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Chapter 1

Introduction

Abstract. We study a generalization of Serre–Tate theory of ordinary abelian varieties and their deformation spaces. This generalization deals with abelian varieties equipped with additional structures. The additional structures can be not only an action of a semisimple algebra and a polarization, but more generally the data given by some "crystalline Hodge cycles" (a p-adic version of a Hodge cycle in the sense of motives). Compared to Serre–Tate ordinary theory, new phenomena appear in the generalized context.

We give an application of the generalized theory to the existence of "good" integral models of those Shimura varieties whose adjoints are products of simple, adjoint Shimura varieties of $D_l^{\mathbf{H}}$ type with $l \geq 4$.

In this monograph we develop a theory of ordinary abelian varieties and p-divisible groups with additional structures. We use it to give applications to the existence of "good" integral models of those Shimura varieties whose adjoints are products of simple, adjoint Shimura varieties of $D_l^{\mathbf{H}}$ type with $l \geq 4$.

The additional structures can be defined by an action of a semisimple algebra, by a polarization, or more generally by some "p-adic Hodge cycles".

In this introduction we first begin by recalling the classical Serre–Tate ordinary theory of abelian varieties and their deformation spaces. In our theory this will correspond to the case of a general linear group, the more general case corresponding to a "p-adic Mumford–Tate group" (i.e., a p-adic reductive group scheme whose generic fibre fixes the Hodge tensors that define the additional structures). After having recalled the classical Serre–Tate ordinary theory, we will give the translation of part of it into our language. Then we will explain the more general objects we consider and we will state our main results. We will end the introduction by mentioning applications and our main motivations.

For reader's convenience, here is a list of standard notions and notations that pertain to (group) schemes. The reader ought to look into it only when needed.

Standard language and notations on (group) schemes. We recall that a group scheme H over an affine scheme $\operatorname{Spec} R$ is called *reductive*, if it is smooth and affine and its fibres are connected and have trivial unipotent radicals (cf. [DG, Vol. III, Exp. XIX, Def. 2.7]). It is known that H is of finite presentation over $\operatorname{Spec} R$, cf. [DG, Vol. III, Exp. XIX, Sect. 2.1 or Rm. 2.9]. If $\operatorname{Spec} R$ is connected, then H is a *semisimple* group scheme (resp. is a *torus*) if and only if one fibre of H is a semisimple group (resp. is a torus) (cf. [DG, Vol. III, Exp. XIX, Cor. 2.6]). Let H^{der}, $Z(H)$, H^{ab}, and H^{ad} be the *derived group scheme* of H, the *center* of H, the *maximal commutative quotient* of H, and the *adjoint group scheme* of H (respectively). Thus we have $H^{\mathrm{ad}} = H/Z(H)$ and $H^{\mathrm{ab}} = H/H^{\mathrm{der}}$, cf. [DG, Vol. III, Exp. XXII, Def. 4.3.6 and Thm. 6.2.1]. Moreover H^{der} and H^{ad} are semisimple group schemes, H^{ab} is a torus, and (cf. [DG, Vol. III, Exp. XXII, Cor. 4.1.7]) $Z(H)$ is a group scheme of *multiplicative type*. Let $Z^0(H)$ be the maximal torus of $Z(H)$; the quotient group scheme $Z(H)/Z^0(H)$ is finite and of multiplicative type.

Let F be a closed subgroup scheme of H; if R is not a field, then we assume F is smooth. Let $\operatorname{Lie}(F)$ be the *Lie algebra* over R of F. As R-modules, we identify $\operatorname{Lie}(F)$ with $\operatorname{Ker}(F(R[x]/x^2) \to F(R))$, where the R-epimorphism $R[x]/(x^2) \twoheadrightarrow R$ takes x to 0. The Lie bracket on $\operatorname{Lie}(F)$ is obtained by taking the total differential of the commutator morphism $F \times_{\operatorname{Spec} R} F \to F$ at identity sections. Often we say F and H are over R and we write $F \times_R F$ instead of $F \times_{\operatorname{Spec} R} F$. For a finite, flat, morphism $\operatorname{Spec} R \to \operatorname{Spec} R_0$ of affine schemes, let $\operatorname{Res}_{R/R_0} F$ be the group scheme over $\operatorname{Spec} R_0$ obtained from F through the *Weil restriction of scalars* (see [BLR, Ch. 7, Sect. 7.6] and [Va5, Subsect. 2.3]).

By a *parabolic subgroup scheme* of H we mean a smooth, closed subgroup scheme P_H of H whose fibres are parabolic subgroup schemes over spectra of fields. The *unipotent radical* U_H of P_H is the maximal unipotent, smooth, normal, closed subgroup scheme of P_H; it has connected fibres. The quotient group scheme P_H/U_H exists and is reductive, cf. [DG, Vol. III, Exp. XXVI, Prop. 1.6]. We identify U_H with the closed subgroup scheme of H^{ad} which is the unipotent radical of the parabolic subgroup scheme $\operatorname{Im}(P_H \to H^{\mathrm{ad}})$ of H^{ad}. By a *Levi subgroup scheme* L_H of P_H we mean a reductive, closed subgroup scheme of P_H such that the natural homomorphism $L_H \to P_H/U_H$ is an isomorphism.

For a free R-module M of finite rank, let $\mathbf{GL}_M$ be the reductive group scheme over $\operatorname{Spec} R$ of linear automorphisms of M. Thus for a commutative R-algebra R_1, $\mathbf{GL}_M(R_1)$ is the group of R_1-linear automorphisms of $M \otimes_R R_1$. If $\operatorname{Lie}(F)$ is a free R-module of finite rank, then in expressions of the form $\mathbf{GL}_{\operatorname{Lie}(F)}$ we view $\operatorname{Lie}(F)$ only as a free R-module; thus we view $\operatorname{Lie}(F)$ as an F-module via the *adjoint representation* $F \to \mathbf{GL}_{\operatorname{Lie}(F)}$. If $f_1, f_2 \in \operatorname{End}_{\mathbf{Z}}(M)$, let $f_1 f_2 := f_1 \circ f_2 \in \operatorname{End}_{\mathbf{Z}}(M)$. Let $M^\vee := \operatorname{Hom}_R(M, R)$.

A bilinear form $\psi_M : M \times M \to R$ on M is called *perfect* if it induces naturally an R-linear isomorphism $M \tilde{\to} M^\vee$. If ψ_M is perfect and alternating, let $\mathbf{GSp}(M, \psi_M)$ and $\mathbf{Sp}(M, \psi_M)$ be the reductive group schemes over R of symplectic similitude isomorphisms and of symplectic isomorphisms (respectively) of (M, ψ_M).

Let $\overline{K}$ be an algebraic closure of a field K. If $T \to S$ is a morphism of schemes and X (resp. X_S or X_i with i as an index) is an S-scheme, let $X_T := X \times_S T$ (resp. X_T or $X_{i,T}$) be the T-scheme which is the pull-back of X (resp. of X_S or X_i). If $T = \operatorname{Spec} R \to S = \operatorname{Spec} R_0$ is a morphism of affine schemes, we often denote X_T and $X_{i,T}$ by X_R and $X_{i,R}$ (respectively). The same type of notations apply for morphisms of schemes and for (morphisms of) *p-divisible groups*. If R is a complete, local ring, we identify the categories of p-divisible groups over $\operatorname{Spf} R$ and $\operatorname{Spec} R$ (cf. [Me, Ch. II, Lem. 4.16]).

1.1. Classical Serre–Tate ordinary theory

Let $p \in \mathbf{N}$ be a prime. Let k be a perfect field of characteristic p. Let $W(k)$ be the ring of *Witt vectors* with coefficients in k. Let $B(k) := W(k)[\frac{1}{p}]$ be the field of fractions of $W(k)$. Let $\sigma := \sigma_k$ be the *Frobenius automorphism* of k, $W(k)$, and $B(k)$. Let r, $d \in \mathbf{N} \cup \{0\}$ with $r \geq d$. In late sixties Serre and Tate developed an *ordinary theory* for p-divisible groups and *abelian varieties* over k. It can be summarized as follows.

Ordinary p-divisible groups. A p-divisible group D over k of *height* r and *dimension* d is called *ordinary* if and only if one of the following five equivalent conditions holds for it:

(a) it is a direct sum of an étale p-divisible group and of a p-divisible group of multiplicative type;

(b) its *Newton polygon* is below the Newton polygon of every other p-divisible group over k of height r and dimension d;

(c) its *Hasse–Witt invariant* is equal to $r - d$;

(d) there exist isomorphisms between $D[p]_{\overline{k}}$ and $\boldsymbol{\mu}_p^d \oplus (\mathbf{Z}/p\mathbf{Z})^{r-d}$;

(e) there exist isomorphisms between $D_{\overline{k}}$ and $\boldsymbol{\mu}_{p^\infty}^d \oplus (\mathbf{Q}_p/\mathbf{Z}_p)^{r-d}$.

Canonical lifts. If D is ordinary, then there exists a unique p-divisible group $D_{W(k)}$ over $W(k)$ which lifts D and which is a direct sum as in (a). It is called the *canonical lift* of D. Via reduction modulo p we have an identity $\mathrm{End}(D_{W(k)}) = \mathrm{End}(D)$ of $\mathbf{Z}_p$-algebras.

Abelian varieties. An abelian variety A over k is called ordinary if its p-divisible group is so. Ordinary abelian varieties are of particular importance as in the reduction modulo p of the moduli spaces of principally polarized abelian varieties, they form open dense subspaces. If A is ordinary, then there exists a unique abelian scheme $A_{W(k)}$ over $W(k)$ which lifts A and whose p-divisible group is a direct sum as in (a); it is called the canonical lift of A. If k is an algebraic closure $\mathbf{F}$ of the field $\mathbf{F}_p$ with p elements, then a theorem of Tate asserts that $A_{W(k)}$ has *complex multiplication*. If k is finite, then $A_{W(k)}$ is the unique lift of A to $W(k)$ to which the *Frobenius endomorphism* of A lifts.

Deformation theory. Let D_{univ} be the *universal p-divisible group* over the deformation space $\mathcal{D}(D)$ of D (see [Il, Thm. 4.8]; based on [Me, Ch. II, Lem. 4.16], we view $\mathcal{D}(D)$ as a scheme). If D is ordinary, then the main properties of D_{univ} are the following four:

(f) we have a short exact sequence $0 \to D^{(1)}_{\mathrm{univ}} \to D_{\mathrm{univ}} \to D^{(0)}_{\mathrm{univ}} \to 0$ of p-divisible groups over $\mathcal{D}(D)$, where $D^{(1)}_{\mathrm{univ}}$ is of multiplicative type and $D^{(0)}_{\mathrm{univ}}$ is étale;

(g) if $k = \overline{k}$, if V is a finite, discrete valuation ring extension of $W(k)$, if $K := V[\frac{1}{p}]$, if D_V is a p-divisible group over V that lifts D, if $T_p^\vee(D_K)$ is the dual of the *Tate module* $T_p(D_K)$ of D_K, then the *p-adic Galois representation* $\mathrm{Gal}(\overline{K}/K) \to \mathbf{GL}_{T_p^\vee(D_K)}(\mathbf{Z}_p)$ factors through the group of $\mathbf{Z}_p$-valued points of a connected, smooth, solvable, closed subgroup scheme of $\mathbf{GL}_{T_p^\vee(D_K)}$;

(h) if $k = \overline{k}$, then the *formal scheme* defined by $\mathcal{D}(D)$ has a canonical structure of a *formal torus* over $\mathrm{Spf}\, W(k)$, the identity section corresponding to the canonical lift of D;

(i) if $k = \overline{k}$, then there exist isomorphisms $\mathcal{D}(D) \tilde{\to} \mathrm{Spec}\, W(k)[[x_1, \ldots, x_{d(r-d)}]]$ that define *canonical coordinates* for $\mathcal{D}(D)$ which are unique up to a suitable action of the group $\mathbf{GL}_{d(r-d)}(\mathbf{Z}_p)$ of automorphisms of the $d(r-d)$ dimensional formal torus over $\mathrm{Spf}\, W(k)$.

The above results on D, $D_{W(k)}$, and $A_{W(k)}$ were published as an appendix to [Me]. The equivalence (a) $\Leftrightarrow$ (b) is also a particular case of a theorem of Mazur (see [Ka2, Thm. 1.4.1]). All the above results on $\mathcal{D}(D)$ are contained in [Ka1,3,4], and [De4]. A variant for principally quasi-polarized p-divisible groups can be deduced easily from the above results and thus it is also part of the classical Serre–Tate ordinary theory. If $k = \overline{k}$, then the formal torus structure of (h) is obtained naturally once one remarks that:

(h.a) the short exact sequence of (f) is the *universal extension* of $D^{(0)}_{\text{univ}} \tilde{\to} (\mathbf{Q}_p/\mathbf{Z}_p)^{r-d}$ by $D^{(1)}_{\text{univ}} \tilde{\to} \boldsymbol{\mu}^d_{p^\infty}$ (here the two isomorphisms are over $\mathcal{D}(D)$);

(h.b) the short exact sequence $0 \to \mathbf{Z}_p^{r-d} \to \mathbf{Q}_p^{r-d} \to (\mathbf{Q}_p/\mathbf{Z}_p)^{r-d} \to 0$ induces a (push forward) coboundary homomorphism $Hom(\mathbf{Z}_p^{r-d}, \boldsymbol{\mu}^d_{p^\infty}) \to Ext^1((\mathbf{Q}_p/\mathbf{Z}_p)^{r-d}, \boldsymbol{\mu}^d_{p^\infty})$ which is an isomorphism between two formal tori over $\operatorname{Spf} W(k)$ of dimension $d(r-d)$.

We recall that a *formal Lie group* $\tilde{\mathcal{F}}$ over $\operatorname{Spf} W(k)$ is a contravariant functor from the category of local, artinian $W(k)$-schemes of residue field k into the category of groups which, when viewed as a functor into the category of sets, is representable by $\operatorname{Spf} \tilde{R}$, where $\tilde{R} = W(k)[[x_1, \ldots, x_m]]$ for some number $m \in \mathbf{N} \cup \{0\}$ called the *dimension* of $\tilde{\mathcal{F}}$ and for some independent variables $x_1, \ldots, x_m$ (see [Me, Ch. 2, Subsect. 1.1.5] and [Fo1]).

1.2. Translation of the classical theory into our language

In Subsection 1.2.1 we recall a few classical results. In Subsections 1.2.2 to 1.2.7 we include intrinsic crystalline interpretations of some parts of Section 1.1. Several interpretations are new and they are meant: (i) to motivate the abstract notions to be introduced in Section 1.3, and (ii) to detail some of the new ideas we will use in this monograph.

Let D be a p-divisible group over k of height r and dimension d. Let (M, φ) be the (contravariant) *Dieudonné module* of D. We recall that M is a free $W(k)$-module of rank r and $\varphi : M \to M$ is a σ-linear endomorphism such that we have $pM \subseteq \varphi(M)$ and $\dim_k(M/\varphi(M)) = d$. Let $\vartheta : M \to M$ be the σ^{-1}-linear endomorphism that is the *Verschiebung map* of (M, φ); we have identities $\vartheta\varphi = \varphi\vartheta = p1_M$. Let $\overline{M} := M/pM$. Let $\overline{\varphi} : \overline{M} \to \overline{M}$ and $\overline{\vartheta} : \overline{M} \to \overline{M}$ be the reductions modulo p of φ and ϑ (respectively). The k-vector space $\operatorname{Ker}(\overline{\varphi}) = \operatorname{Im}(\overline{\vartheta})$ has dimension d. We recall that the *classical Dieudonné theory* says that the category of p-divisible groups over k is antiequivalent to the category of (contravariant) Dieudonné modules over k (see [Fo1, Ch. III, Prop. 6.1 iii)], etc.).

1.2.1. Lifts. Each lift of D to a p-divisible group $D_{W(k)}$ over $W(k)$ defines naturally a direct summand F^1 of M which modulo p is $\operatorname{Ker}(\overline{\varphi})$ (see [Me], [BBM], etc.); the direct summand F^1 is called the *Hodge filtration* of $D_{W(k)}$.

We have $\varphi^{-1}(M) = M + \frac{1}{p}F^1$. We refer to the triple (M, F^1, φ) as a *filtered Dieudonné module* over k and to F^1 as a *lift* of (M, φ). It is well known that:

(a) if $p \geq 3$ or if $p = 2$ and either D or its Cartier dual D^{t} is connected, then the correspondence $D_{W(k)} \leftrightarrow F^1$ is a bijection.

The case $p \geq 3$ is a consequence of the *Grothendieck–Messing deformation theory* (see [Me]). The general case is (for instance) a consequence of [Fo1, Ch. IV, Prop. 1.6]. In loc. cit. it is proved more generally that the following two properties hold:

(b) if $p \geq 3$, then the category of p-divisible groups over $W(k)$ is antiequivalent to the category of filtered Dieudonné modules over k;

(c) if $p = 2$, then the category of p-divisible groups over $W(k)$ that are connected (resp. that have connected Cartier duals) is antiequivalent to the category of filtered Dieudonné modules over k that do not have Newton polygon slope 0 (resp. do not have Newton polygon slope 1).

We emphasize that strictly speaking, [Fo1, Ch. IV, Prop. 1.6] is stated in terms of *Honda triples* $(M, \varphi(\frac{1}{p}F^1), \varphi)$ and not in terms of filtered Dieudonné modules (M, F^1, φ). The classical Hasse–Witt invariant $HW(D)$ of D (or of (M, φ)) can be defined as

$$HW(D) := \dim_k(\cap_{m \in \mathbf{N}} \overline{\varphi}^m(\overline{M})).$$

It is the multiplicity of the Newton polygon slope 0 of (M, φ).

1.2.2. End objects of Dieudonné modules. One can easily translate the theory of ordinary p-divisible groups over k recalled in Section 1.1, in terms of the Dieudonné module (M, φ). We want to make such a translation intrinsic, expressed only in terms of the algebraic group $\mathbf{GL}_M$ (or rather of its Lie algebra associated naturally to $\mathrm{End}_{W(k)}(M)$) and not in terms of the $\mathbf{GL}_M$-module M. Later on in our theory, we will replace $\mathbf{GL}_M$ by a reductive, closed subgroup scheme of it. Next we will have a look back at Section 1.1 from such an intrinsic point of view.

The canonical identification $\mathrm{End}_{B(k)}(M[\frac{1}{p}]) = M[\frac{1}{p}] \otimes_{B(k)} M^\vee[\frac{1}{p}]$ allows us to view $\mathrm{End}_{B(k)}(M[\frac{1}{p}])$ as the $B(k)$-vector space of an F-isocrystal over k. More precisely, we will denote also by φ the σ-linear automorphism of $\mathrm{End}_{B(k)}(M[\frac{1}{p}])$ that takes an element $x \in \mathrm{End}_{B(k)}(M[\frac{1}{p}])$ to $\varphi \circ x \circ \varphi^{-1} \in \mathrm{End}_{B(k)}(M[\frac{1}{p}])$. Let $\Psi : \mathrm{End}_{W(k)}(M) \to \mathrm{End}_{W(k)}(M)$ be the σ-linear endomorphism defined by the rule: for $x \in \mathrm{End}_{W(k)}(M)$ we have

$$\Psi(x) := p(\varphi(x)) = p\varphi \circ x \circ \varphi^{-1} = \varphi \circ x \circ \vartheta \in \mathrm{End}_{W(k)}(M).$$

The pair $(\mathrm{End}_{W(k)}(M), \Psi)$ is the tensor product of the Dieudonné modules of D and D^{t}.

Let $\overline{\Psi} : \mathrm{End}_k(\overline{M}) \to \mathrm{End}_k(\overline{M})$ be the reduction modulo p of Ψ. The integer

$$\mathbf{I}(D) := \dim_k(\cap_{m\in\mathbf{N}}\overline{\Psi}^m(\mathrm{End}_k(\overline{M})))$$

computes the multiplicity of the Newton polygon slope 0 of Ψ and thus also the multiplicity of the Newton polygon slope -1 of $(\mathrm{End}_{B(k)}(M[\frac{1}{p}]), \varphi)$. Based on this and the last sentence of the previous paragraph, we get the identity

$$\mathbf{I}(D) = HW(D)HW(D^{\mathrm{t}}).$$

In particular, if D has a quasi-polarization (i.e., if D is isogenous to D^{t} and therefore we have $r = 2d$), then we have $HW(D) = \sqrt{\mathbf{I}(D)}$. Moreover, we easily get that

$$D \text{ is ordinary} \Longleftrightarrow \mathbf{I}(D) = d(r-d).$$

1.2.3. Characterization of p-torsion subgroup schemes in terms of $(\mathrm{End}_k(\overline{M}), \overline{\Psi})$. For $g \in \mathbf{GL}_M(W(k))$, let $\overline{g}^{\mathrm{ad}}$ be its image in $\mathbf{GL}_M^{\mathrm{ad}}(k)$. Let D_1 be the p-divisible group over k whose Dieudonné module is $(M, g\varphi)$. If $k = \overline{k}$, then the σ-linear endomorphism Ψ "encompasses" both maps φ, $\vartheta : M \to M$. We exemplify this property modulo p.

Claim. *If $k = \overline{k}$, then there exists an isomorphism $D[p]\tilde{\to}D_1[p]$ if and only if there exists an element $\overline{h} \in \mathbf{GL}_M^{\mathrm{ad}}(k)$ such that as σ-linear endomorphisms of* $\mathrm{End}_k(\overline{M})$ *we have an identity $\overline{h}\overline{\Psi} = \overline{g}^{\mathrm{ad}}\overline{\Psi}\overline{h}$ (here we need $k = \overline{k}$ as Ψ can not "differentiate" between φ and its $\mathbf{G}_m(W(k))$-multiples).*

We check here the "if" part of the Claim. Let $h \in \mathbf{GL}_M(W(k))$ be an element such that we have $\overline{h}^{\mathrm{ad}} = \overline{h}$. As we have $\overline{h}\overline{\Psi} = \overline{g}^{\mathrm{ad}}\overline{\Psi}\overline{h}$, the reduction modulo p of h normalizes $\mathrm{Ker}(\overline{\Psi}) = \{\overline{x} \in \mathrm{End}_k(\overline{M}) | \overline{x}(\mathrm{Im}(\overline{\vartheta})) \subseteq \mathrm{Ker}(\overline{\varphi})\}$ and therefore it normalizes $\mathrm{Ker}(\overline{\varphi}) = \mathrm{Im}(\overline{\vartheta})$. This implies that $(M, h\varphi h^{-1})$ is a Dieudonné module. Thus by replacing φ with $h\varphi h^{-1}$, we can assume that $\overline{h} \in \mathbf{GL}_M^{\mathrm{ad}}(k)$ is the identity element; therefore $\overline{\Psi} = \overline{g}^{\mathrm{ad}}\overline{\Psi}$. We define $\overline{M_1} := \varphi(\frac{1}{p}F^1)/\varphi(F^1)$ and $\overline{M_0} := \mathrm{Im}(\overline{\varphi})$. We have a direct sum decomposition $\overline{M} = \overline{M_1} \oplus \overline{M_0}$ of k-vector spaces, cf. the identity $\varphi^{-1}(M) = M + \frac{1}{p}F^1$. As $\overline{\Psi} = \overline{g}^{\mathrm{ad}}\overline{\Psi}$, the element $\overline{g}^{\mathrm{ad}}$ fixes $\mathrm{Im}(\overline{\Psi})$. But $\mathrm{Im}(\overline{\Psi})$ is the direct summand $\mathrm{Hom}_k(\overline{M_1}, \overline{M_0})$ of

$$\mathrm{End}_k(\overline{M}) = \mathrm{End}_k(\overline{M_1}) \oplus \mathrm{End}_k(\overline{M_0}) \oplus \mathrm{Hom}_k(\overline{M_1}, \overline{M_0}) \oplus \mathrm{Hom}_k(\overline{M_0}, \overline{M_1})$$

and thus the centralizer of $\mathrm{Im}(\overline{\Psi})$ in $\mathrm{End}_k(\overline{M})$ is $\mathrm{Im}(\overline{\Psi}) \oplus k1_{\overline{M}}$. Therefore there exists an element $x \in \mathrm{End}_{W(k)}(M)$ such that $\overline{g}^{\mathrm{ad}}$ is the image in

$\mathbf{GL}_M^{\mathrm{ad}}(k)$ of the element $1_M + \Psi(x) \in \mathbf{GL}_M(W(k))$. Let $g_1 := 1_M + px \in \mathbf{GL}_M(W(k))$. By replacing $g\varphi$ with

$$g_1 g\varphi g_1^{-1} = g_1 g\varphi g_1^{-1}\varphi^{-1}\varphi = g_1 g(1_M + \Psi(x))^{-1}\varphi,$$

g gets replaced by the element $g_1 g(1_M + \Psi(x))^{-1}$ whose image in $\mathbf{GL}_M^{\mathrm{ad}}(k)$ is the identity element. Therefore we can assume that $\bar{g}^{\mathrm{ad}} \in \mathbf{GL}_M^{\mathrm{ad}}(k)$ is the identity element. Thus, as $D_1[p]$ depends only on g modulo p, we can assume that $g \in Z(\mathbf{GL}_M)(W(k))$. In this last case, as $k = \bar{k}$ there exists an isomorphism $(M, g\varphi)\tilde{\to}(M, \varphi)$ defined by an element of $Z(\mathbf{GL}_M)(W(k))$ and therefore in fact we have $D\tilde{\to}D_1$.

1.2.4. New p-divisible groups via Lie algebras. Until Section 1.3 we will work over an arbitrary perfect field k of characteristic p and we take D to be ordinary and $D_{W(k)}$ to be its canonical lift. We have $\varphi(F^1) = pF^1$. Let F^0 be the direct supplement of F^1 in M such that we have $\varphi(F^0) = F^0$. Let P^+ and P^- be the maximal parabolic subgroup schemes of $\mathbf{GL}_M$ that normalize F^1 and F^0 (respectively). The map Ψ leaves invariant both $\mathrm{Lie}(P^+)$ and $\mathrm{Lie}(P^-)$. For $* \in \{+, -\}$ let U^* be the unipotent radical of P^* and let S^* be the solvable, closed subgroup scheme of $\mathbf{GL}_M$ generated by U^* and by the image of the cocharacter $\mu : \mathbf{G}_m \to \mathbf{GL}_M$ that fixes F^0 and that acts as the inverse of the identical character of $\mathbf{G}_m$ on F^1. We remark that U^* is constructed canonically in terms of the F-isocrystal $(\mathrm{End}_{B(k)}(M[\frac{1}{p}]), \varphi)$, as $\mathrm{Lie}(U^*)[\frac{1}{p}]$ is the Newton polygon slope $*1$ part of it. The cocharacter $\mu : \mathbf{G}_m \to \mathbf{GL}_M$ is *minuscule* and it is a *p-adic Hodge cocharacter* analogue to the classical Hodge cocharacters.

We identify naturally $\mathrm{Lie}(U^+)$ with $\mathrm{Hom}_{W(k)}(F^0, F^1)$, $\mathrm{Lie}(U^-)$ with $\mathrm{Hom}_{W(k)}(F^1, F^0)$, $\mathrm{Lie}(S^+)$ with $\mathrm{Hom}_{W(k)}(F^0, F^1) \oplus W(k)1_{F^1}$, and $\mathrm{Lie}(S^-)$ with $\mathrm{Hom}_{W(k)}(F^1, F^0) \oplus W(k)1_{F^1}$ (the last two identities are of $W(k)$-modules). The $W(k)$-module

$$E := \mathrm{End}_{W(k)}(F^0) \oplus \mathrm{Hom}_{W(k)}(F^1, F^0) \oplus \{x \in \mathrm{End}_{W(k)}(F^1) | \text{trace of } x \text{ is } 0\}$$

is left invariant by Ψ. If p does not divide $d = \mathrm{rk}_{W(k)}(F^1)$, then we have a direct sum decomposition $\mathrm{End}_{W(k)}(M) = E \oplus \mathrm{Lie}(S^+)$ left invariant by Ψ. In general (i.e., regardless of the fact that p does or does not divide d), we can identify $(\mathrm{Lie}(S^+), \Psi)$ with $(\mathrm{End}_{W(k)}(M)/E, \Psi)$ in such a way that $\mathrm{Hom}_{W(k)}(F^0, F^1)$ gets identified with its image in $\mathrm{End}_{W(k)}(M)/E$ and 1_{F^1} maps to the image in $\mathrm{End}_{W(k)}(M)/E$ of an arbitrary endomorphism $t \in \mathrm{End}_{W(k)}(F^1)$ whose trace is 1 if $d \neq 0$; note that the $W(k)$-linear isomorphism $\mathrm{Lie}(S^+)\tilde{\to}\mathrm{End}_{W(k)}(M)/E$ that produces such an identification is not in general induced naturally by the inclusion $\mathrm{Lie}(S^+) \hookrightarrow \mathrm{End}_{W(k)}(M)$.

Let $D^+_{W(k)}$ be the p-divisible group of multiplicative type over $W(k)$ whose filtered Dieudonné module is $(\mathrm{Lie}(U^+), \mathrm{Lie}(U^+), \varphi)$. We call it the *positive p-divisible group* over $W(k)$ of either D or (M, φ). The p-divisible group $D^{+(0)}$ over k whose Dieudonné module is $(\mathrm{Lie}(S^+), \varphi)$ is called the *standard non-negative p-divisible group* of either D or (M, φ); it is the direct sum of $\mathbf{Q}_p/\mathbf{Z}_p$ (over k) and of the special fibre D^+ of $D^+_{W(k)}$.

Let $D^-_{W(k)}$ be the étale p-divisible group over $W(k)$ whose filtered Dieudonné module is $(\mathrm{Lie}(U^-), 0, \Psi)$. We call it the *negative p-divisible group* over $W(k)$ of either D or (M, φ). The p-divisible group $D^{-(0)}$ over k whose Dieudonné module is $(\mathrm{Lie}(S^-), \Psi)$ is called the *standard non-positive p-divisible group* of either D or (M, φ); it is the direct sum of $\boldsymbol{\mu}_{p^\infty}$ (over k) and of the special fibre D^- of $D^-_{W(k)}$.

1.2.5. Deformation theory revisited. Let $R := W(k)[[x_1, \ldots, x_{d(r-d)}]]$. Let Φ_R be the Frobenius lift of R that is compatible with σ and that takes x_i to x_i^p for all $i \in \{1, \ldots, d(r-d)\}$. Deforming D is equivalent to deforming $D^{+(0)}$. The easiest way to see this is to start with a *versal deformation* of D over $\mathrm{Spec}\, R$ and to show that the deformation of $D^{+(0)}$ we get naturally by identifying $\mathrm{Lie}(S^+)$ with $\mathrm{End}_{W(k)}(M)/E$ is versal. At the level of *filtered F-crystals* over R/pR, this goes as follows.

Let $u^-_{\mathrm{univ}} : \mathrm{Spec}\, R \to U^-$ be a formally étale morphism which modulo the ideal $\mathcal{I} := (x_1, \ldots, x_{d(r-d)})$ of R defines the identity section of U^-. We also identify u^-_{univ} with an R-linear automorphism of $M \otimes_{W(k)} R$ which modulo $\mathcal{I}$ is 1_M. *Faltings deformation theory* (see [Fa2, Thm. 10]) and [Me, Ch. II, Lem. 4.16] assure us that there exists a p-divisible group over $\mathrm{Spec}\, R$ such that: (i) its reduction modulo $\mathcal{I}$ is $D_{W(k)}$, and (ii) its filtered F-crystal over R/pR is the quadruple

$$\mathfrak{C} := (M \otimes_{W(k)} R, F^1 \otimes_{W(k)} R, u^-_{\mathrm{univ}}(\varphi \otimes \Phi_R), \nabla),$$

where ∇ is an integrable, nilpotent modulo p *connection* on $M \otimes_{W(k)} R$ that is uniquely determined by $u^-_{\mathrm{univ}}(\varphi \otimes \Phi_R)$. It is easy to see that the Kodaira–Spencer map of ∇ is an isomorphism. Thus we can naturally identify $\mathcal{D}(D)$ with $\mathrm{Spec}\, R$.

Similarly (via the mentioned $W(k)$-linear isomorphism $\mathrm{Lie}(S^+) \tilde{\to} \mathrm{End}_{W(k)}(M)/E$) we have a natural identification $\mathcal{D}(D) = \mathrm{Spec}\, R$ under which the filtered F-crystal of the deformation of $D^{+(0)}$ over $\mathcal{D}(D^{+(0)})$ is the following quadruple

$$\begin{aligned}\mathfrak{C}^{+(0)} :=& ((\mathrm{End}_{W(k)}(M)/E) \otimes_{W(k)} R, \mathrm{Hom}_{W(k)}(F^0, F^1) \otimes_{W(k)} R,\\ & u^-_{\mathrm{univ}}(\varphi \otimes \Phi_R), \nabla^{+(0)}),\end{aligned}$$

where $\nabla^{+(0)}$ is the connection on $(\mathrm{End}_{W(k)}(M)/E) \otimes_{W(k)} R$ induced naturally by ∇. The group scheme U^- acts naturally on $\mathrm{End}_{W(k)}(M)/E$ via inner conjugation and passage to quotients and therefore u^-_{univ} acts accordingly on $(\mathrm{End}_{W(k)}(M)/E) \otimes_{W(k)} R$.

We explain why $\mathfrak{C}^{+(0)}$ defines indeed a versal deformation of $D^{+(0)}$. Let U^{-+} be the unipotent radical of the maximal parabolic subgroup scheme of $\mathbf{GL}_{\mathrm{End}_{W(k)}(M)/E}$ that normalizes the image of $W(k)t$ in $\mathrm{End}_{W(k)}(M)/E$ (see Subsection 1.2.4 for t). The natural homomorphism $U^- \to \mathbf{GL}_{\mathrm{End}_{W(k)}(M)/E}$ is a monomorphism that induces an isomorphism $q : U^- \tilde{\to} U^{-+}$. At the level of matrices this is equivalent to the following statement: if C is a commutative $W(k)$-algebra and if $x \in \mathrm{Hom}_{W(k)}(F^1, F^0) \otimes_{W(k)} C \setminus 0$, then for $n_C := 1_{M \otimes_{W(k)} C} + x = \left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right) \in U^-(C)$ there exists $y = \left(\begin{smallmatrix} 0 & y \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{Hom}_{W(k)}(F^0, F^1) \otimes_{W(k)} C$ such that the element

$$n_C y n_C^{-1} - y = \begin{pmatrix} -yx & 0 \\ -xyx & xy \end{pmatrix} \in \mathrm{End}_{W(k)}(M) \otimes_{W(k)} C$$

has the property that the trace of $-yx \in \mathrm{End}_{W(k)}(F^1) \otimes_{W(k)} C$ is nonzero. Here the 2×2 (block) matrices are with respect to the direct sum decomposition $M = F^1 \oplus F^0$ and thus their entries are matrices themselves.

Let $u^{-+}_{\mathrm{univ}} := q \circ u^-_{\mathrm{univ}} : \mathrm{Spec}\, R \to U^{-+}$; it is a formally étale morphism which modulo $\mathcal{I}$ defines the identity section of U^{-+}. This implies that $\mathfrak{C}^{+(0)}$ defines indeed a versal deformation of $D^{+(0)}$.

Thus we can identify naturally $\mathcal{D}(D)$ with $\mathrm{Spec}\, R = \mathcal{D}(D^{+(0)})$. But as we have $D^{+(0)} = \mathbf{Q}_p/\mathbf{Z}_p \oplus D^+$, the classical Serre–Tate ordinary theory tells us that the formal scheme defined by $\mathcal{D}(D^{+(0)})$ has a canonical structure of a formal Lie group $\mathcal{L}^+$ over $\mathrm{Spf}\, W(k)$. We recall that if C is a local, artinian $W(k)$-algebra of residue field k, then the addition operation on the set $\mathcal{D}(D^{+(0)})(C) = \mathcal{L}^+(C)$ is defined via the Yoneda addition law for short exact sequences of p-divisible groups over C that are of the following form

$$0 \to D^+_C \to \tilde{D}^{+(0)}_C \to \mathbf{Q}_p/\mathbf{Z}_p \to 0.$$

It is well known that $\mathcal{L}^+$ is isomorphic to the formal Lie group of $D^+_{W(k)}$ (see Lemma 5.3.3; see also the property 1.1 (h.b)).

Due to the identification $\mathcal{D}(D) = \mathcal{D}(D^{+(0)})$, the formal scheme defined by $\mathcal{D}(D)$ has also a natural structure of a formal Lie group $\mathcal{L}$ which can be identified with $\mathcal{L}^+$.

1.2.6. Sums of lifts. Let z_3 be the addition of two $\mathrm{Spf}\, W(k)$-valued points z_1 and z_2 of the formal scheme defined by $\mathcal{D}(D)$ with respect to $\mathcal{L}$. For $i \in \{1, 2, 3\}$ let F^1_i be the lift of (M, φ) that corresponds to z_i. There exists a unique element $u_i \in \mathrm{Ker}(U^-(W(k)) \to U^-(k))$ such that we have $u_i(F^1) = F_i^1$. Let $s_i := u_i - 1_M \in \mathrm{Lie}(U^-)$. Let z_i^+ be the Spf $W(k)$-valued point of $\mathcal{L}^+$ defined by z_i. Let $u_i^+ := q(W(k))(u_i) \in U^{-+}(W(k))$. The filtered Dieudonné module associated to z_i^+ is $(\mathrm{End}_{W(k)}(M)/E, u_i^+(\mathrm{Hom}_{W(k)}(F^0, F^1)), \varphi)$, cf. the constructions of Subsection 1.2.5. To the addition of short exact sequences mentioned in Subsection 1.2.5 (applied over quotients C of $W(k)$), corresponds the multiplication formula $u_3^+ = u_1^+ u_2^+$. As q is an isomorphism, we get $u_3 = u_1 u_2$. In other words the lift F_3^1 is the "*sum*" of F_1^1 and F_2^1 i.e., we have

(1)
$$s_3 = s_1 + s_2 \text{ (additively) or equivalently } u_3 = u_1 u_2 \text{ (multiplicatively).}$$

If $p \geq 3$ or if $p = 2$ and $d \in \{0, r\}$, then z_3 is uniquely determined by u_3 (cf. property 1.2.1 (a)). If $p = 2$ and $1 \leq d \leq r-1$, then it is well known that z_3 is uniquely determined by u_3 up to a 2-torsion point of $\mathcal{L}(\mathrm{Spf}\, W(k))$ (this is also a direct consequence of Proposition 9.5.1). Formula (1) guarantees that in the case when $k = \bar{k}$, the formal Lie group structure $\mathcal{L}$ on the formal scheme defined by $\mathcal{D}(D)$ is the one defined by the classical Serre–Tate deformation theory (see properties 1.1 (h.a) and (h.b)). This holds even if $p = 2$ and $1 \leq d \leq r - 1$, as each automorphism of formal schemes $\mathcal{L} \tilde{\to} \mathcal{L}$ that respects identity sections and whose composite with the square endomorphism $[2] : \mathcal{L} \to \mathcal{L}$ is an endomorphism of formal Lie groups, is in fact an automorphism of formal Lie groups.

1.2.7. Duality of language: positive versus negative. The trace form on $\mathrm{End}_{W(k)}(M)$ restricts to a perfect bilinear form

$$b : \mathrm{Lie}(U^-) \times \mathrm{Lie}(U^+) \to W(k)$$

i.e., the natural $W(k)$-linear map $\mathrm{Lie}(U^-)^\vee \to \mathrm{Lie}(U^+)$ defined by b is a $W(k)$-linear isomorphism. This $W(k)$-linear isomorphism allows us to identify naturally $D^+_{W(k)}$ with the Cartier dual $(D^-_{W(k)})^{\mathrm{t}}$ of $D^-_{W(k)}$, cf. properties 1.2.1 (b) and (c). As in Subsections 1.2.4 and 1.2.5, one checks that deforming D is equivalent to deforming $D^{-(0)}$. The main advantage of working with $D^{-(0)}$ is that it involves simpler group theoretical arguments (like we do not have to consider $W(k)$-linear isomorphisms $\mathrm{Lie}(S^+) \tilde{\to} \mathrm{End}_{W(k)}(M)/E$) which are more adequate for suitable generalizations. The main two disadvantages of working with $D^{-(0)}$ are: (i) we have to use the bilinear form b in order to identify $D^+_{W(k)}$ with $(D^-_{W(k)})^{\mathrm{t}}$ and (ii) we have to work with Ψ instead of with φ.

1.3. Generalized language

1.3.1. Goal. The goal of the monograph is to generalize the results recalled in Section 1.1 about ordinary p-divisible groups over k and their canonical lifts, to:

• the **abstract context** of *Shimura p-divisible objects* over k (see this section and the next one), and to

• the **geometric context** of good *moduli spaces* of principally polarized abelian schemes endowed with specializations of *Hodge cycles* (see Sections 1.5 and 1.6).

For generalizing the notions of ordinariness and canonical lifts to the abstract context, we do not require any deformation theory and accordingly we do not construct explicitly abstract *formal deformation spaces.* Thus not to make this monograph too long, we do not present here the generalization of properties 1.1 (h) and (i) and of the deformation theories of [dJ], [Fa2, Sect. 7], and [Zi2] to the abstract context. Moreover, in order to avoid repetitions, the greatest part of the generalization of properties 1.1 (f) to (i) will be stated only for the geometric context and will be a restricted generalization in the case of properties 1.1 (h) and (i).

The formal deformation spaces over $\mathrm{Spf}\, W(\bar{k})$ we get in the geometric context are provided by completions of local ring of integral models and admit as well (due to Faltings deformation theory) a simple and concrete direct description. Moreover they are expected to have natural structures of commutative formal Lie groups as well as some other *nilpotent structures.* In this monograph we mainly restrict to the case when these two types of structures are expected to coincide (i.e., in the so called commutative case): in this case we show that the formal deformation spaces do have canonical structures of commutative formal Lie groups over $\mathrm{Spf}\, W(\bar{k})$ that are isomorphic to formal Lie groups of a very specific type of p-divisible groups over $W(\bar{k})$.

Any future generalization of the property 1.1 (i) to the most general geometric case will have to rely on a comprehensive theory of connections on these formal deformation spaces; therefore in this monograph we mainly use connections only to understand their Kodaira–Spencer maps. This theory of connections will allow us in future work to define different natural structures on the formal deformation spaces (including as well the "forcing" to get formal tori) that will ease and enrich the study of these formal deformation spaces.

See Section 1.7 for applications. See Section 1.8 for extra literature that pertains to Sections 1.3 to 1.7. See Section 1.9 for our main motivation. See Section 1.10 for more details on how Chapters 2 to 9 are organized. For a,

$b \in \mathbf{Z}$ with $b \geq a$, let

$$[a, b]_{\mathbf{Z}} := \{a, a+1, \ldots, b\} = [a, b] \cap \mathbf{Z}.$$

We now follow [Va9, Subsect. 1.2] to introduce the language that will allow us to generalize the classical Serre–Tate ordinary theory. To motivate it, let $D_{W(k)}$ be the p-divisible group of an abelian scheme A over $W(k)$ and let (M, F^1, φ) be its filtered Dieudonné module. Often the abelian variety $A_{\overline{B(k)}}$ is endowed with a set of some type of cycles (algebraic, Hodge, crystalline, etc.) normalized by the Galois group $\mathrm{Gal}(\overline{B(k)}/B(k))$ and this leads to the study of quadruples of the form (M, F^1, φ, G), where G is a flat, closed subgroup scheme of $\mathbf{GL}_M$ such that the Lie algebra $\mathrm{Lie}(G_{B(k)})$ is normalized by φ and we have a direct sum decomposition $M = F^1 \oplus F^0$ with the property that the image of a cocharacter $\mathbf{G}_m \to \mathbf{GL}_M$ obtained as in Subsection 1.2.4, is contained in G. We think of $G_{\overline{B(k)}}$ as the subgroup of $\mathbf{GL}_{M \otimes_{W(k)} \overline{B(k)}}$ that fixes the crystalline realizations of these cycles. In this monograph we study the case when G is a reductive, closed subgroup scheme of $\mathbf{GL}_M$. Moving from the geometric context of abelian schemes to a more general and abstract context "related to" *Shimura varieties*, we have the following basic definition.

1.3.2. Basic definition. A *Shimura filtered p-divisible object* over k in the *range* $[a, b]$ is a quadruple

$$(M, (F^i(M))_{i \in [a,b]_{\mathbf{Z}}}, \varphi, G),$$

where

– M is a free $W(k)$-module of finite rank,

– $(F^i(M))_{i \in [a,b]_{\mathbf{Z}}}$ is a decreasing and exhaustive filtration of M by direct summands,

– φ is a σ-linear automorphism of $M[\frac{1}{p}]$, and

– G is a reductive, closed subgroup scheme of $\mathbf{GL}_M$,

such that there exists a direct sum decomposition $M = \oplus_{i=a}^{b} \tilde{F}^i(M)$ for which the following three properties hold:

(a) we have an identity $F^i(M) = \oplus_{j=i}^{b} \tilde{F}^j(M)$ for all $i \in [a, b]_{\mathbf{Z}}$ and moreover $\varphi^{-1}(M) = \oplus_{i=a}^{b} p^{-i} \tilde{F}^i(M)$ (equivalently and moreover $\varphi^{-1}(M) = \sum_{i=a}^{b} p^{-i} F^i(M)$);

(b) the cocharacter of $\mathbf{GL}_M$ that acts on $\tilde{F}^i(M)$ via the $-i$-th power of the identity character of $\mathbf{G}_m$, factors through G and this factorization

$\mu : \mathbf{G}_m \to G$ is either a minuscule cocharacter or factors through $Z^0(G)$ i.e., we have a direct sum decomposition

$$\mathrm{Lie}(G) = \oplus_{j\in\{-1,0,1\}} \tilde{F}^j(\mathrm{Lie}(G))$$

such that $\mu(\beta)$ acts (via the adjoint representation) on $\tilde{F}^j(\mathrm{Lie}(G))$ as the multiplication with β^{-j} for all $j \in \{-1, 0, 1\}$ and all $\beta \in \mathbf{G}_m(W(k))$;

(c) denoting also by φ the σ-linear automorphism of $\mathrm{End}_{B(k)}(M[\frac{1}{p}])$ that takes $x \in \mathrm{End}_{B(k)}(M[\frac{1}{p}])$ to $\varphi \circ x \circ \varphi^{-1} \in \mathrm{End}_{B(k)}(M[\frac{1}{p}])$, we have $\varphi(\mathrm{Lie}(G)[\frac{1}{p}]) = \mathrm{Lie}(G)[\frac{1}{p}]$ i.e., $(\mathrm{Lie}(G)[\frac{1}{p}], \varphi)$ is an F-subisocrystal of $(\mathrm{End}_{B(k)}(M[\frac{1}{p}]), \varphi)$.

We call the triple (M, φ, G) a *Shimura p-divisible object* over k in the *range* $[a, b]$. Following [Pi], we refer to μ as a *Hodge cocharacter* of (M, φ, G). We refer to the quadruple $(M, (F^i(M))_{i\in[a,b]_{\mathbf{Z}}}, \varphi, G)$ or to the filtration $(F^i(M))_{i\in[a,b]_{\mathbf{Z}}}$ (of M) as a *lift* of (M, φ, G) or as the lift of (M, φ, G) defined by μ. If $(a, b) = (0, 1)$, then we do not mention $F^0(M) = M$ and thus we also refer to (M, φ, G) (resp. to $(M, F^1(M), \varphi, G)$) as a *Shimura* (resp. as a *Shimura filtered*) *F-crystal* over k and to $(M, F^1(M), \varphi, G)$ or to $F^1(M)$ as a *lift* of (M, φ, G). Often we do not mention over k or in the range $[a, b]$. For $i \in [a, b]_{\mathbf{Z}}$ let $\varphi_i : F^i(M) \to M$ be the restriction of $p^{-i}\varphi$ to $F^i(M)$. Triples of the form $(M, (F^i(M))_{i\in[a,b]_{\mathbf{Z}}}, \varphi)$ show up in [La] and [Wi]. If $m \in \mathbf{N}$, then the reduction modulo p^m of $(M, (F^i(M))_{i\in[a,b]_{\mathbf{Z}}}, (\varphi_i)_{i\in[a,b]_{\mathbf{Z}}})$ is an object of the category $\mathcal{MF}_{[a,b]}(W(k))$ used in [FL], [Wi], and [Fa1]. This and the fact that Definition 1.3.2 can be adapted over arbitrary regular, formally smooth $W(k)$-schemes, justifies our terminology "p-divisible object" (it extrapolates the terminology object used in [Fa1]). Splittings $M = \oplus_{i=a}^{b} \tilde{F}^i(M)$ of the lift $(F^i(M))_{i\in[a,b]_{\mathbf{Z}}}$ of (M, φ, G) show up first in [Wi].

See [Va9, Subsect. 1.2] for the main reasons we use the above language and not the rational language of isocrystals with a $\triangle$-structure used in [Ko1] and [RR, Def. 3.3] (here $\triangle$ stands for a suitable $\mathbf{Q}_p$-form of $G_{B(k)}$). In particular, our language allows us to work over all perfect fields without assuming the existence of a suitable $\mathbf{Z}_p$ structure of (M, φ, G) and to consider different invariants and properties which in general depend on the choice of the $W(k)$-lattice M of $M[\frac{1}{p}]$ (such as the isomorphism class of the G-module M, etc.).

1.3.3. Parabolic subgroup schemes. Let

$$F^0(\mathrm{Lie}(G)) := \tilde{F}^0(\mathrm{Lie}(G)) \oplus \tilde{F}^1(\mathrm{Lie}(G)).$$

Let P be the normalizer of $(F^i(M))_{i\in[a,b]_{\mathbf{Z}}}$ in G; it is a parabolic subgroup scheme of G whose Lie algebra is $F^0(\mathrm{Lie}(G))$ (cf. Subsection 2.5.3).

By the *non-negative parabolic subgroup scheme* of (M,φ,G) we mean the parabolic subgroup scheme

$$P_G^+(\varphi)$$

of G with the properties that $\mathrm{Lie}(P_G^+(\varphi))[\frac{1}{p}]$ is normalized by φ and that all Newton polygon slopes of $(\mathrm{Lie}(P_G^+(\varphi))[\frac{1}{p}],\varphi)$ (resp. of $(\mathrm{Lie}(G)[\frac{1}{p}]/\mathrm{Lie}(P_G^+(\varphi))[\frac{1}{p}],\varphi))$ are non-negative (resp. are negative). By the *Levi subgroup* of (M,φ,G) we mean the unique Levi subgroup

$$L_G^0(\varphi)_{B(k)}$$

of $P_G^+(\varphi)_{B(k)}$ with the property that $\mathrm{Lie}(L_G^0(\varphi)_{B(k)})$ is normalized by φ. In Subsections 2.6.1 and 2.6.2 we will argue following [Va9, Subsect. 2.3] the existence of the subgroup schemes $P_G^+(\varphi)$ and $L_G^0(\varphi)_{B(k)}$. In Subsection 2.6.2 we also check that all Newton polygon slopes of $(\mathrm{Lie}(L_G^0(\varphi)_{B(k)}),\varphi)$ are 0 and that $\mathrm{Lie}(L_G^0(\varphi)_{B(k)})$ is the maximal $B(k)$-vector subspace of $\mathrm{Lie}(P_G^+(\varphi)_{B(k)})$ which has this property.

1.3.4. A new Hasse–Witt type of invariant. We consider the product decomposition $G_k^{\mathrm{ad}}=\prod_{i\in I}G_{i,k}$ of G_k^{ad} into simple, adjoint groups over k, cf. [Ti1, Subsubsect. 3.1.2]. It lifts naturally to a product decomposition

$$G^{\mathrm{ad}}=\prod_{i\in I}G_i \tag{2}$$

of G^{ad} into simple, adjoint group schemes over $W(k)$, cf. [DG, Vol. III, Exp. XXIV, Prop. 1.21]. We have $\mathrm{Lie}(G^{\mathrm{ad}})[\frac{1}{p}]=[\mathrm{Lie}(G),\mathrm{Lie}(G)][\frac{1}{p}]$. Thus $\mathrm{Lie}(G^{\mathrm{ad}})=\oplus_{i\in I}\mathrm{Lie}(G_i)$ is naturally a Lie subalgebra of $\mathrm{Lie}(G)[\frac{1}{p}]$.

By the *FSHW shift* of (M,φ,G) we mean the σ-linear endomorphism

$$\Psi:\mathrm{Lie}(G^{\mathrm{ad}})\to\mathrm{Lie}(G^{\mathrm{ad}})$$

which for $i\in I$ takes $x\in\mathrm{Lie}(G_i)$ to $\varphi(x)$ or to $p(\varphi(x))=\varphi(px)$ depending on the fact that the composite of $\mu:\mathbf{G}_m\to G$ with the projection epimorphism $G\twoheadrightarrow G_i$ is or is not trivial. The existence of the map Ψ is checked in Subsection 2.4.3. Let $\overline{\Psi}:\mathrm{Lie}(G_k^{\mathrm{ad}})\to\mathrm{Lie}(G_k^{\mathrm{ad}})$ be the reduction modulo p of Ψ; we call it the *FSHW map* of (M,φ,G). By the *FSHW invariant* of (M,φ,G) we mean the following dimension

$$\mathbf{I}(M,\varphi,G):=\dim_k(\cap_{m\in\mathbf{N}}\mathrm{Im}(\overline{\Psi}^m))=\dim_k(\mathrm{Im}(\overline{\Psi}^{\dim(G_k^{\mathrm{ad}})}))\in\mathbf{N}\cup\{0\}.$$

Here *FSHW* stands for Faltings–Shimura–Hasse–Witt. If for each element $i\in I$ the composite of $\mu:\mathbf{G}_m\to G$ with the projection epimorphism

$G \twoheadrightarrow G_i$ is not trivial, then Ψ is a usual *Tate twist* of φ (i.e., for each element $x \in \mathrm{Lie}(G^{\mathrm{ad}})$ we have $\Psi(x) = p\varphi(x)$).

1.3.5. Two different notions of ordinariness and of canonical lifts. We say $(M, (F^i(M))_{i\in[a,b]_{\mathbf{Z}}}, \varphi, G)$ is a *Uni-canonical lift* (of (M, φ, G)) if $L^0_G(\varphi)_{B(k)}$ is a subgroup of $P_{B(k)}$. If moreover $P^+_G(\varphi)$ is a (closed) subgroup scheme of P, then we say $(M, (F^i(M))_{i\in[a,b]_{\mathbf{Z}}}, \varphi, G)$ is a *Sh-canonical lift* (of (M, φ, G)).

We say (M, φ, G) is *Uni-ordinary* (resp. *Sh-ordinary*) if it has a lift that is a Uni-canonical (resp. that is a Sh-canonical) lift.

Here Uni stands for unique and Sh stands for Shimura. If $G = \mathbf{GL}_M$ and $0 \le a \le b \le 1$, then it is easy to see that the notions Uni-ordinary and Sh-ordinary (resp. Uni-canonical lift and Sh-canonical lift) coincide with the usual notion ordinary (resp. canonical lift) introduced in Section 1.1.

The notion Uni-ordinariness is more general than the notion Sh-ordinariness. Uni-canonical lifts which are not Sh-canonical lifts typically show up when $\varphi \otimes \sigma_{\overline{k}}$ produces a non-trivial permutation of the simple factors of $\mathrm{Lie}(G^{\mathrm{ad}}_{B(\overline{k})})$ (see Subsections 4.3.3 to 4.3.5 for several examples and see Section 7.5 for the simplest geometric examples).

Next we introduce two extra concepts that pertain to Shimura p-divisible objects.

1.3.6. Definitions. (a) By an *endomorphism* of $(M, (F^i(M))_{i\in[a,b]_{\mathbf{Z}}}, \varphi, G)$ (resp. of (M, φ, G)) we mean an element $e \in F^0(\mathrm{Lie}(G))$ (resp. $e \in \mathrm{Lie}(G)$) that is fixed by φ.

(b) Let $g_1, g_2 \in G(W(k))$. We say $(M, g_1\varphi, G)$ and $(M, g_2\varphi, G)$ are *inner isomorphic*, if there exists an element $h \in G(W(k))$ such that we have $hg_1\varphi = g_2\varphi h$. We also refer to such an h as an *inner isomorphism* between $(M, g_1\varphi, G)$ and $(M, g_2\varphi, G)$.

1.3.7. Characterization of Uni-canonical lifts in terms of endomorphisms. We have $L^0_G(\varphi)_{B(k)} \leqslant P_{B(k)}$ if and only if we have an inclusion $\mathrm{Lie}(L^0_G(\varphi)_{B(k)}) \subseteq \mathrm{Lie}(P_{B(k)})$, cf. [Bo, Ch. II, Sect. 7.1]. Thus $L^0_G(\varphi)_{B(k)} \leqslant P_{B(k)}$ if and only if $\mathrm{Lie}(L^0_G(\varphi)_{B(\overline{k})}) \subseteq \mathrm{Lie}(P_{B(\overline{k})})$. But

$$\mathfrak{L} := \mathrm{Lie}(G_{W(\overline{k})}) \cap \{x \in \mathrm{Lie}(L^0_G(\varphi)_{B(\overline{k})}) | \varphi \otimes \sigma_{\overline{k}}(x) = x\}$$

is the Lie algebra over $\mathbf{Z}_p$ of endomorphisms of $(M \otimes_{W(k)} W(\overline{k}), \varphi \otimes \sigma_{\overline{k}}, G_{W(\overline{k})})$ and moreover $\mathrm{Lie}(L^0_G(\varphi)_{B(\overline{k})})$ is $B(\overline{k})$-generated by $\mathfrak{L}$. Thus $L^0_G(\varphi)_{B(k)} \leqslant P_{B(k)}$ if and only if we have $\mathfrak{L} \subseteq \mathrm{Lie}(P_{B(\overline{k})})$ and therefore if and only if the inclusion $\mathfrak{L} \subseteq \mathrm{Lie}(P_{W(\overline{k})}) = F^0(\mathrm{Lie}(G)) \otimes_{W(k)} W(\overline{k})$ holds. Thus $(M, (F^i(M))_{i\in[a,b]_{\mathbf{Z}}}, \varphi, G)$ is a Uni-canonical lift if and only if each endomorphism of $(M \otimes_{W(k)} W(\overline{k}), \varphi \otimes \sigma_{\overline{k}}, G_{W(\overline{k})})$ is as well an endomorphism of $(M \otimes_{W(k)} W(\overline{k}), (F^i(M) \otimes_{W(k)} W(\overline{k}))_{i\in[a,b]_{\mathbf{Z}}}, \varphi \otimes \sigma_{\overline{k}}, G_{W(\overline{k})})$.

1.3.8. Definitions. Let $\tilde{M}$ be a free $W(k)$-module of finite rank $d_{\tilde{M}}$. Let $(F^i(\tilde{M}))_{i\in[a,b]_{\mathbf{Z}}}$ be a decreasing and exhaustive filtration of $\tilde{M}$ by direct summands. Let $\tilde{\varphi}$ be a σ-linear automorphism of $\tilde{M}[\frac{1}{p}]$ such that we have $\tilde{\varphi}^{-1}(\tilde{M}) = \sum_{i=a}^{b} p^{-i}F^i(\tilde{M})$. If $a \geq 0$, we refer to the triple $(\tilde{M}, (F^i(\tilde{M}))_{i\in[a,b]_{\mathbf{Z}}}, \tilde{\varphi})$ as a filtered F-crystal over k.

(a) We say $(\tilde{M}, (F^i(\tilde{M}))_{i\in[a,b]_{\mathbf{Z}}}, \tilde{\varphi})$ is *circular* if there exists a $W(k)$-basis $\tilde{\mathcal{B}} := \{e_1, \ldots, e_{d_{\tilde{M}}}\}$ for $\tilde{M}$ such that for all $j \in [1, d_{\tilde{M}}]_{\mathbf{Z}}$ we have

$$\tilde{\varphi}(e_j) = p^{m_j} e_{j+1},$$

where $e_{d_{\tilde{M}+1}} := e_1$ and $e_j \in F^{m_j}(\tilde{M}) \setminus F^{m_j+1}(\tilde{M})$ with $F^{b+1}(\tilde{M}) := 0$ and with $m_j \in [a,b]_{\mathbf{Z}}$. If moreover the function $f_{\tilde{\mathcal{B}}} : \mathbf{Z}/d_{\tilde{M}}\mathbf{Z} \to [a,b]_{\mathbf{Z}}$ defined by $f_{\tilde{\mathcal{B}}}([j]) = m_j$ is not periodic, then we say $(\tilde{M}, (F^i(\tilde{M}))_{i\in[a,b]_{\mathbf{Z}}}, \tilde{\varphi})$ is *circular indecomposable*.

(b) We say $(\tilde{M}, (F^i(\tilde{M}))_{i\in[a,b]_{\mathbf{Z}}}, \tilde{\varphi})$ is *cyclic* if there exists a direct sum decomposition $(\tilde{M}, (F^i(\tilde{M}))_{i\in[a,b]_{\mathbf{Z}}}, \tilde{\varphi}) = \oplus_{l\in\tilde{L}}(\tilde{M}_l, (F^i(\tilde{M}) \cap \tilde{M}_l)_{i\in[a,b]_{\mathbf{Z}}}, \tilde{\varphi}_l)$ such that for each $l \in \tilde{L}$ the triple $(\tilde{M}_l, (F^i(\tilde{M}) \cap M_l)_{i\in[a,b]_{\mathbf{Z}}}, \tilde{\varphi}_l)$ is circular indecomposable.

(c) We assume that $0 \leq a \leq b \leq 1$ and that there exists a p-divisible group $\tilde{D}_{W(k)}$ over $W(k)$ whose filtered Dieudonné module is $(\tilde{M}, F^1(\tilde{M}), \tilde{\varphi})$. We say that $\tilde{D}_{W(k)}$ or $(\tilde{M}, F^1(\tilde{M}), \tilde{\varphi})$ is circular indecomposable (resp. circular), if $(\tilde{M}, (F^i(\tilde{M}))_{i\in\{0,1\}}, \tilde{\varphi})$ is circular indecomposable (resp. is circular). We say $\tilde{D}_{W(k)}$ is cyclic if it is a direct sum of p-divisible groups over $W(k)$ that are circular indecomposable.

1.4. Main results on Shimura p-divisible objects

In this section we state our main results on the objects we have defined in the previous one. Let (M, φ, G) be a Shimura p-divisible object over k in the range $[a,b]$. For each element $g \in G(W(k))$ the triple $(M, g\varphi, G)$ is as well a Shimura p-divisible object over k. One refers to $\{(M, g\varphi, G)|g \in G(W(k))\}$ as a *family* of Shimura p-divisible object over k in the range $[a,b]$. See (2) for the product decomposition $G^{\mathrm{ad}} = \prod_{i\in I} G_i$. We have the following three basic results (most of them are stated in terms of the family $\{(M, g\varphi, G)|g \in G(W(k))\}$, due to the sake of flexibility and of simplifying the proofs).

1.4.1. Theorem. *Let $g \in G(W(k))$. The following two statements are equivalent:*

(a) *the triple $(M, g\varphi, G)$ is Sh-ordinary;*

(b) *for each element* $g_1 \in G(W(k))$ *the Newton polygon of* $(M, g_1\varphi)$ *is above the Newton polygon of* $(M, g\varphi)$.

Moreover, if $(M, g\varphi, G)$ *is Sh-ordinary, then:*

(c) *for each element* $g_1 \in G(W(k))$ *we have an inequality* $\mathbf{I}(M, g_1\varphi, G) \leq \mathbf{I}(M, g\varphi, G)$.

If G is *quasi-split* i.e., it has a Borel subgroup scheme (for instance, this holds if either k is a finite field or $k = \overline{k}$), then Theorem 3.2.4 identifies all situations in which the statements 1.4.1 (c) and 1.4.1 (a) are in fact equivalent (i.e., 1.4.1 (c) $\Rightarrow$ 1.4.1 (a)). In particular, if $G = \mathbf{GL_M}$, then $b - a \in \{0, 1\}$ (cf. property 1.3.2 (b)) and the statements 1.4.1 (c) and 1.4.1 (a) are equivalent (cf. Theorem 3.2.4).

1.4.2. Theorem. *Let* $d_M := \mathrm{rk}_{W(k)}(M)$. *The following four statements are equivalent:*

(a) *the triple* (M, φ, G) *is Uni-ordinary;*

(b) *there exists a unique lift* $(F_0^i(M))_{i\in[a,b]_{\mathbf{Z}}}$ *of* (M, φ, G) *such that the Levi subgroup* $L_G^0(\varphi)_{B(k)}$ *of* (M, φ, G) *normalizes* $F_0^i(M)[\frac{1}{p}]$ *for all* $i \in [a, b]_{\mathbf{Z}}$*;*

(c) *there exists a unique Hodge cocharacter* $\mu_0 : \mathbf{G}_m \to G_{W(\overline{k})}$ *of* $(M \otimes_{W(k)} W(\overline{k}), \varphi \otimes \sigma_{\overline{k}}, G_{W(\overline{k})})$ *such that there exists a* $W(\overline{k})$*-basis* $\{e_1, \ldots, e_{d_M}\}$ *for* $M \otimes_{W(k)} W(\overline{k})$ *with the property that for all* $j \in [1, d_M]_{\mathbf{Z}}$ *the* $W(\overline{k})$*-span of* e_j *is normalized by* $\mathrm{Im}(\mu_0)$ *and we have* $\varphi \otimes \sigma_{\overline{k}}(e_j) = p^{n_j} e_{\pi_0(j)}$, *where* $n_j \in [a, b]_{\mathbf{Z}}$ *and where* π_0 *is a permutation of the set* $[1, d_M]_{\mathbf{Z}}$*;*

(d) *we have a direct sum decomposition of* F*-crystals over* k

$$(M, p^{-a}\varphi) = \oplus_{\gamma\in\mathbf{Q}}(M_\gamma, p^{-a}\varphi)$$

with the properties that: (i) for each rational number γ, *either* $M_\gamma = 0$ *or the only Newton polygon slope of* $(M_\gamma, p^{-a}\varphi)$ *is* γ, *and (ii) there exists a unique lift* $(F_1^i(M))_{i\in[a,b]_{\mathbf{Z}}}$ *of* (M, φ, G) *such that we have an identity*

$$F_1^i(M) = \oplus_{\gamma\in\mathbf{Q}}(F_1^i(M) \cap M_\gamma) \quad \forall i \in [a, b]_{\mathbf{Z}}.$$

If (M, φ, G) *is Uni-ordinary, then we have* $F_0^i(M) = F_1^i(M)$ *for all* $i \in [a, b]_{\mathbf{Z}}$. *If* (M, φ, G) *is Uni-ordinary and* $k = \overline{k}$, *then* $(M, (F_0^i(M))_{i\in[a,b]_{\mathbf{Z}}}, \varphi)$ *is cyclic.*

1.4.3. Theorem. *Let* $\overline{g}^{\mathrm{ad}} \in G^{\mathrm{ad}}(k)$ *be the image of an element* $g \in G(W(k))$. *We assume that* (M, φ, G) *is Sh-ordinary and that* $k = \overline{k}$. *Then statements 1.4.1 (a) and (b) are also equivalent to any one of the following two additional statements:*

(a) *the Shimura p-divisible objects (M,φ,G) and $(M,g\varphi,G)$ are inner isomorphic;*

(b) *there exists an element $\overline{h}\in G^{\mathrm{ad}}(k)$ such that as σ-linear endomorphisms of* $\mathrm{Lie}(G^{\mathrm{ad}}_k)$ *we have an identity $\overline{h\Psi}=\overline{g}^{\mathrm{ad}}\overline{\Psi h}$.*

1.4.4. Corollary. *There exists an open, Zariski dense subscheme $\mathcal{U}$ of G^{ad}_k such that for each perfect field $\tilde{k}$ that contains k and for every element $g_{\tilde{k}}\in G(W(\tilde{k}))$, the triple $(M\otimes_{W(k)}W(\tilde{k}),g_{\tilde{k}}(\varphi\otimes\sigma_{\tilde{k}}),G_{W(\tilde{k})})$ is Sh-ordinary if and only if we have $\overline{g}^{\mathrm{ad}}_{\tilde{k}}\in\mathcal{U}(\tilde{k})$.*

If $G=\mathbf{GL}_M$, then $b-a\in\{0,1\}$ and up to a Tate twist we recover the classical context of Section 1.1. Thus Subsection 1.3.7 and the above results generalize the interpretations of ordinary p-divisible groups and their canonical lifts in terms of (filtered or truncated modulo p) Dieudonné modules. We now include an example that illustrates many of the above notions and results.

1.4.5. Example. We assume that there exists a direct sum decomposition $M=M_0\oplus M_1\oplus M_2$ with the property that there exist $W(k)$-bases $\mathcal{B}_i=\{e^i_j|j\in[1,4]_{\mathbf{Z}}\}$ for M_i such that we have $\varphi(e^i_j)=p^{n_{i,j}}e^{i+1}_j$, where the upper index $i\in\{0,1,2\}$ is taken modulo 3 and where $n_{i,j}\in\{0,1\}$ is non-zero if and only if we have $j-i\leq 1$. Thus $(a,b)=(0,1)$. The Newton polygon slopes of (M,φ) are 0, $\frac{1}{3}$, $\frac{2}{3}$, and 1, all multiplicities being 3. Let $\tilde{F}^0(M)$ be the $W(k)$-span of $\{e^i_j|0\leq i\leq 2, 2\leq j-i\leq 4-i\}$. Let $\tilde{F}^1(M)=F^1(M)$ be the $W(k)$-span of $\{e^i_j|0\leq i\leq 2, 1-i\leq j-i\leq 1\}$. Let $F^0(M):=M=\tilde{F}^0(M)\oplus\tilde{F}^1(M)$ and $G:=\mathbf{GL}_{M_0}\times_{W(k)}\mathbf{GL}_{M_1}\times_{W(k)}\mathbf{GL}_{M_2}$. Let $\mu:\mathbf{G}_m\to G$ be the cocharacter that fixes $\tilde{F}^0(M)$ and that acts on $\tilde{F}^1(M)=F^1(M)$ via the inverse of the identical character of $\mathbf{G}_m$. The triple $(M,F^1(M),\varphi,G)$ is a Shimura filtered F-crystal.

The parabolic subgroup scheme $P^+_G(\varphi)$ of G is the Borel subgroup scheme of G that normalizes the $W(k)$-span of $\{e^i_j|0\leq i\leq\tilde{i}, j\in[1,4]_{\mathbf{Z}}\}$ for all all $\tilde{i}\in\{0,1,2\}$. Thus $P^+_G(\varphi)$ normalizes $F^1(M)$. Therefore (M,φ,G) is Sh-ordinary and $F^1(M)$ is its Sh-canonical lift.

As σ-linear endomorphisms of $\mathrm{Lie}(G^{\mathrm{ad}})$, we have $\frac{1}{p}\Psi=\varphi$. Thus $\mathbf{I}(M,\varphi,G)$ computes the multiplicity of the Newton polygon slope -1 of $(\mathrm{Lie}(G^{\mathrm{ad}}),\varphi)$ and therefore it is 3. For $i\in\{0,1,2\}$ and j_1, $j_2\in[1,4]_{\mathbf{Z}}$, let $e^i_{j_1,j_2}\in\mathrm{End}_{W(k)}(M_i)$ be such that for $j_3\in[1,4]_{\mathbf{Z}}$ we have $e^i_{j_1,j_2}(e^i_{j_3})=\delta_{j_2,j_3}e^i_{j_1}$. If $j_1\neq j_2$, then we have $e^i_{j_1,j_2}\in\mathrm{Lie}(G^{\mathrm{ad}})$. We have $\Psi(e^i_{4,1})=e^{i+1}_{4,1}$ and $\Psi(e^i_{1,4})=pe^{i+1}_{1,4}$. Let $\mathfrak{n}^-$ (resp. $\mathfrak{n}^+$) be the $W(k)$-submodule of $\mathrm{Lie}(G^{\mathrm{ad}})$ generated by all elements fixed by Ψ^3 (resp. by $\frac{1}{p^3}\Psi^3$). We have

$$\mathfrak{n}^-=\oplus_{i=0}^2W(k)e^i_{4,1}\quad\text{and}\quad\mathfrak{n}^+=\oplus_{i=0}^2W(k)e^i_{1,4}.$$

The normalizer $\tilde{P}_G^-(\varphi)$ (resp. $\tilde{P}_G^+(\varphi)$) of $\mathfrak{n}^-$ (resp. of $\mathfrak{n}^+$) in G is the parabolic subgroup scheme of G whose Lie algebra is the $W(k)$-span of

$$\{e_{j_1,j_2}^i | 0 \leq i \leq 2, \text{ either } 1 \leq j_2 \leq j_1 \leq 4 \text{ or } (j_1, j_2) = (2,3)\}$$

(resp. of $\{e_{j_1,j_2}^i | 0 \leq i \leq 2, \text{ either } 1 \leq j_1 \leq j_2 \leq 4 \text{ or } (j_1, j_2) = (3,2)\}$). Thus $P_G^+(\varphi)$ is a proper subgroup scheme of $\tilde{P}_G^+(\varphi)$. Let $\tilde{L}_G^0(\varphi)$ be the Levi subgroup scheme of either $\tilde{P}_G^-(\varphi)$ or $\tilde{P}_G^+(\varphi)$, whose Lie algebra is the $W(k)$-span of $\{e_{j,j}^i, e_{2,3}^i, e_{3,2}^i | 0 \leq i \leq 2, j \in [1,4]_{\mathbf{Z}}\}$. The group scheme $\tilde{L}_G^0(\varphi)^{\mathrm{ad}}$ is a $\mathbf{PGL}_2^3$ group scheme, the cocharacter μ factors through $\tilde{L}_G^0(\varphi)$, and the triple $(M, \varphi, \tilde{L}_G^0(\varphi))$ is also Sh-ordinary.

Let $w \in \tilde{L}_G^0(\varphi)(W(k))$ be such that it fixes e_j^i for $(i,j) \notin \{(0,2),(0,3)\}$ and it interchanges e_2^0 and e_3^0. The Newton polygon slopes of $(M, w\varphi)$ are 0, $\frac{1}{2}$, and 1 with multiplicities 3, 6, and 3 (respectively). As w fixes $\mathfrak{n}^-$, $\mathfrak{n}^-$ is $W(k)$-generated by all elements of $\mathrm{Lie}(G^{\mathrm{ad}})$ fixed by $(w\Psi)^3$. By applying this over $\overline{k}$ we get that $\mathbf{I}(M, w\varphi, G) = 3$. But the triples $(M, w\varphi, G)$ and $(M, w\varphi, \tilde{L}_G^0(\varphi))$ are not Sh-ordinary, as $e_{3,2}^i \in \mathrm{Lie}(P_{\tilde{L}_G^0(\varphi)}^+(w\varphi))$ and as $e_{3,2}^i(F^1(M)) \otimes_{W(k)} k \not\subseteq F^1(M) \otimes_{W(k)} k$. Let $g \in G(W(k))$ be such that we have $\mathbf{I}(M, g\varphi, G) = 3$. Up to an inner isomorphism we can assume that $g \in \tilde{L}_G^0(\varphi)(W(k))$, cf. property 3.3 (b). For $i \in \{0,1,2\}$, the intersection $\tilde{L}_G^0(\varphi) \cap \mathbf{GL}_{M_i}$ normalizes $F^1(M) \cap M_i$ if and only if $i \neq 1$ (i.e., μ has a non-trivial image only in that $\mathbf{PGL}_2$ factor of $\tilde{L}_G^0(\varphi)^{\mathrm{ad}}$ which corresponds to $i = 1$). This implies that $\mathbf{I}(M, g\varphi, \tilde{L}_G^0(\varphi))$ is the multiplicity of the Newton polygon slope $-\frac{1}{3}$ of $(\mathrm{Lie}(\tilde{L}_G^0(\varphi)), g\varphi)$. In particular, we have $\mathbf{I}(M, w\varphi, \tilde{L}_G^0(\varphi)) = 0 < 3 = \mathbf{I}(M, \varphi, \tilde{L}_G^0(\varphi))$.

In general, for $g \in \tilde{L}_G^0(\varphi)(W(k))$ the triple $(M, g\varphi, G)$ is Sh-ordinary if and only if the triple $(M, g\varphi, \tilde{L}_G^0(\varphi))$ is Sh-ordinary (cf. Theorem 3.7.1 (e)).

1.4.6. Remark. As Theorems 1.4.1 and 3.2.4 point out, one can not always use only one $FSHW$ invariant in order to "detect" Sh-ordinariness. However, as Example 1.4.5 points out, always one can use a well defined sequence of $FSHW$ invariants in order to "detect" Sh-ordinariness. Example 1.4.5 also points out that the proofs of Theorems 1.4.1, 3.2.4, and 1.4.3 will involve an induction on $\dim(G_k)$ and will appeal to *opposite parabolic subgroup schemes* of G, such as $\tilde{P}_G^-(\varphi)$ and $\tilde{P}_G^+(\varphi)$.

We now shift our attention from the abstract context of Sections 1.3 and 1.4 to a geometric context that involves abelian schemes and Shimura varieties of *Hodge type*.

1.5. Standard Hodge situations

We use the standard terminology of [De5] of Hodge cycles on an abelian scheme A_Z over a reduced $\mathbf{Q}$–scheme Z. Thus we write each Hodge cycle v

on A_Z as a pair

$$(v_{dR}, v_{ét}),$$

where v_{dR} and $v_{ét}$ are the de Rham and the étale (respectively) component of v. The étale component $v_{ét}$ as its turn has an l-component $v_{ét}^l$, for each prime $l \in \mathbf{N}$. For instance, if Z is the spectrum of a field ‡ and if $\overline{‡}$ is a fixed algebraic closure of ‡, then $v_{ét}^p$ is a suitable $\mathrm{Gal}(\overline{‡}/‡)$-invariant tensor of the tensor algebra of $H^1_{ét}(A_{\overline{Z}}, \mathbf{Q}_p) \oplus (H^1_{ét}(A_{\overline{Z}}, \mathbf{Q}_p))^\vee \oplus \mathbf{Q}_p(1)$, where $\overline{Z} := \mathrm{Spec}\, \overline{‡}$ and where $\mathbf{Q}_p(1)$ is the usual Tate twist. If moreover $\overline{‡}$ is a subfield of $\mathbf{C}$, then we also use the *Betti realization* of v: it corresponds to v_{dR} (resp. to $v_{ét}^l$) via the standard isomorphism that relates the Betti cohomology with $\mathbf{Q}$–coefficients of $A_Z \times_Z \mathrm{Spec}\, \mathbf{C}$ with the de Rham (resp. the $\mathbf{Q}_l$ étale) cohomology of $A_{\overline{Z}}$ (see [De5]).

For generalities on *Shimura pairs*, on their adjoints, *reflex fields*, and *canonical models*, and on *injective maps* between them we refer to [De2,3], [Mi1,2], and [Va3, Subsects. 2.1 to 2.10]. For different types of Shimura varieties we refer to [De3], [Mi2], and [Va3, Subsect. 2.5]. We now follow [Va9, Subsect. 5.1] to introduce *standard Hodge situations*. They generalize the *standard PEL situations* used in [Zi1], [LR], [Ko2], and [RZ]; however, the things are more technical than in these references due to the passage from tensors of degree 2 (polarizations and endomorphisms) to tensors of arbitrary degree on which no special properties are imposed. Here "tensors" refer to different cohomological realizations of Hodge cycles on abelian varieties.

1.5.1. Basic notations. We start with an injective map of Shimura pairs

$$f : (G_{\mathbf{Q}}, X) \hookrightarrow (\mathbf{GSp}(W, \psi), S).$$

The pair (W, ψ) is a symplectic space over $\mathbf{Q}$ and S is the set of all homomorphisms $\mathrm{Res}_{\mathbf{C}/\mathbf{R}} \mathbf{G}_m \to \mathbf{GSp}(W, \psi)_{\mathbf{R}}$ that define *Hodge* $\mathbf{Q}$*–structures* on W of type $\{(-1, 0), (0, -1)\}$ and that have either $2\pi i\psi$ or $-2\pi i\psi$ as polarizations. One calls $(\mathbf{GSp}(W, \psi), S)$ a *Siegel modular pair* and $(G_{\mathbf{Q}}, X)$ a Shimura pair of Hodge type. The group $G_{\mathbf{Q}}$ is reductive over $\mathbf{Q}$ and is identified via f with a subgroup of $\mathbf{GSp}(W, \psi)$. The set X is a $G_{\mathbf{Q}}(\mathbf{R})$-conjugacy class of homomorphisms $\mathrm{Res}_{\mathbf{C}/\mathbf{R}} \mathbf{G}_m \to G_{\mathbf{R}}$ whose composites with $f_{\mathbf{R}}$ belong to S. Both S and X have canonical structures of hermitian symmetric domains, cf. [De3, Cor. 1.1.17].

Let L be a $\mathbf{Z}$–lattice of W which is self-dual with respect to ψ (i.e., such that ψ induces a perfect, alternating form $\psi : L \times L \to \mathbf{Z}$). Let $\mathbf{Z}_{(p)}$ be the localization of $\mathbf{Z}$ at its prime ideal (p). Let $L_{(p)} := L \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$. We assume that p and L are such that:

(♯) *the schematic closure* $G_{\mathbf{Z}_{(p)}}$ *of* $G_{\mathbf{Q}}$ *in* $\mathbf{GSp}(L_{(p)},\psi)$ *is a reductive group scheme over* $\mathbf{Z}_{(p)}$.

Let $K_p := \mathbf{GSp}(L,\psi)(\mathbf{Z}_p)$. Let

$$H_p := G_{\mathbf{Z}_{(p)}}(\mathbf{Z}_p) = G_{\mathbf{Q}}(\mathbf{Q}_p) \cap K_p.$$

Let $r := \frac{\dim_{\mathbf{Q}}(W)}{2} \in \mathbf{N}$. Let $\mathbf{A}_f$ (resp. $\mathbf{A}_f^{(p)}$) be the $\mathbf{Q}$–algebra of finite adèles (resp. of finite adèles with the p-component omitted). We have $\mathbf{A}_f = \mathbf{Q}_p \times \mathbf{A}_f^{(p)}$.

Let $x \in X$. Let $\mu_x : \mathbf{G}_m \to G_{\mathbf{C}}$ be the Hodge cocharacter that defines the Hodge $\mathbf{Q}$–structure on W defined by x. We have a direct sum decomposition $W \otimes_{\mathbf{Q}} \mathbf{C} = F_x^{-1,0} \oplus F_x^{0,-1}$ such that $\mathbf{G}_m$ acts via μ_x trivially on $F_x^{0,-1}$ and through the identity character on $F_x^{-1,0}$.

The reflex field $E(G_{\mathbf{Q}},X)$ of $(G_{\mathbf{Q}},X)$ is the subfield of $\mathbf{C}$ that is the field of definition of the $G_{\mathbf{Q}}(\mathbf{C})$-conjugacy class of cocharacters of $G_{\mathbf{C}}$ defined by (any) μ_x; it is a number field. The canonical model $\mathrm{Sh}(G_{\mathbf{Q}},X)$ of $(G_{\mathbf{Q}},X)$ over $E(G_{\mathbf{Q}},X)$ is a Shimura variety of Hodge type. Let v be a prime of $E(G_{\mathbf{Q}},X)$ that divides p and let $k(v)$ be its residue field. Let $O_{(v)}$ be the localization of the ring of integers of $E(G_{\mathbf{Q}},X)$ with respect to v. As $G_{\mathbf{Z}_p}$ is a reductive group scheme, the prime v is unramified over p (cf. [Mi2, Cor. 4.7 (a)]). Our main data is the triple

$$(f,L,v)$$

for which (♯) holds, to be called a *potential standard Hodge situation*.

For the notion of an *integral canonical model* of a quadruple of the form $(G_{\mathbf{Q}},X,H_p,v)$ we refer to [Va3, Defs. 3.2.3 6) and 3.2.6] (see also [Mi1, Sect. 2] and [Mi2]). Let

$$\mathcal{M}$$

be the $\mathbf{Z}_{(p)}$-scheme which parameterizes isomorphism classes of principally polarized abelian schemes over $\mathbf{Z}_{(p)}$-schemes that are of relative dimension r and that have compatible level s symplectic similitude structure for all $s \in \mathbf{N}\setminus p\mathbf{N}$. The scheme $\mathcal{M}$ together with the natural action of $\mathbf{GSp}(W,\psi)(\mathbf{A}_f^{(p)})$ on it, is an integral canonical model of $(\mathbf{GSp}(W,\psi),S,K_p,p)$ (see [Va3, Ex. 3.2.9 and Subsect. 4.1] or [Mi1, Thm. 2.10]). These structures and this action are defined naturally via (L,ψ) (see [Va3, Subsect. 4.1]) and the previous sentence makes sense as we can identify naturally $\mathcal{M}_{\mathbf{Q}}$ with $\mathrm{Sh}(\mathbf{GSp}(W,\psi),S)/K_p$.

As $Z(\mathbf{GSp}(L_{(p)},\psi))(\mathbf{Z}_{(p)})$ is a discrete subgroup of $Z(\mathbf{GSp}(W,\psi))(\mathbf{A}_f^{(p)})$, we have $\mathcal{M}_{\mathbf{Q}}(\mathbf{C}) = \mathbf{GSp}(L_{(p)},\psi)(\mathbf{Z}_{(p)})\backslash(S \times \mathbf{GSp}(W,\psi)(\mathbf{A}_f^{(p)}))$ (cf. [Mi2, Prop. 4.11]). We also have

$$\mathrm{Sh}(G_{\mathbf{Q}},X)/H_p(\mathbf{C}) = G_{\mathbf{Z}_{(p)}}(\mathbf{Z}_{(p)})\backslash[X \times (G_{\mathbf{Q}}(\mathbf{A}_f^{(p)})\backslash\overline{Z(G_{\mathbf{Z}_{(p)}})(\mathbf{Z}_{(p)})})],$$

where $\overline{Z(G_{\mathbf{Z}_{(p)}})(\mathbf{Z}_{(p)})}$ is the topological closure of $Z(G_{\mathbf{Z}_{(p)}})(\mathbf{Z}_{(p)})$ in $Z(G_{\mathbf{Q}})(\mathbf{A}_f^{(p)})$ (cf. [Mi2, Prop. 4.11]). We have a morphism $\mathrm{Sh}(G_{\mathbf{Q}}, X)/H_p \to \mathcal{M}_{E(G_{\mathbf{Q}},X)}$ of $E(G_{\mathbf{Q}}, X)$-schemes whose pull-back to $\mathbf{C}$ is defined by the natural embedding $X \times G_{\mathbf{Q}}(\mathbf{A}_f^{(p)}) \hookrightarrow S \times \mathbf{GSp}(W, \psi)(\mathbf{A}_f^{(p)})$ via natural passages to quotients, cf. [De2, Cor. 5.4]. The last two sentences imply that $\overline{Z(G_{\mathbf{Z}_{(p)}})(\mathbf{Z}_{(p)})} = Z(G_{\mathbf{Z}_{(p)}})(\mathbf{Z}_{(p)})$ (cf. also [De2, Cor. 2.1.11]). Thus we have

$$\mathrm{Sh}(G_{\mathbf{Q}}, X)/H_p(\mathbf{C}) = G_{\mathbf{Z}_{(p)}}(\mathbf{Z}_{(p)})\backslash(X \times G_{\mathbf{Q}}(\mathbf{A}_f^{(p)})).$$

We easily get that $\mathrm{Sh}(G_{\mathbf{Q}}, X)_{\mathbf{C}}/H_p$ is a closed subscheme of $\mathcal{M}_{\mathbf{C}}$. Thus $\mathrm{Sh}(G_{\mathbf{Q}}, X)/H_p$ is a closed subscheme of $\mathcal{M}_{E(G_{\mathbf{Q}},X)}$. Let

$$\mathcal{N}$$

be the normalization of the schematic closure of $\mathrm{Sh}(G_{\mathbf{Q}}, X)/H_p$ in $\mathcal{M}_{O_{(v)}}$. Let

$$(\mathcal{A}, \Lambda_{\mathcal{A}})$$

be the pull-back to $\mathcal{N}$ of the *universal principally polarized abelian scheme* over $\mathcal{M}$. Let

$$(v_\alpha)_{\alpha\in\mathcal{J}}$$

be a family of tensors in $\sqcup_{n\in\mathbf{N}} W^{\vee\otimes n} \otimes_{\mathbf{Q}} W^{\otimes n}$ such that $G_{\mathbf{Q}}$ is the subgroup of $\mathbf{GL}_W$ that fixes v_α for all $\alpha \in \mathcal{J}$. As $G_{\mathbf{Q}}$ is reductive and $Z(\mathbf{GL}_W) \leqslant G_{\mathbf{Q}}$, the existence of the family $(v_\alpha)_{\alpha\in\mathcal{J}}$ is implied by [De5, Prop. 3.1 c)].

The choice of L and $(v_\alpha)_{\alpha\in\mathcal{J}}$, allows a moduli interpretation of $\mathrm{Sh}(G_{\mathbf{Q}}, X)$ (see [De2,3], [Mi2], and [Va3, Subsect. 4.1 and Lem. 4.1.3]). For instance, the set of complex points $\mathrm{Sh}(G_{\mathbf{Q}}, X)(\mathbf{C}) = G_{\mathbf{Q}}(\mathbf{Q})\backslash(X \times G_{\mathbf{Q}}(\mathbf{A}_f))$ is the set of isomorphism classes of principally polarized abelian varieties over $\mathbf{C}$ that are of dimension r, that carry a family of Hodge cycles indexed by the set $\mathcal{J}$, that have compatible level s symplectic similitude structure for all $s \in \mathbf{N}$, and that satisfy some extra axioms.

This interpretation endows the abelian scheme $\mathcal{A}_{E(G_{\mathbf{Q}},X)}$ with a family $(w_\alpha^{\mathcal{A}})_{\alpha\in\mathcal{J}}$ of Hodge cycles whose Betti realizations can be described as follows. Let

$$w = [x_w, g_w] \in \mathrm{Sh}(G_{\mathbf{Q}}, X)/H_p(\mathbf{C}) = G_{\mathbf{Z}_{(p)}}(\mathbf{Z}_{(p)})\backslash(X \times G_{\mathbf{Q}}(\mathbf{A}_f^{(p)}))$$

where $x_w \in X$ and $g_w \in G_{\mathbf{Q}}(\mathbf{A}_f^{(p)})$. Let L_{g_w} be the $\mathbf{Z}$-lattice of W such that we have $L_{g_w} \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}} = g_w(L \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}})$; here we view g_w as an element of $G_{\mathbf{Q}}(\mathbf{A}_f)$. Let $(A_w, \lambda_{A_w}) := w^*((\mathcal{A}, \Lambda_{\mathcal{A}})_{E(G_{\mathbf{Q}},X)})$. Then A_w is the complex abelian variety whose analytic space is

$$A_w(\mathbf{C}) = L_{g_w}\backslash W \otimes_{\mathbf{Q}} \mathbf{C}/F_{x_w}^{0,-1},$$

the principal polarization λ_{A_w} of A_w is defined by a suitable $\mathbf{G}_m(\mathbf{Z}_{(p)})$-multiple of ψ, and $w^*(w_\alpha^{\mathcal{A}})$ is the Hodge cycle on A_w whose Betti realization is the tensor v_α of the tensor algebra of $W^\vee \oplus W = (L_{g_w}^\vee \otimes_{\mathbf{Z}} \mathbf{Q}) \oplus (L_{g_w} \otimes_{\mathbf{Z}} \mathbf{Q})$.

Let $L_p := L \otimes_{\mathbf{Z}} \mathbf{Z}_p = L_{(p)} \otimes_{\mathbf{Z}_{(p)}} \mathbf{Z}_p$. Let $\psi^\vee$ be the perfect, alternating form on either $L_{(p)}^\vee$ or $L_p^\vee$ that is defined naturally by ψ. We have natural identifications

$$H^1_{\acute{e}t}(A_w, \mathbf{Z}_p) = L_{g_w}^\vee \otimes_{\mathbf{Z}} \mathbf{Z}_p = L^\vee \otimes_{\mathbf{Z}} \mathbf{Z}_p = L_p^\vee$$

under which: (i) the p-component of the étale component of $w^*(w_\alpha^{\mathcal{A}})$ is v_α, and (ii) the perfect form on $L^\vee \otimes_{\mathbf{Z}} \mathbf{Z}_p$ that corresponds naturally to λ_{A_w} is a $\mathbf{G}_m(\mathbf{Z}_{(p)})$-multiple of $\psi^\vee$.

Let $'$ be the involution of $\mathrm{End}_{\mathbf{Z}_{(p)}}(L_{(p)})$ defined by the identity $\psi(b(x), y) = \psi(x, b'(y))$, where $b \in \mathrm{End}_{\mathbf{Z}_{(p)}}(L_{(p)})$ and $x,\ y \in L_{(p)}$. Let $\mathcal{B} := \{b \in \mathrm{End}_{\mathbf{Z}_{(p)}}(L_{(p)}) | b \text{ fixed by } G_{\mathbf{Z}_{(p)}}\}$. We say (f, L, v) or $(f, L, v, \mathcal{B})$ is a standard PEL situation if the following axiom holds:

(PEL) *$\mathcal{B}$ is normalized by $'$, $\mathcal{B} \otimes_{\mathbf{Z}_{(p)}} W(\mathbf{F})$ is a product of matrix $W(\mathbf{F})$-algebras, and $G_{\mathbf{Q}}$ is the identity component of the subgroup of $\mathbf{GSp}(W, \psi)$ that fixes all elements of $\mathcal{B}[\frac{1}{p}]$.*

If moreover $G_{\mathbf{Q}}$ is the subgroup of $\mathbf{GSp}(W, \psi)$ that fixes all elements of $\mathcal{B}[\frac{1}{p}]$, then we also refer to (f, L, v) or to $(f, L, v, \mathcal{B})$ as a *standard moduli PEL situation.*

1.5.2. The abstract Shimura F-crystal. The isomorphism $L \tilde{\to} L^\vee$ induced by ψ allows us to identify $G_{\mathbf{Z}_p}$ with a closed subgroup scheme of $\mathbf{GL}_{L_p^\vee}$.

Let T_0 be a maximal torus of a Borel subgroup scheme B_0 of $G_{\mathbf{Z}_p}$. Let

$$\mu_0 : \mathbf{G}_m \hookrightarrow T_{0,W(k(v))}$$

be an injective cocharacter such that the following condition holds (cf. [Mi2, Cor. 4.7 (b)]):

(a) under a monomorphism $W(k(v)) \hookrightarrow \mathbf{C}$ that extends the composite monomorphism $O_{(v)} \hookrightarrow E(G_{\mathbf{Q}}, X) \hookrightarrow \mathbf{C}$, it is $G_{\mathbf{Q}}(\mathbf{C})$-conjugate to the Hodge cocharacters $\mu_x : \mathbf{G}_m \to G_{\mathbf{C}}$.

We will choose μ_0 such that moreover we have the following property:

(b) if $M_0 := L_p^\vee \otimes_{\mathbf{Z}_p} W(k(v)) = F_0^1 \oplus F_0^0$ is the direct sum decomposition such that $\mathbf{G}_m$ acts via μ_0 trivially on F_0^0 and as the inverse of the identical character of $\mathbf{G}_m$ on F_0^1, then $B_{0,W(k(v))}$ normalizes F_0^1.

We emphasize that μ_0 is uniquely determined by properties (a) and (b).

Let $\varphi_0 := (1_{L_p^\vee} \otimes \sigma_{k(v)}) \circ \mu_0(\frac{1}{p}) : M_0 \to M_0$. We have $\varphi_0^{-1}(M_0) = \frac{1}{p} F_0^1 \oplus F_0^0$ and $\varphi_0(\mathrm{Lie}(G_{B(k(v))})) = \mathrm{Lie}(G_{B(k(v))})$. Thus the quadruple

$(M_0, F_0^1, \varphi_0, G_{W(k(v))})$ is a Shimura filtered F-crystal and the Hodge cocharacter μ_0 defines the lift F_0^1. We refer to the triple

$$\mathcal{C}_0 := (M_0, \varphi_0, G_{W(k(v))})$$

as the *abstract Shimura F-crystal* of (f, L, v). Up to a $W(k(v))$-linear automorphism of M_0 defined by an element of $G_{\mathbf{Z}_p}(\mathbf{Z}_p)$, $\mathcal{C}_0$ does not depend on the choices of T_0 and B_0 (as all pairs of the form (T_0, B_0) are $G_{\mathbf{Z}_p}(\mathbf{Z}_p)$-conjugate, cf. Subsection 2.3.8 (d)).

1.5.3. Definition. The potential standard Hodge situation (f, L, v) is called a standard Hodge situation if the following two conditions hold:

(a) the $O_{(v)}$-scheme $\mathcal{N}$ is regular and formally smooth;

(b) for each perfect field k of characteristic p and for every $W(k)$-valued point $z \in \mathcal{N}(W(k))$, the quadruple $(M, F^1, \varphi, \tilde{G})$ is a Shimura filtered F-crystal; here (M, F^1, φ) is the filtered Dieudonné module of the

$$p-\text{divisible group } D_z \text{ of } A := z^*(\mathcal{A})$$

and $\tilde{G}$ is the schematic closure in $\mathbf{GL}_M$ of the subgroup of $\mathbf{GL}_{M[\frac{1}{p}]} = \mathbf{GL}_{H^1_{dR}(A/W(k))[\frac{1}{p}]}$ that fixes the de Rham component t_α of $z^*_{E(G_{\mathbf{Q}},X)}(w_\alpha^{\mathcal{A}})$ for all $\alpha \in \mathcal{J}$.

The de Rham component t_α is a tensor of the tensor algebra of $M[\frac{1}{p}] \oplus M^\vee[\frac{1}{p}]$ which is fixed by φ, cf. [Va9, Cor. 5.1.7].

1.5.4. Shimura F-crystals attached to points of $\mathcal{N}(k)$. We assume that (f, L, v) is a standard Hodge situation and we use the notations of Subsection 1.5.3. We refer to the quadruple $(M, F^1, \varphi, \tilde{G})$ as the *Shimura filtered F-crystal attached* to $z \in \mathcal{N}(W(k))$. Let $y \in \mathcal{N}(k)$ be the point defined naturally by z. The triple $(M, \varphi, (t_\alpha)_{\alpha\in\mathcal{J}})$ depends only on the point $y \in \mathcal{N}(k)$ and not on the lift $z \in \mathcal{N}(W(k)$ of y, cf. [Va9, p. 69] or Proposition 6.2.7 (d). Thus we call $(M, \varphi, \tilde{G})$ as the *Shimura F-crystal attached* to the point $y \in \mathcal{N}(k)$.

1.5.5. Remarks. (a) If the condition 1.5.3 (a) holds, then $\mathcal{N}$ is an integral canonical model of $(G_{\mathbf{Q}}, X, H_p, v)$ (cf. [Va3, Cor. 3.4.4]). If $p \geq 5$, then the condition 1.5.3 (a) holds (cf. [Va3, Subsect. 3.2.12, Prop. 3.4.1, and Thm. 6.4.2] and its corrections in [Va7, Appendix]). See [Va3, Subsects. 6.5 and 6.6] for the first general ways to construct standard Hodge situations. For instance, if (f, L, v) is a potential standard Hodge situation and if $p \geq \max\{5, r\}$ (see Subsection 1.5.1 for r), then (f, L, v) is a standard Hodge situation (cf. [Va3, Thm. 5.1, Rm. 5.8.2, and Cor. 5.8.6]). In [Va12] it is proved that each potential standard Hodge situation is in fact a standard Hodge situation. In [Ki] it is claimed that a potential standard Hodge situation is in fact a standard Hodge situation provided for $p = 2$ the 2-rank of each abelian variety over $\mathbf{F}$ which is a pull-back of $\mathcal{A}$ is 0.

(b) A standard PEL situation is a standard Hodge situation, provided either $p \geq 3$ or it is a standard moduli PEL situation (see [Zi1], [LR], [Ko2], [RZ], and [Va3, Ex. 5.6.3]; the assumption that $\mathcal{B}[\frac{1}{p}]$ is a $\mathbf{Q}$–simple algebra used in [Ko2, Sect. 5] for $p \geq 2$ and thus also in [Va3, Ex. 4.3.11] with $p \geq 3$, can be eliminated).

1.5.6. Shimura types. We review the Shimura types. Let $(G^{\mathrm{ad}}_{\mathbf{Q}}, X^{\mathrm{ad}})$ be the *adjoint Shimura pair* of $(G_{\mathbf{Q}}, X)$; thus X^{ad} is the $G^{\mathrm{ad}}_{\mathbf{Q}}(\mathbf{R})$-conjugacy class of the composite of any element $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}\mathbf{G}_m \to G_{\mathbf{R}}$ of X with the natural epimorphism $G_{\mathbf{R}} \twoheadrightarrow G^{\mathrm{ad}}_{\mathbf{R}}$. In this subsection we assume that the adjoint group $G^{\mathrm{ad}}_{\mathbf{Q}}$ is non-trivial. Let $(\tilde{G}_{\mathbf{Q}}, \tilde{X})$ be a simple factor of $(G^{\mathrm{ad}}_{\mathbf{Q}}, X^{\mathrm{ad}})$; thus $\tilde{G}_{\mathbf{Q}}$ is a simple group over $\mathbf{Q}$ that is a direct factor of $G^{\mathrm{ad}}_{\mathbf{Q}}$. Let $\tilde{\mathcal{X}}$ be the Lie type of the simple factors of $\tilde{G}_{\mathbf{C}}$. The existence of the cocharacters $\mu_x : \mathbf{G}_m \to G_{\mathbf{C}}$ that act on $W \otimes_{\mathbf{Q}} \mathbf{C}$ via the trivial and the identical character of $\mathbf{G}_m$ implies that (see [Sa], [De3], [Se1], [Pi2, Table 4.2], etc.):

(i) $\tilde{\mathcal{X}}$ is a classical Lie type;

(ii) if $\mathfrak{f}$ is a simple factor of $\mathrm{Lie}(\tilde{G}^{\mathrm{ad}}_{\mathbf{C}})$, then the weight ϖ of each simple $\mathfrak{f}$-submodule of $W \otimes_{\mathbf{Q}} \mathbf{C}$, is a *minuscule weight*.

We recall that ϖ can be: any fundamental weight if $\tilde{\mathcal{X}} = A_\ell$, the fundamental weight ϖ_ℓ if $\tilde{\mathcal{X}} = B_\ell$, the fundamental weight ϖ_1 if $\tilde{\mathcal{X}} = C_\ell$, and any one of the three fundamental weights ϖ_1, $\varpi_{\ell-1}$, ϖ_ℓ if $\tilde{\mathcal{X}} = D_\ell$ with $\ell \geq 4$; see either the mentioned references or [Bou3, Ch. VIII, Subsect. 7.3, Rm.]. If $\tilde{\mathcal{X}}$ is A_ℓ, B_ℓ, or C_ℓ with $\ell \in \mathbf{N}$, then we say $(\tilde{G}_{\mathbf{Q}}, \tilde{X})$ is of $\tilde{\mathcal{X}}$ type. We assume now that $\tilde{\mathcal{X}}$ is D_ℓ with $\ell \geq 4$. Then $(\tilde{G}_{\mathbf{Q}}, \tilde{X})$ is of either $D^{\mathbf{H}}_\ell$ or $D^{\mathbf{R}}_\ell$ type, cf. [De3, Table 2.3.8]. If $(\tilde{G}_{\mathbf{Q}}, \tilde{X})$ is of $D^{\mathbf{H}}_\ell$ (resp. of $D^{\mathbf{R}}_\ell$) type, then each non-compact, simple factor of the identity component of the real Lie group $\tilde{G}_{\mathbf{R}}(\mathbf{R})$ is isogenous to $\mathbf{SO}^*(2\ell)$ (resp. to the identity component of $\mathbf{SO}(2, 2\ell-2)$) and the converse of this holds for $\ell \geq 5$ (see [He, Ch. X, Sect. 2] for the classical semisimple, real Lie groups $\mathbf{SO}^*(2\ell)$ and $\mathbf{SO}(2, 2\ell-2)$).

We mention that besides the Shimura types mentioned above, we have precisely three extra ones. The three extra Shimura types are D^{mixed}_ℓ with $\ell \geq 4$, E_6, and E_7. Abstract Theorems 1.4.1 to 1.4.3 apply to all of them.

We have the following concrete (i.e., geometric) form of Section 1.4.

1.6. Main results on standard Hodge situations

Let (f, L, v) be a standard Hodge situation. Let $(\mathcal{A}, \Lambda_{\mathcal{A}})$ be the principally polarized abelian scheme over $\mathcal{N}$ introduced in Section 1.5. Let $y \in \mathcal{N}(k)$. Let $z \in \mathcal{N}(W(k))$ be a lift of y. Let the quintuple $(M, F^1, \varphi, (t_\alpha)_{\alpha\in\mathcal{J}}, \tilde{G})$ be as in Definition 1.5.3; thus the triple $(M, \varphi, \tilde{G})$ is the Shimura F-crystal attached to the point $y \in \mathcal{N}(k)$.

Let $\mathcal{C}_0 := (M_0, \varphi_0, G_{W(k(v))})$ be the abstract Shimura F-crystal of (f, L, v), cf. Subsection 1.5.2. Let $U_0^+ := U^+_{G_{W(k(v))}}(\varphi_0)$ be the unipotent radical of the parabolic subgroup scheme $P^+_{G_{W(k(v))}}(\varphi_0)$ of $G_{W(k(v))}$, cf. Subsection 1.3.3. Let

$$\mathfrak{I}_0 \in \mathbf{N} \cup \{0\}$$

be the nilpotent class of U_0^+.

1.6.1. Theorem. *There exists an open, Zariski dense, $G_{\mathbf{Q}}(\mathbf{A}_f^{(p)})$-invariant subscheme $\mathcal{O}$ of $\mathcal{N}_{k(v)}$ such that for a point $y \in \mathcal{N}(k)$ we have $y \in \mathcal{O}(k)$ if and only if one of the following four equivalent statements holds:*

(a) *the Shimura F-crystal $(M, \varphi, \tilde{G})$ attached to y is Sh-ordinary;*

(b) *the Newton polygon of (M, φ) is the Newton polygon $\mathfrak{N}_0$ of $\mathcal{C}_0$ (i.e., of (M_0, φ_0));*

(c) *there exists a Lie isomorphism $\mathrm{Lie}(\tilde{G}^{\mathrm{ad}}_{W(\bar{k})}) \tilde{\to} \mathrm{Lie}(G^{\mathrm{ad}}_{W(\bar{k})})$ with the properties that: (i) it is defined naturally by a $\overline{B(\bar{k})}$-linear isomorphism $M \otimes_{W(k)} \overline{B(\bar{k})} \tilde{\to} M_0 \otimes_{W(k(v))} \overline{B(\bar{k})}$ which takes t_α to v_α for all $\alpha \in \mathcal{J}$, and (ii) its reduction modulo p defines an isomorphism between the $FSHW$ maps of the triples $(M \otimes_{W(k)} W(\bar{k}), \varphi \otimes \sigma_{\bar{k}}, \tilde{G}_{W(\bar{k})})$ and $\mathcal{C}_0 \otimes \bar{k} := (M_0 \otimes_{W(k(v))} W(\bar{k}), \varphi_0 \otimes \sigma_{\bar{k}}, G_{W(\bar{k})})$;*

(d) *the Newton polygon of $(\mathrm{Lie}(\tilde{G}^{\mathrm{ad}}), \varphi)$ is the Newton polygon of $(\mathrm{Lie}(G^{\mathrm{ad}}_{W(k(v))}), \varphi_0)$.*

If $\mathfrak{I}_0 \leq 2$, then the above four statements are also equivalent to the following one:

(e) *we have an equality $\mathbf{I}(M, \varphi, \tilde{G}) = \mathbf{I}(\mathcal{C}_0)$ (equivalently, we have an inequality $\mathbf{I}(M, \varphi, \tilde{G}) \geq \mathbf{I}(\mathcal{C}_0)$).*

Moreover, if $k = \bar{k}$ and if y, $y_1 \in \mathcal{N}(k)$ are two points that factor through the same connected component of $\mathcal{O}$, then there exists an isomorphism $(M, \varphi) \tilde{\to} (M_1, \varphi_1)$ defined by a $W(k)$-linear isomorphism $M \tilde{\to} M_1$ that takes t_α to $t_{1,\alpha}$ for all $\alpha \in \mathcal{J}$; here $(M_1, \varphi_1, (t_{1,\alpha})_{\alpha\in\mathcal{J}})$ is the triple that defines the Shimura F-crystal $(M_1, \varphi_1, \tilde{G}_1)$ attached to the point $y_1 \in \mathcal{N}(k)$.

1.6.2. Theorem. *Let $y \in \mathcal{N}(k)$ be such that its attached Shimura F-crystal $(M, \varphi, \tilde{G})$ is Uni-ordinary. If $p = 2$ we assume that either $k = \overline{k}$ or (M, φ) has no integral Newton polygon slopes. Let $n_y \in \mathbf{N} \cup \{0\} \cup \{\infty\}$ be such that the number of $W(k)$-valued points $z \in \mathcal{N}(W(k))$ which lift y and whose attached Shimura filtered F-crystals $(M, F^1, \varphi, \tilde{G})$ are the Uni-canonical lift of $(M, \varphi, \tilde{G})$, is precisely n_y.*

Then we have $n_y \in \mathbf{N}$. If $p \geq 3$ or if $p = 2$ and (M, φ) has no integral Newton polygon slopes, then $n_y = 1$ and the resulting $W(k)$-valued point $z \in \mathcal{N}(W(k))$ is the unique $W(k)$-valued point $z_1 \in \mathcal{N}(W(k))$ that lifts y and such that the p-divisible group D_{z_1} of $z_1^*(\mathcal{A})$ is a direct sum of p-divisible groups over $W(k)$ whose special fibres have only one Newton polygon slope; if moreover $k = \overline{k}$, then the p-divisible group D_z of $z^*(\mathcal{A})$ is cyclic.

1.6.3. Theorem. *We assume that $k = \mathbf{F}$ and that the $W(k)$-valued point $z \in \mathcal{N}(W(k))$ is such that $(M, F^1, \varphi, \tilde{G})$ is a Uni-canonical lift. We have the following four properties:*

(a) *Then $A = z^*(\mathcal{A})$ has complex multiplication.*

(b) *Let $e_{\mathbf{F}}$ be an endomorphism of $A_{\mathbf{F}}$ whose crystalline realization $c_{\mathbf{F}} \in \mathrm{End}_{W(k)}(M)$ is an endomorphism of $(M, \varphi, \tilde{G})$. If $p = 2$ we assume that either $c_{\mathbf{F}}$ has a zero image in $\mathrm{End}_{W(k)}(M/4M)$ or (M, φ) has no integral Newton polygon slopes. Then $e_{\mathbf{F}}$ lifts to an endomorphism of A.*

(c) *Let k_0 be a finite field. Let $\tilde{A}_{k_0}$ be an abelian variety over k_0 such that $\tilde{A}_{\mathbf{F}}$ is $A_{\mathbf{F}} = y^*(\mathcal{A})$. Let $\Pi \in \mathrm{End}(A_{\mathbf{F}})$ be the extension to $\mathbf{F}$ of the Frobenius endomorphism of $\tilde{A}_{k_0}$. Let $z_0 : \mathrm{Spec}\, V \to \mathcal{N}$ be a lift of y, where V is a finite, discrete valuation ring extension of $W(\mathbf{F})$. Let $s \in \mathbf{N} \cup \{0\}$ and $t \in \mathbf{N}$. We assume that the endomorphism $p^s\Pi^t$ of $A_{\mathbf{F}} = y^*(\mathcal{A})$ lifts to an endomorphism of the abelian scheme $z_0^*(\mathcal{A})$ over V.*

If the condition $\mathbf{W}$ for z_0 (to be defined in Subsection 6.5.5) holds, then the abelian scheme $z_0^*(\mathcal{A})$ has complex multiplication. Condition $\mathbf{W}$ for z_0 holds if $V = W(\mathbf{F})$ (therefore, if $V = W(\mathbf{F})$, then $z_0^*(\mathcal{A})$ has complex multiplication).

(d) *Referring to (c), if moreover $V = W(\mathbf{F})$ (resp. $V = W(\mathbf{F})$, $p = 2$, and $s = 0$), then z_0 is a Uni-canonical lift (resp. is the unique $W(k)$-valued point $z_1 \in \mathcal{N}(W(k))$ which lifts y and for which the p-divisible group D_{z_1} of $z_1^*(\mathcal{A})$ is cyclic).*

1.6.4. Definitions. We call every point $y \in \mathcal{N}(k)$ as in Theorem 1.6.2 as a *Uni-ordinary point* (of either $\mathcal{N}_{k(v)}$ or $\mathcal{N}$). We call every $W(k)$-valued point $z \in \mathcal{N}(W(k))$ as in Theorem 1.6.2 as a *Uni-canonical lift* (of either $y \in \mathcal{N}(k)$ or $\mathcal{N}$). If moreover $y \in \mathcal{O}(k) \subseteq \mathcal{N}(k)$, then we also refer to y as a *Sh-ordinary point* (of either $\mathcal{N}_{k(v)}$ or $\mathcal{N}$) and to z as a *Sh-canonical lift* (of either $y \in \mathcal{O}(k)$ or $\mathcal{N}$). We also refer to $\mathcal{O}$ as the *Sh-ordinary locus* of $\mathcal{N}_{k(v)}$.

1.6.5. Remark. The simplest standard Hodge situations for which there exist Uni-ordinary points which are not Sh-ordinary points involve *Hilbert–Blumenthal varieties* of dimension at least 3 (see Section 7.5).

1.6.6. On $\mathfrak{I}_0$. We can have $\mathfrak{I}_0 \geq 2$ only if there exists a simple factor $(\tilde{G}_{\mathbf{Q}}, \tilde{X})$ of $(G^{\mathrm{ad}}_{\mathbf{Q}}, X^{\mathrm{ad}})$ that is either of A_ℓ type with $\ell \geq 2$ or of $D^{\mathbf{H}}_\ell$ type with $\ell \geq 4$ (see Subsection 3.4.7 and Corollary 3.6.3 for an abstract version of this; see also the property 9.5.2 (l)). In particular, we always have $\mathfrak{I}_0 = 1$ if the group $G^{\mathrm{ad}}_{\mathbf{Q}}$ is non-trivial and all simple factors of $(G^{\mathrm{ad}}_{\mathbf{Q}}, X^{\mathrm{ad}})$ are of B_ℓ $(\ell \geq 1)$, C_ℓ $(\ell \geq 3)$, or $D^{\mathbf{R}}_\ell$ $(\ell \geq 4)$ type. If the group $G^{\mathrm{ad}}_{\mathbf{Q}}$ is simple (thus we have $\tilde{G}_{\mathbf{Q}} = G^{\mathrm{ad}}_{\mathbf{Q}}$) and if $(G^{\mathrm{ad}}_{\mathbf{Q}}, X^{\mathrm{ad}})$ is of A_ℓ type with $\ell \geq 2$ (resp. of $D^{\mathbf{H}}_\ell$ type with $\ell \geq 4$), then one can check that regardless of what the prime v is, we have $\mathfrak{I}_0 \in [1, \ell]_{\mathbf{Z}}$ (resp. $\mathfrak{I}_0 \in \{1, 2\}$); in addition, we have plenty of situations in which $\mathfrak{I}_0$ is an arbitrary number in $[1, \ell]_{\mathbf{Z}}$ (resp. in $\{1, 2\}$).

1.6.7. On Condition W. Condition **W** for $z_0 : \operatorname{Spec} V \to \mathcal{N}$ as in statement 1.6.3 (c) pertains to certain *Bruhat decompositions* (see Subsection 6.5.5) and it is easy to see that it is also a necessary condition in order that the abelian scheme $z_0^*(\mathcal{A})$ over V of the statement 1.6.3 (c) has complex multiplication. Roughly speaking, the condition **W** for z_0 holds if there exist enough endomorphisms of $(M, \varphi, \tilde{G})$ that are crystalline realizations of endomorphisms of $z_0^*(\mathcal{A})$. Condition **W** for z_0 always holds if y is a Sh-ordinary point, cf. proof of Corollary 9.6.4.

1.7. On applications

Different examples, complements, and applications of Theorems 1.6.1 to 1.6.3 are gathered in Chapters 7 to 9. We will mention here only five such complements and applications; to state them, in this section we take $k = \overline{k}$.

In Section 7.2 we define *Dieudonné truncations* modulo positive, integral powers of p of Shimura F-crystals attached to k-valued points of $\mathcal{N}$ and we include a modulo p variant of the last paragraph of Theorem 1.6.1 (see Corollary 7.2.2).

In Section 8.2 we complete for $p \geq 5$ (the last step of) the proof of the existence of integral canonical models in unramified mixed characteristic $(0, p)$ of Shimura varieties whose adjoints are products of simple, adjoint Shimura varieties of some $D^{\mathbf{H}}_\ell$ type with $\ell \geq 4$. In other words, the Zariski density part of Theorem 1.6.1 and the statement 1.6.3 (a) allow us to apply [Va3, Lem. 6.8.1 and Criteria 6.8.2] for proving (see Subsection 8.2.4) the postponed result [Va3, Thm. 6.1.2*] for these Shimura varieties.

In Sections 9.1, 9.2, and 9.4 we generalize properties 1.1 (f) to (i) to the context of a standard Hodge situation (f, L, v). These generalizations are rooted on the notions of Subsection 1.3.5 and on Faltings deformation theory of [Fa2, Sect. 7]. We detail on the generalization of the property 1.1 (h) and on complements to it.

To $\mathcal{C}_0\otimes k = (M_0\otimes_{W(k(v))} W(k), \varphi_0\otimes\sigma_k, G_{W(k)})$ we attach a commutative formal Lie group $\mathcal{F}_1$ over $\mathrm{Spf}\, W(k)$ and a connected, nilpotent group scheme $\mathcal{F}_2$ over $\mathrm{Spec}\, W(k)$ of nilpotent class $\mathfrak{I}_0$ (see Sections 5.3 and 5.4). For instance, $\mathcal{F}_1$ is the formal Lie group of the p-divisible group $D^+_{W(k)}$ over $W(k)$ whose filtered Dieudonné module is

$$(\mathrm{Lie}(U^+_{0,W(k)}), \tilde{F}^1_0(\mathrm{Lie}(G_{W(k)})), \varphi_0\otimes\sigma),$$

where $\tilde{F}^1_0(\mathrm{Lie}(G_{W(k)})) := \{x \in \mathrm{Lie}(G_{W(k)}) | x(F^1_0 \otimes_{W(k(v))} W(k)) = 0 \text{ and } x(M_0\otimes_{W(k(v))} W(k)) \subseteq F^1_0\otimes_{W(k(v))} W(k)\}$; we call $D^+_{W(k)}$ the positive p-divisible group over $W(k)$ of $\mathcal{C}_0\otimes k$. Moreover, $\mathcal{F}_2$ is a modification of the projective limit of the finite, flat group schemes $\mathrm{proj.lim.}_{m\to\infty}\mathcal{F}_1[p^m]$ over $W(k)$ that pays attention to the Lie structure of $\mathrm{Lie}(U^+_{0,W(k)})$. We view $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) as the *étale* (resp. as the *crystalline*) *possible way* to define or to look at the formal deformation space of $\mathcal{C}_0\otimes k$. We have $\mathcal{F}_2 = \mathrm{proj.lim.}_{m\to\infty}\mathcal{F}_1[p^m]$ if and only if U^+_0 is commutative. See Examples 5.2.7 to 5.2.10 and the property 9.5.2 (b) for different examples that pertain to the structure of $D^+_{W(k)}$ (which does depend on $\mathfrak{I}_0$).

Let $y \in \mathcal{O}(k) \subseteq \mathcal{N}(k)$. Let $\mathcal{D}_y = \mathrm{Spf}\, R$ be the formal scheme of the completion R of the local ring of the k-valued point of $\mathcal{N}_{W(k)}$ defined by y. See Theorem 1.6.2 for $n_y \in \mathbf{N}$. One expects that: (i) $\mathcal{D}_y$ has n_y natural structures of a formal Lie group isomorphic to $\mathcal{F}_1$ and (ii) there exist formal closed subschemes of $\mathcal{D}_y$ which have n_y natural structures of a nilpotent, finite, flat formal group scheme associated naturally to $\mathcal{F}_2$ and to a fixed $m\in\mathbf{N}$. Theorem 9.4 proves such expectations in the so called commutative case when $\mathfrak{I}_0 \le 1$ (i.e., when U^+_0 is commutative) and in Section 9.8 we deal with some specific properties of the case $\mathfrak{I}_0 \ge 2$.

For simplicity, in this and the next paragraph we will refer only to the commutative case; thus we have $\mathfrak{I}_0 \le 1$. If $p \ge 3$, let $z \in \mathcal{N}(W(k))$ be the Sh-canonical lift of y. If $p = 2$, then we choose a Sh-canonical lift $z \in \mathcal{N}(W(k))$ of y (cf. Theorem 1.6.2). Then $\mathcal{D}_y$ has a unique canonical formal Lie group structure defined by the following two properties: (i) the origin is defined by z, and (ii) the addition of two $\mathrm{Spf}\, W(k)$-valued points of $\mathcal{D}_y$ is defined as in Subsection 1.2.6 via sums of lifts. If $p = 2$, then we have $n_y = 2^{m_0(1)}$, where $m_0(1)$ is the multiplicity of the Newton polygon slope 1 of $(\mathrm{Lie}(G^{\mathrm{ad}}_{W(k(v))})[\frac{1}{p}], \varphi_0)$ (see Proposition 9.5.1). Also if $p = 2$, then the n_y formal Lie group structures on $\mathcal{D}_y$ that correspond to the n_y possible choices of z, differ only by translations through 2-torsion $\mathrm{Spf}\, W(k)$-valued points (see property 9.5.2 (d)). For $p = 2$, the methods used in this monograph can not show that we can always choose z such that the p-divisible group D_z of $z^*(\mathcal{A})$ is cyclic (see [Va12] for a proof of this); thus, if $n_y > 1$, in this monograph we can not single out in general any such 2-torsion $\mathrm{Spf}\, W(k)$-valued point and this explains why we mention everywhere n_y canonical structures.

See Proposition 9.6.1 and Definition 9.6.2 for the $\mathrm{Spf}\, V$-valued points version of sums of lifts and see Corollary 9.6.5 and Remarks 9.6.6 for $\mathrm{Spf}\, V$-valued torsion points; here V is a finite, discrete valuation ring extension of $W(k)$. See Subsection 5.4.1 and Proposition 9.7.1 for different functorial aspects of the formal Lie groups $\mathcal{F}_1$ and of the mentioned canonical formal Lie group structures on $\mathcal{D}_y$. Let v^{ad} be the prime of the reflex field $E(G^{\mathrm{ad}}_{\mathbf{Q}}, X^{\mathrm{ad}})$ of $(G^{\mathrm{ad}}_{\mathbf{Q}}, X^{\mathrm{ad}})$ divided by v. If the residue field $k(v^{\mathrm{ad}})$ of v^{ad} is $\mathbf{F}_p$, then up to $\mathrm{Spf}\, W(k)$-valued points of order two (they can exist only if $p = 2$), all the canonical formal Lie group structures on $\mathcal{D}_y$ coincide and $\mathcal{F}_1$ is a formal torus (see property 9.5.2 (a)); this result for the case when $k(v) = k(v^{\mathrm{ad}}) = \mathbf{F}_p$ was first obtained in [No].

1.8. Extra literature

The *FSHW* maps were introduced in [Va1, Subsubsect. 5.5.4] following a suggestion of Faltings and this explains our terminology. They are the adjoint Lie analogues of *truncated Barsotti–Tate groups* of level 1 over k, cf. Claim of Subsection 1.2.3, the statement 1.4.3 (b), etc. To our knowledge, [Va1, Subsect. 5.5 and 5.6] is the first place where Bruhat decompositions and *Weyl elements* are used in the study of $\overline{\Psi}$ and thus implicitly of truncated Barsotti–Tate groups of level 1 over k (see property (15) of Proposition 3.5.1); see also [Mo1], [MW], [Va8], etc. The notion Uni-ordinariness is a new concept. The Sh-canonical lifts were first mentioned in [Va3, Subsects. 1.6 and 1.6.2]; previously to [Va3,4], they were used only either for the case of ordinary p-divisible groups or for cases pertaining to H^2 crystalline realizations of hyperkähler varieties (like K3 surfaces). In the generality of Theorem 1.6.2, for $p \geq 5$ they were introduced in the first version [Va2] of this monograph (for $p \geq 2$ see also [Va4]). See Remark 4.2.2 for a link between Theorem 1.4.1 and [RR]. See also [Va6, Props. 4.3.1 and 4.3.2] for general ordinary p-divisible objects and their canonical lifts.

The Shimura F-crystal $\mathcal{C}_0$ was introduced in [Va1, Subsect. 5.3 g)] (see also [RR, Thm. 4.2] for a general abstract version of it). For Siegel modular varieties the Zariski density part of Theorem 1.6.1 was checked previously in [Kob], [NO], [FC], [Va1], [We1], [Oo1,2], etc. All these references used different methods. Under slight restrictions (that were not required and are eliminated here), the Zariski density part of Theorem 1.6.1 was obtained in [Va1, Thm. 5.1]. Loc. cit. is correct only when $\mathfrak{I}_0 \leq 2$, as it does not make a distinction between $P_G^+(\varphi)$'s and $\tilde{P}_G^+(\varphi)$'s parabolic subgroup schemes as Example 1.4.5 does; but the modifications required to eliminate this restriction involve only a trivial induction (cf. also Remark 1.4.6). In [We1] a variant of the Zariski density part of Theorem 1.6.1 is obtained for standard PEL situations: it defines $\mathcal{O}$ via 1.6.1 (b) and for $p = 2$ it deals only with standard moduli PEL situations. Paper [We2] uses a variant of 1.6.1 (c) (that pertains to truncated Barsotti–Tate groups of level 1), in order to refine [We1] for $p \geq 3$ and for the so call C case of moduli standard PEL situations. Positive p-divisible groups over $W(k)$ like $D^+_{W(k)}$ and the nilpotent class $\mathfrak{I}_0$ were introduced in [Va4] (see also [Va10, Sect. 4]).

We (resp. B. Moonen) reported first on Theorem 1.6.1 and [Va1,2] (resp. on [Va1, Sect. 5]) in Durham (resp. in Münster) July (resp. April) 1996. The paper [Mo2] obtains some results analogous to ours for standard PEL situations with $p > 2$ or for standard moduli PEL situations with $p = 2$.

1.9. Our main motivation

The generalized Serre–Tate theories of Sections 1.4 and 1.6 will play for Shimura varieties the same role played by the classical Serre–Tate theory for Siegel modular varieties and for p-divisible groups. For instance, the generalizations of many previous works centered on ordinariness (like [De1], [FC, Ch. VII, Sect. 4], [No], etc.) are in the immediate reach. But our five main reasons to develop these two theories are the following ones.

(a) To use them in connection to the combinatorial *conjecture* of *Langlands–Rapoport* that describes the set $\mathcal{N}(\mathbf{F})$ together with the natural actions on it of $G_{\mathbf{Q}}(\mathbf{A}_f^{(p)})$ and of the Frobenius automorphism of $\mathbf{F}$ whose fixed field is $k(v)$ (see [LR], [Mi1], and [Pf]). This conjecture is a key ingredient toward the understanding of the *zeta functions* of quotients of finite type of $\mathcal{N}_{E(G_{\mathbf{Q}},X)}$ and of different *trace functions* that pertain to $\mathbf{Q}_l$-*local systems* on smooth quotients of $\mathcal{N}$ (see [LR], [Ko2], and [Mi1]; here $l \in \mathbf{N} \setminus \{p\}$ is a prime). Not to make this monograph too long, we only list here its parts that are related to the conjecture (see already [Va13]). Subsections 1.6.3 (a), 4.3.1 (b), 6.2.8, 7.4.1, 8.1, 9.6.5, and 9.6.6 form *all* that one requires to prove this conjecture for the subset $\mathcal{O}(\mathbf{F})$ of $\mathcal{N}(\mathbf{F})$ (i.e., generically); this will generalize [De1] and the analogous generic parts of [Zi1] and [Ko2].

(b) In [Va12] we use them to prove that for $p \geq 2$ every potential standard Hodge situation is a standard Hodge situation.

(c) To provide intrinsic methods that would allow to extend Theorems 1.6.1 to 1.6.3 as well as (a) and (b) to all integral canonical models of

Shimura varieties of *preabelian type* (a Shimura variety is said to be of preabelian type if its adjoint Shimura variety is isomorphic to the adjoint Shimura variety of a Shimura variety of Hodge type).

(d) To apply Section 1.4 to construct integral canonical models of Shimura varieties of special type (i.e., of Shimura varieties which are not of preabelian type and thus whose adjoint Shimura varieties have simple factors of D_ℓ^{mixed}, E_6, or E_7 types).

(e) They extend to all types of polarized varieties endowed with Hodge cycles whose different moduli spaces are open subvarieties of (quotients of) Shimura varieties of preabelian type (like *polarized hyperkähler varieties*; see [An] for more examples).

1.10. More on the overall organization

In Chapter 2 we include preliminaries on reductive group schemes and Newton polygons. In Section 3.1 we list basic properties of Hodge cocharacters. See Section 3.2 for Theorem 3.2.4 and for a general outline of the proofs of Theorems 1.4.1 and 3.2.4. The outline is carried out in Sections 3.3 to 3.8. In Sections 4.1 and 4.2 we prove Theorems 1.4.2 and 1.4.3 (respectively). See Subsection 4.2.1 for the proof of Corollary 1.4.4. In Chapter 5 we introduce formal Lie groups and nilpotent group scheme of Sh-ordinary p-divisible objects over $\overline{k}$. In Chapter 6 we prove Theorems 1.6.1 to 1.6.3. In Chapters 7 to 9 we include examples, complements, and applications of Theorems 1.6.1 to 1.6.3 (see their beginnings and Sections 1.7 and 1.8).

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