Lectures on Non-Linear Wave Equations

Second Edition

Christopher D. Sogge
Department of Mathematics
Johns Hopkins University

International Press
www.intlpress.com

Lectures on Non-Linear Wave Equations, 2nd Edition

Christopher D. Sogge (Johns Hopkins University)

2010 Mathematics Subject Classification. Primary 35L70, Secondary 42B25.

ISBN 978-1-57146-279-4

Paperback reissue 2013.
Previously published in 2008 under ISBN 978-1-57146-173-5 (hardcover).

Printed in the United States of America.

PREFACE TO THE SECOND EDITION

In the dozen years since I wrote the first edition of this book, there have been many important developments in the subject of nonlinear wave equations. Since I wanted the book to remain one that could be reasonably covered in a one semester graduate course, I was of course not able to present all of these. I also decided to mainly expand on advances in the topics covered in the earlier edition, rather than adding new ones. In particular, for the above reasons, I decided to omit any treatment of the very important advances in the area of nonlinear Schrödinger equations in Euclidean space, as well as the seminal work that has been done on nonlinear wave and Schrödinger equations on manifolds.

In the current edition, more so than the previous one, the material basically splits into two halves.

The first is the now classical energy integral method, and it is covered in the first two chapters of the book. The main techniques are based on combining L^2-estimates (energy) with L^∞-bounds (dispersive) to prove theorems. Since the first edition was written, some of the arguments have been simplified a bit, but otherwise the, by now, classical material covered in the text has not changed much. On the other hand, as we note in the historical comments, there has been much new work on low regularity existence problems. Much work remains in this important area.

The second main topic of the book, which is covered in the last four chapters, concerns L^p-estimates for the wave equation and their applications. These go by the name of Strichartz estimates, and the most important of these involve mixed norms, $L^q_t L^r_x$. In the first edition, we presented a fairly complete treatment of these when the spatial dimension is three. After its publication, sharp estimates in all dimensions for both the wave equation and the Schrödinger equation in Minkowski space was obtained by Keel and Tao. One of the main new features of this edition of the book is a proof of the Keel-Tao endpoint Strichartz estimates. We closely follow their elegant arguments, but we also present the important Christ-Kiselev lemma to show how the bounds for the homogeneous equation imply those for the inhomogeneous equation. Another development in the subject was the proof of the Strauss conjecture regarding small-amplitude wave equations with power nonlinearities. The higher dimensional versions of this conjecture was settled by Georgiev, Lindblad and the author by proving certain weighted Strichartz estimates. We show how this is done to obtain sharp results in three dimensions (which is a theorem of John) and then discuss the higher dimensional case and the improved bounds in this case that are due to Tataru. The last topic that is covered in the book, Grillakis' theorem about global existence for the energy-critical wave equation is essentially unchanged from the first edition.

As before, I am very grateful for the help of many people in preparing this book. This edition was based on a course that I taught at Johns Hopkins in the spring semester of 2007. I also lectured on much of the new material in this edition at the Zhejiang University in Hangzhou shortly afterwards. I would like to acknowledge the valuable input that I received from everyone who attended the lectures. I am especially grateful to my colleague and friend, Makoto Nakamura for very carefully going through a draft and suggesting many improvements. His very thorough reading of early drafts made the task of preparing this new edition much easier. I would also like to thank Jin-Cheng Jiang for several helpful comments and suggestions, as well as for his help in preparing the figures that have been incorporated in the text. I am also greatful to Jason Metcalfe for helpful suggestions and advice.

This work was prepared using $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX and was supported in part by the NSF.

Baltimore C. D. Sogge

PREFACE

These notes are based on a course I gave at UCLA in the fall of 1994. I tried to make the course self-contained, presenting as background basic facts about the solution of the linear wave equation as well as the basic tools from harmonic analysis that were used. The heart of the course concerned three types of problems in the theory of nonlinear wave equations, that, to varying degrees, have non-trivial overlap with analysis.

I first presented results concerning existence for certain quasilinear wave equations, usually with small Cauchy data. The global results relied on energy estimates, Sobolev's theorem, as well as Klainerman's generalized Sobolev inequalities which make use of vector fields preserving the equation $\Box u = 0$. As a preview of things to come, we also presented a recent low-regularity local existence theorem of Klainerman and Machedon which is based on a variation of Strichartz's restriction theorem for the light cone.

The next topic we covered involved various results concerning semilinear wave equations with small data. The first one presented was a remarkable theorem of John which says that in $\mathbb{R}^{1+3}_+$ the equation $\Box u = |u|^\kappa$ always has a global solution for small smooth compactly data if $\kappa > 1 + \sqrt{2}$, while, conversely, if $\kappa < 1 + \sqrt{2}$ there can be blow-up even for arbitrarily small data. We followed John's argument for the blow-up part of the theorem, but for the positive part we used a somewhat different argument which relies on the Hardy-Littlewood maximal theorem. After this, we presented some local and global existence theorems involving sharp regularity assumptions on the data. The proof uses mixed-norm variants of the Strichartz estimate mentioned before.

The last topic covered involves global existence results for arbitrary smooth data for equations of the form $\Box u = -|u|^{\kappa-1}u$ in $\mathbb{R}^{1+3}_+$. The sign of the nonlinearity is easily seen to be crucial. Using the earlier mixed-norm estimates we can prove a classical theorem of Jörgens saying that there is global existence for the "subcritical" range of $\kappa < 5$. This argument also gives a result of Rauch saying that there is global existence for the "critical" case $\kappa = 5$ if the data has small energy. Removing this assumption for this case is delicate, and we shall do this using an argument of Struwe based on a Morawetz-Pohožaev identity. This general result for the critical wave equation in $\mathbb{R}^{1+3}_+$ is due to Grillakis.

As I pointed out before, my main goal in preparing these informal notes was to try to provide the students with a self-contained treatment of certain problems in nonlinear wave equations. Because of focusing on this (and my ignorance), I am afraid that my treatment of historical background may be at best inadequate. For this reason, I refer the reader to the excellent notes of Hörmander [5], John [8] and Strauss [4]. These works also supplement mine since they were used while preparing the course. For background concerning the literature about generalized Strichartz inequalities the reader should consult the excellent survey article of Ginibre and Velo [4].

Finally, it is a great pleasure to thank the many people who have helped me in this endeavor. Most of all, I would like to thank everyone who participated in the course and offered many useful comments and criticisms, including A. Chang, I. Laba, G. Simonett and W. Wang. I also would like to thank S. Klainerman for his suggestions, and H. Lindblad, M. Machedon and H. Smith for going through portions of the notes. I am especially grateful to S. Cuccagna for thoroughly reading through the entire manuscript. Lastly, I would like to thank D.H. Phong for encouraging me to undertake this project.

This work was prepared using $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX and was supported in part by the NSF.

Los Angeles C. D. Sogge

CONTENTS

INDEX

INDEX OF NOTATION

About the Author

Christopher Sogge received his PhD from E.M. Stein at Princeton University in 1985. He has been an NSF Postdoctoral Fellow (1985-1988), a Sloan Research Fellow (1988-1989), a Guggenheim Fellow (2005-2006), and a recipient of a Presidential Young Investigator Award (1988-1993). He has held positions at the University of Chicago (1985-1989) and UCLA (1989-1996), and he currently is a Professor at the Johns Hopkins University. Professor Sogge's research interests include Fourier analysis, partial differential equations, and geometry.