groups and automorphic forms, number theory, harmonic analysis on reductive Lie groups. Much of it was done here.... This whole general field had become such an active*

[1] *Seminar on Differential Geometry,* Annals of Mathematics Studies, 102, pp. 3–71, Princeton University Press, 1982.
[2] Ibid., pp. 669–706.
[3] From "The School of Mathematics at the Institute for Advanced Study", in "A Century of Mathematics in America, Part III", pp. 119–147, *Hist. Math.*, 3, Amer. Math. Soc., Providence, Rhode Island, 1989.

> *and important part of "core mathematics" that it was all to the good. However, that was not matched by activities of similar scope in other areas and created some imbalance, accentuated by Atiyah's resignation in 1972. For reasons already explained, in our view it was not in the best interest of the school in the long run and to correct it by increasing activities in other areas became a concern. There are two obvious means to try to remedy this: the special programs and new faculty appointments... The financial situation of the Institute was worrisome... But we resume both as soon as possible: Enrico Bombieri came to the faculty in 1977 and Shing-Tung Yau in 1980, broadening greatly its coverage. We had also to wait until 1977 for the programs but have had one almost every year from then on... 1979–1980 was the year of the biggest program to date, on differential geometry and analysis, in particular nonlinear PDE. The number of seminars was somewhat overwhelming. Several were concentrated at the end of the week, so as to make it easier for people in neighboring (in a rather wide sense including New York and Philadelphia) institutions to participate. Roughly speaking, the main activities were subdivided in three parts: differential geometry, minimal submanifolds, and mathematical physics, with seminar coordinators L. Simon for the second one, S.-T. Yau for the other two. A remarkable feature of the third one (devoted to relativity, the positive mass conjecture, gauge theories, quantum gravity) was the cooperation between mathematicians and physicists, probably a first here since the early days. Two volumes of notes[4] resulted from this program.*

A few years later, Yau wrote another major survey, "Nonlinear analysis in geometry" (*Enseign. Math.* (2) 33, pp. 109–158, 1987). It gives a broad overview of the role of nonlinear differential equations in differential geometry, algebraic geometry, topology and mathematical physics. It conveys a panoramic view of the rapidly expanding subject of geometric analysis.

With the continuing growth of geometric analysis, other major surveys were needed in the late 1990s. The paper "Review of geometry and analysis" (*Asian Journal of Mathematics*, vol. 4, pp. 235–278, International Press, 2000) gives a comprehensive overview of areas of mathematics and physics related to Calabi-Yau manifolds and mirror symmetry.

In a more recent survey,[5] Yau gave a rather complete historical survey of the field of geometric analysis as one of the major players of this important field. He gave many important observations and philosophical points of views on both problems and methods used to attack them.

One of his most recent substantial surveys is "A survey of Calabi-Yau manifolds" (*Surveys in Differential Geometry*, vol. 13, International Press, 2009). It provides a comprehensive introduction to the history and results of research on Calabi-Yau manifolds, recent developments in this area and their applications.

He has also written several influential books and edited many others.

One particularly influential book is a joint book with Rick Schoen, *Lectures on Differential Geometry* (International Press), first published in 1994 and translated into Chinese. An expanded Chinese version was published in 2004. One important feature is that it goes directly to the most important and active topics of the subject and hence guides the reader to get into research quickly. This book has educated generations of differential geometers and will continue to be an indispensable book for many years to come. A closely related book by Schoen and Yau is *Lectures on Harmonic Maps* (International Press, 1997). It also helps the reader to learn some of the most important recent results on, and applications of, harmonic maps.

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[4] One of the two volumes alluded to here is *Seminar on Differential Geometry*.

[5] "Perspectives on geometric analysis", *Surveys in Differential Geometry,* vol. 10, pp. 275–379, International Press, 2006.

The book *Seminar on Differential Geometry,* mentioned above, has also been very influential to the development of geometric analysis since the 1980s.

Five books on mirror symmetry edited (or co-edited) by Yau have been very influential to the emergence of the string theory. The first one, *Essays on mirror manifolds*, published in 1992, was edited by Yau and based on the workshop organized by him in 1991 at MSRI, Berkeley. This is the first book that drew the attention of mathematicians, especially algebraic geometers, to string theory, and has played a decisive role in the trend of mathematical physics.

In some sense, the "expository writing" of Yau might also be interpreted broadly to include many conferences he has organized and various forms of interaction with others. The most prominent one is probably the conference series, *Current Developments in Mathematics* (International Press) and the resulting book series.

He also founded the book series *Surveys in Differential Geometry* (International Press), each of which consists of substantial surveys on subjects by some of the most active experts. Yau has contributed many papers to them.

There are over 30 volumes in the two series *Current Developments in Mathematics* and *Surveys in Differential Geometry*, and together they provide a panoramic view of the contemporary mathematics. Their impact on the development of mathematics is huge.

It is important to note that Yau has supervised more than sixty-five Ph.D. students up to now and has reached out to mathematics students and mathematicians at various levels through many mathematics centers and initiatives organized by him.

On top of all of these research, teaching, writing, conferences and math centers, Yau has also launched and edited many journals both in pure and applied mathematics. For example, he turned the *Journal of Differential Geometry* into a leading global journal, which has published the best works of Simon Donaldson, Michael Friedman, Richard Hamilton, Richard Schoen etc. What is special in these two volumes The two volumes which comprise the present work, *Selected Expository Works of Shing-Tung Yau with Commentary,* contain—in addition to all the expository papers written by Yau up to now and his commentaries on them—commentaries by other experts on some of Yau's work and some questions raised by him, which in turn complement those commentaries on his work which appear in the book *Geometry and Analysis, No. 1* (Advanced Lectures in Mathematics, vol. 17, International Press and Higher Education Press, 2011).

The *Selected Expository Works of Shing-Tung Yau with Commentary* is organized as follows: The commentaries by others are grouped together at the end of the second volume. All the expository papers by Yau are arranged according to the years of publication, and each paper is preceded by a commentary written by Yau.

Yau has written detailed commentaries on every expository paper in these volumes. They contain both mathematical backgrounds, practical circumstances, which reflect his career, his travels, his interaction with other mathematicians, and most importantly the development of his mathematics and his views on the field. In short, he recounts his firsthand experiences.

Together, these commentaries provide a most unusual scientific biography of an unusual mathematician. Reading them will allow one to enter into his mathematical world, to appreciate more of his enormous output and contributions and to understand better his perspective.[6]

[6] For readers who are interested in commentaries of mathematicians, it might be helpful to look up the collected works of Klein and Brouwer. In 1882-83, Klein and Poincaré announced or rather claimed proofs of the uniformization of Riemann surfaces of finite type (i.e., those obtained from compact Riemann surfaces by removing finitely many points), using the method of continuity. There were some gaps, and Brouwer's major work in topology fixed some of them. Later in 1907-1908, Koebe and Poincaré offered different proofs of the uniformization theorem for general Riemann surfaces. The uniformization theorem is probably the most beautiful and important theo-

These commentaries are especially written for these volumes, and we should stress that they represent the personal opinions of Yau.

As mentioned before, Yau's lists of open problems have had a huge impact on the broad area of geometric analysis. One special feature of the current two volumes is that these well-known open problem lists have been updated with the help of many people. Besides providing a record of the status of these problems, they might also point to new directions.

Yau has written many hundreds of research papers. Some of his highly original papers were published in journals and books that are not easily accessible or widely known. We have taken this chance to include seven such papers. They are of both historical and current interest since they contain some novel ideas and have opened up new research directions. We believe that they will continue to inspire the reader.

Besides these few papers, no other original research paper of Yau is included here. A complete list of all his mathematics publications up to April 16, 2013, according to MathSciNet, is included in these volumes together with his curriculum vitae. Although Yau has worked on many subjects in mathematics and mathematical physics, there are still some underlying themes to be found at any point in his career. The reader can find correspondence or connections between the original research papers on the list and the expository papers here.

To summarize, Yau is a man of mathematics. He is always driven to create and develop mathematics, and to share it with others. It is safe to say that without his contributions, his vision, writings, and teachings, the broad subject of geometry would not have attained its current mature state.

We hope that these two volumes will adequately reflect both his mathematics and his perspective on mathematics. Since Yau is one of the most active originators of and participants in the geometric analysis, these two volumes can also be viewed as an eyewitness account of this important subject of mathematics, and they should provide a valuable reference and guide to both geometers and non-geometers at all stages of their careers.

We also hope that they are interesting and enjoyable books to be read at different times and levels.

Acknowledgments

On behalf of Prof. Shing-Tung Yau, we would like to thank all the people who have written commentaries included in the second volume or have contributed to updates on the open problems on the two lists compiled by Prof. Yau. We would also like to thank the following people for their help in preparing these two volumes: Po-Ning Chen, Chun-Chien Cheng, Yushan Deng, Mabel Liu, Ping-Zen Ong, Liping Wang, Hung-Hsi Wu and Hao Xu. It is clear that without their efforts, these two volumes would not have been finished in such a timely fashion.

Lizhen Ji
Ann Arbor, June 2014

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rem in the last one hundred (or even one hundred and fifty years), and the history surrounding its proofs is both complicated and inspiring. Interaction and some heated arguments between these major mathematicians involved, and their different perspectives on mathematics, were well recorded in the commentaries of the collected works of Klein and Brouwer, which help explain the evolution of mathematical ideas.