Semi-Classical Analysis

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Preface

Semi-classical analysis

There are a number of excellent texts available on the topic of this monograph, among them Dimassi and Sjostrand's "Spectral Asymptotics in the Semi-classical Analysis" [DiSj], Zworski's "Lectures on Semi-classical Analysis" [Zwor], Martinez's "An introduction to Semi-classical and Microlocal Analysis " [Mart], Didier Robert's "Autour de l'Approximation Semi-classique" [Did] and Colin de Verdiere's, "Méthodes Semi-classiques et Théorie Spectral", [Col]. The focus of this monograph, however, is an aspect of this subject which is somewhat less systematically developed in the texts cited above than it will be here: In semi-classical analysis many of the basic results involve asymptotic expansions in which the terms can be computed by symbolic techniques and the focus of these notes will be the "symbol calculus" that this creates. In particular, the techniques involved in this symbolic calculus have their origins in symplectic geometry and the first seven chapters of this monograph will, to a large extent, be a discussion of this underlying symplectic geometry.

Another feature which, to some extent, differentiates this monograph from the texts above is an emphasis on the *global* aspects of this subject: We will spend a considerable amount of time showing that the objects we are studying are coordinate invariant and hence make sense on manifolds; and, in fact, we will try, in so far as possible, to give intrinsic coordinate free descriptions of these objects. In particular, although one can find an excellent account of the global symbol calculus of Fourier integral operators in Hörmander's seminal paper "Fourier integral operators I", the adaptation of this calculus to the semi-classical setting with all the i's dotted and t's crossed is not entirely trivial, and most of chapters 6 and 7 will be devoted to this task.

This emphasis on globality will also be reflected in our choice of topics in the later chapters of this book, for instance: wave and heat trace formulas for globally defined semi-classical differential operators on manifolds and equivariant versions of these results involving Lie group actions. (Also, apropos of Lie groups, we will devote most of Chapter 12 to discussing semi-classical aspects of the representation theory of these groups.)

We will give a more detailed description of these later chapters (and, in fact, of the whole book) in Section 4 of this preface. However before we do so we will attempt to describe in a few words what "semi-classical" analysis is concerned with and what role symplectic geometry plays in this subject.

The Bohr correspondence principle

One way to think of semi-classical analysis is as an investigation of the mathematical implications of the Bohr correspondence principle: the assertion that classical mechanics is the limit, as $\hbar$ tends to zero, of quantum mechanics.[1] To illustrate how this principle works, let's consider a physical system consisting of a single point particle, p, of mass, m, in $\mathbb{R}^n$ acted on by a conservative force $F = -\nabla V$, $V \in \mathcal{C}^\infty(\mathbb{R}^n)$. The total energy of this system (kinetic plus potential) is given by $H(x,\xi) = \frac{1}{2m}|\xi|^2 + V(x)$, where x is the position and ξ the momentum of p, and the motion of this system in phase space is described by the Hamilton–Jacobi equations

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial \xi}(x,\xi) \\ \frac{d\xi}{dt} &= -\frac{\partial H}{\partial x}(x,\xi) \end{aligned} \tag{1}$$

The quantum mechanical description of this system on the other hand is given by the Schrödinger equation

$$ih\frac{\partial}{\partial t}\varphi = -\frac{h^2}{2m}\Delta\varphi + V\varphi \tag{2}$$

whose L^2 normalized solution, $\int |\varphi|^2\,dx = 1$, gives one a probability measure $\mu_t = |\varphi(x,t)|^2\,dx$ that describes the "probable" position of the state described by ϕ at time t.

Of particular interest are the steady state solutions of (2). If we assume for simplicity that the eigenvalues $\lambda_k(\hbar)$ of the Schrödinger operator are discrete and the corresponding L^2 normalized eigenfunctions are $\varphi_k(x)$ then the functions, $e^{-i\frac{t\lambda_k}{\hbar}}\varphi_k(x)$, are steady state solutions of (2) in the sense that the measures $\mu_k = |\varphi_k(x,t)|^2\,dx$ are independent of t. The $\lambda_k(\hbar)$'s are, by definition the energies of these steady state solutions, and the number of states with energies lying on the interval $a < \lambda < b$ is given by

$$N(a,b,\hbar) = \sharp\{a < \lambda_k(\hbar) < b\}\,. \tag{3}$$

On the other hand a crude semi-classical method for computing this number of states is to invoke the Heisenberg uncertainty principle

$$|\delta x_i \delta \xi_i| \geq 2\pi\hbar \tag{4}$$

and the Pauli exclusion principle (which can be interpreted as saying that no two of these states can occupy the same position in phase space) to conclude that the maximum number of classical states with energies on the interval $a < H < b$

[1] Mathematicians are sometimes bothered by this formulation of the BCP since $\hbar$ is a fixed constant, i.e., is (approximately) 10^{-27} erg secs., (a conversion factor from the natural units of inverse seconds to the conventional unit of ergs) not a parameter that one can vary at will. However, unlike e and π, it is a *physical* constant: in the world of classical physics in which quantities are measured in ergs and secs, it is negligibly small, but in the world of subatomic physics it's not. Therefore the transition from quantum to semi-classical can legitimately be regarded as an "$\hbar$ tends to zero" limit.

is approximately equal to the maximal number of disjoint rectangles lying in the region, $a < H(x, \xi) < b$ and satisfying the volume constraint imposed by (4). For $\hbar$ small the number of such rectangles is approximately

$$\left(\frac{1}{2\pi\hbar}\right)^n \operatorname{vol}(a < H(x,\xi) < b) \tag{5}$$

so as $\hbar$ tends to zero

$$(2\pi\hbar)^n N(a,b,\hbar) \to \operatorname{vol}(a < H(x,\xi) < b)\,. \tag{6}$$

We will see in Chapter 10 of this monograph that the empirical derivation of this "Weyl law" can be made rigorous and is valid, not just for the Schrödinger operator, but for a large class of semi-classical and classical differential operators as well.

The symplectic category

We recall that a symplectic manifold is a pair (M, ω) where M is a $2n$-dimensional manifold and $\omega \in \Omega^2(M)$ a closed two-form satisfying $\omega_p^n \neq 0$ for all $p \in M$. Given a symplectic manifold (M, ω) we will denote by M^- the symplectic manifold, $(M, -\omega)$ and given two symplectic manifolds, M_i, $i = 1, 2$ we will denote by $M_1 \times M_2$ the product of these two manifolds equipped with the symplectic form

$$\omega = (pr_1)^*\omega_1 + (pr_2)^*\omega_2\,.$$

Finally, given a $2n$-dimensional symplectic manifold, (M, ω), we'll call an n-dimensional submanifold, Λ of M *Lagrangian* if the inclusion map, $\iota_\Lambda : \Lambda \to M$ satisfies $\iota_\Lambda^*\omega = 0$, i.e. ω vanishes when restricted to Λ. Of particular importance for us will be Lagrangian submanifolds of the product manifold, $M_1^- \times M_2$, and these we will call *canonical relations*.

With these notations in place we will define the *symplectic category* to be the category whose objects are symplectic manifolds and whose morphisms are canonical relations: i.e. given symplectic manifolds, M_1 and M_2, we will define a morphism of M_1 into M_2 to be a canonical relation, Γ, in $M_1^- \times M_2$. (We will use double arrow notation, $\Gamma : M_1 \twoheadrightarrow M_2$ for these morphisms to distinguish them from a more conventional class of morphisms, symplectic maps.)

To make these objects and morphisms into a category we have to specify a composition law for pairs of morphisms, $\Gamma_i : M_i \twoheadrightarrow M_{i+1}$ $i = 1, 2$ and this we do by the recipe

$$(m_1, m_3) \in \Gamma \Leftrightarrow (m_1, m_2) \in \Gamma_1 \quad \text{and} \quad (m_2, m_3) \in \Gamma_2 \tag{7}$$

for some $m_2 \in M_2$. Unfortunately the "Γ" defined by this recipe is not always a canonical relation (or even a manifold) but it is if one imposes some transversality conditions on Γ_1 and Γ_2 which we'll spell out in detail in Chapter 4.

The fundamental notion in our approach to semi-classical analysis is a "quantization operation" for canonical relations. We're not yet in position to discuss this quantization operation in general. (This will be the topic of Chapters 8–11 of this monograph.) But we'll briefly discuss an important special case: Let X be a manifold and let $M = T^*X$ be the cotangent bundle of X equipped with its standard symplectic form (the two-form, ω, which, in standard cotangent coordinates, is given by, $\sum dx_i \wedge d\xi_i$). A Lagrangian manifold Λ of M is *horizontal* if the cotangent fibration, $\pi(x,\xi) = x$, maps Λ bijectively onto X. Assuming X is simply connected, this condition amounts to the condition

$$\Lambda = \Lambda_\varphi \tag{8}$$

where φ is a real-valued $\mathcal{C}^\infty$ function on X and

$$\Lambda_\varphi = \{(x,\xi) \in T^*X\,,\ \xi = d\varphi_x\}\,. \tag{9}$$

Now let $M_i = T^*X_i$, $i = 1,2$ and let $\Gamma : M_1 \twoheadrightarrow M_2$ be a canonical relation. Then

$$\Gamma^\sharp = \{(x_1, -\xi_1, x_2, \xi_2)\,,\ (x_1, \xi_1, x_2, \xi_2) \in \Gamma\}$$

is a Lagrangian submanifold of the product manifold

$$M_1 \times M_2 = T^*(X_1 \times X_2)$$

and hence if $\Gamma^\sharp$ is horizontal it is defined as above by a real-valued $\mathcal{C}^\infty$ function $\varphi(x_1, x_2)$ on $X_1 \times X_2$. We will quantize Γ by associating with it the set of linear operators

$$T_h : \mathcal{C}_0^\infty(X_1) \to \mathcal{C}^\infty(X_2) \tag{10}$$

of the form

$$T_\hbar f(x_2) = \int e^{i\frac{\varphi(x_1,x_2)}{\hbar}} a(x_1, x_2, \hbar) f(x_1)\, dx_1 \tag{11}$$

where $a(x_1, x_2, \hbar)$ is in $\mathcal{C}^\infty(X_1 \times X_2 \times \mathbb{R})$ and $\hbar$ is positive parameter (our stand-in for Planck's constant). These "semi-classical Fourier integral operators" are the main topic of this monograph, and our goal in Chapters 8–11 will be to show that their properties under composition, taking transposes, computing traces, etc. are governed symbolically by symplectic properties of their corresponding canonical relations. In fact, we will show that the symbolic calculus that describes the leading asymptotics of these operators in the $\hbar \to 0$ limit can be entirely described by constructing (as we will do in Chapter 7) an "enhanced symplectic category" consisting of pairs (Γ, σ) where Γ is a canonical relation and σ a section of the "pre-quantum line bundle" on Γ.

The plan of attack, part 1

Chapter 1 of this monograph will essentially be a fleshed out version of this preface. We will show how one can construct solutions of hyperbolic partial differential equations for short time intervals, modulo error terms of order $O(\hbar)$, by reducing this problem to a problem involving solutions of the Hamilton–Jacobi equation (1). Then, using an embryonic version of the symbol theory mentioned above we will show that these "solutions modulo $O(\hbar)$" can be converted into "solutions modulo $O(\hbar^\infty)$". We will also show that this method of solving (2) breaks down when the solution of the associated classical equation (1) develops caustics, thus setting the stage for the much more general approach to this problem described in Chapter 8 where methods for dealing with caustics and related complications are developed in detail.

In Chapter 1 we will also discuss in more detail the Weyl law (6). (At this stage we are not prepared to prove (6) in general but we will show how to prove it in two simple illuminating special cases.)

Chapter 2 will be a short crash course in symplectic geometry in which we will review basic definitions and theorems. Most of this material can be found in standard references such as [AM], [Can] or [GSSyT], however the material at the end of this section on the Lagrangian Grassmannian and Maslov theory is not so readily accessible and for this reason we've treated it in more detail.

In Chapter 3 we will, as preparation for our "categorical" approach to symplectic geometry, discuss some prototypical examples of categories. The category of finite sets and relations, and the linear symplectic category (in which the objects are symplectic vector spaces and the morphisms are linear canonical relations). The first of these examples are mainly introduced for the propose of illustrating categorical concepts; however the second example will play an essential role in what follows. In particular, the fact that the linear symplectic category is a true category: that the composition of *linear* canonical relations is always well defined, will be a key ingredient in our construction of a symbol calculus for semi-classical Fourier integral operators.

Chapter 4 will begin our account of the standard non-linear version of this category, the symplectic "category" itself.[2] Among the topics we will discuss are composition operations, a factorization theorem of Weinstein (which asserts that every canonical relation is the composition of an immersion and submersion), an embedding result (which shows that the standard differential category of $\mathcal{C}^\infty$ manifolds, and $\mathcal{C}^\infty$ maps is a subcategory of the symplectic category) and other examples of such subcategories. In particular, one important such subcategory is the *exact* symplectic category, whose objects are pairs, (M, α) where α is a one-form on M whose exterior derivative is symplectic. In this category the Lagrangian submanifolds, Λ, of M will also be required to be exact, i.e. to satisfy $\iota_\Lambda^* \alpha = d\varphi_\Lambda$ for some $\varphi_\Lambda \in \mathcal{C}^\infty(\Lambda)$. (In Chapter 8 when we associate oscillatory integrals with Lagrangian submanifolds, Λ, of T^*X the fixing of this φ_Λ will enable us to avoid the presence of troublesome undefined oscillatory factors in these integrals.)

We will also describe in detail a number of examples of canonical relations that will frequently be encountered later on. To give a brief description of some of examples in this preface let's denote by "pt." the "point-object" in the symplectic category: the unique-up-to-symplectomorphism connected symplectic manifold of dimension zero, and regard a Lagrangian submanifold of a symplectic manifold, M, as being a morphism

$$\Lambda : \text{pt.} \twoheadrightarrow M\,.$$

In addition given a canonical relation $\Gamma : M_1 \twoheadrightarrow M_2$ let's denote by $\Gamma^t : M_2 \twoheadrightarrow M_1$ the transpose canonical relation; i.e. require that $(m_2, m_1) \in \Gamma^t$ if $(m_1, m_2) \in \Gamma$.

[2]Many of the ideas discussed in this chapter are directly or indirectly inspired by Alan Weinstein's 1981 *Bulletin* article "Symplectic geometry", not the least of these being the term, "category", for a collection of morphisms for which there are simple, easy-to-verify criteria for composability.

Example 1. Let X and Y be manifolds and $f : X \to Y$ a $\mathcal{C}^\infty$ map. Then

$$\Gamma_f : T^*X \twoheadrightarrow T^*Y \tag{12}$$

is the canonical relation defined by

$$(x, \xi, y, \eta) \in \Gamma_f \Leftrightarrow y = f(x) \quad \text{and} \quad \xi = df_x^* \eta\,. \tag{13}$$

The correspondence that associates Γ_f to f gives us the embedding of the differential category into the symplectic category that we mentioned above. Moreover we will see in Chapter 8 that Γ_f and its transpose have natural quantizations: Γ_f^t as the pull-back operation

$$f^* : \mathcal{C}^\infty(Y) \to \mathcal{C}^\infty(X) \tag{14}$$

and Γ_f as the transpose of this operation on distributions.

Example 2. If $\pi : Z \to X$ is a $\mathcal{C}^\infty$ fibration the distributional transpose of (14) maps $\mathcal{C}_0^\infty(Z)$ into $\mathcal{C}_0^\infty(X)$ and hence defines a fiber integration operation

$$\pi_* : \mathcal{C}_0^\infty(Z) \to \mathcal{C}_0^\infty(X) \tag{15}$$

about which we will have more to say when we preview the "quantum" chapters of this monograph in the next section.

Example 3. Let Z be a closed submanifold of $X_1 \times X_2$ and let π_i be the projection of Z onto X_i. Then by quantizing $\Gamma_{\pi_2} \times \Gamma_{\pi_1}^t$ we obtain a class of Fourier integral operators which play a fundamental role in integral geometry: generalized Radon transforms.

Example 4. The identity map of T^*X onto itself. We will show in Chapter 8 that the entity in the quantum world that corresponds to the identity map is the algebra of semi-classical pseudodifferential operators (about which we will have a lot more to say below!)

Example 5. The symplectic involution

$$\Gamma : T^*\mathbb{R}^n \to T^*\mathbb{R}^n \qquad (x, \xi) \to (\xi, -x)\,. \tag{16}$$

This is the horizontal canonical relation in $(T^*\mathbb{R}^n)^- \times T^*\mathbb{R}^n$ associated with the Lagrangian manifold Λ_φ where $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is the function, $\varphi(x, y) = -x \cdot y$.

If one quantizes Γ by the recipe (11) taking $a(x, y, \hbar)$ to be the constant function $(2\pi\hbar)^{-n}$ one gets the semi-classical Fourier transform

$$F_\hbar f(x) = (2\pi\hbar)^{-n/2} \int e^{-i\frac{x \cdot y}{\hbar}} f(y)\, dy\,. \tag{17}$$

(See Chapter 5, §15 and Chapter 8, §9.)

This operator will play an important role in our "local" description of the algebra of semi-classical pseudodifferential operators when the manifold X in Example 3 is an open subset of $\mathbb{R}^n$.

Example 6. (Generating functions) Given a Lagrangian manifold, $\Lambda \subseteq T^*X$, a fiber bundle $\pi : Z \to X$ and a function $\varphi \in \mathcal{C}^\infty(Z)$, we will say that φ is a *generating function* for Λ with respect to the fibration, π, if Λ is the composition of the relations, $\Lambda_\varphi : pt \twoheadrightarrow T^*Z$ and $\Gamma_\pi : T^*Z \twoheadrightarrow T^*X$. In the same spirit, if $\Gamma : T^*X \twoheadrightarrow T^*Y$ is a canonical relation, $\pi : Z \to X \times Y$ is a fiber bundle and $\varphi \in \mathcal{C}^\infty(Z)$ we will say that φ is a generating function for Γ with respect to π if it is a generating function for the associated Lagrangian manifold, $\Gamma^\sharp$ in $T^*(X \times Y)$. These functions will play a key role in our definition of Fourier integral operators in Chapter 8, and in Chapter 5 we will give a detailed account of their properties. In particular, we will show that locally every Lagrangian manifold is definable by a generating function and we will also prove a uniqueness result which says that locally any generating function can be obtained from any other by a sequence of clearly defined "Hörmander moves". We will also prove a number of functorial properties of generating functions: e.g. show that if

$$\Gamma_i : T^*X_i \twoheadrightarrow T^*X_{i+1} \qquad i = 1, 2$$

are canonical relations and (Z_i, π_i, φ_i) generating data for Γ_i, then if Γ_i and Γ_2 are composable, the φ_i's are composable as well in the sense that there is a simple procedure for constructing from the φ_i's a generating function for $\Gamma_2 \circ \Gamma_1$. Finally in the last part of Chapter 5 we will investigate the question, "Do global generating functions exist?" This question is one of the main unanswered open questions in present-day symplectic topology; so we will not be able to say much about it; however we will show that if one tries to construct a global generating function by patching together local generating functions one encounters a topological obstacle: the vanishing of a cohomology class in $H^1(\Lambda, \mathbb{Z})$. This cohomology class, the Maslov class, puts in its appearance in this theory in other contexts as well. In particular, the line bundle on Λ associated with the mod 4 reduction of this cohomology class is a main ingredient in the leading symbol theory of semi-classical Fourier integral operators.

The other main ingredient in this symbol theory is *half-densities*. These will be discussed in Chapter 6, and in Chapter 7 we will show how to "enhance" the symplectic category by replacing canonical relations by pairs, (Γ, σ) where Γ is a canonical relation and σ a half-density on Γ, and by showing that the composition law for canonical relations that we discussed above extends to a composition law for these pairs. (In §7.8 we will add a further complication to this picture by replacing the σ's by $\sigma \otimes m$'s where m is a section of the Maslov bundle.)

The plan of attack, part 2

Section 4 was an overview of Chapters 1–7, the symplectic or "classical" half of this monograph, We'll turn next to the material in the next five chapters, the application of these results to semi-classical analysis. Let $(\Lambda, \varphi_\Lambda)$ be an exact Lagrangian submanifold of T^*X. If Λ is horizontal, i.e. of the form (8)–(9) one can associate with Λ the space of oscillatory functions

$$\mu \in I^k(X; \Lambda) \Leftrightarrow \mu = \hbar^k a(x, \hbar) e^{i \frac{\varphi(x)}{\hbar}} \tag{18}$$

where a is a $\mathcal{C}^\infty$ function on $X \times \mathbb{R}$ and φ_Λ is the pull-back of φ to Λ. More generally if Λ is defined by generating data, (Z, π, φ) and φ and φ_Λ are compatible in an appropriate sense (see Section 8.1) we will define the elements of $I^k(X;\Lambda)$ to be oscillatory functions of the form

$$\mu = \hbar^{k-d/2}\pi_*\left(a(z,\hbar)e^{i\frac{\varphi(z)}{\hbar}}\right) \tag{19}$$

where d is the fiber dimension of Z, $a(z,\hbar)$ is a $\mathcal{C}^\infty$ function on $Z\times\mathbb{R}$ and π_* is the operator (15)[3]

More generally if $(\Lambda,\varphi_\Lambda)$ is an arbitrary exact Lagrangian manifold in T^*X one can define $I^k(X;\Lambda)$ by patching together local versions of this space. (As we mentioned in §4, φ_Λ plays an important role in this patching process. The compatibility of φ_Λ with local generating data avoids the presence of a lot of undefined oscillatory factors in the elements of $I^k(X;\Lambda)$ that one obtains from this patching process.)

Some of our goals in Chapter 8 will be:

1. To show that the space $I^k(X;\Lambda)$ is well-defined. (Doing so will rely heavily on the uniqueness theorem for generating functions proved in Chapter 5).

2. To show that if $\mathbb{L}_\Lambda$ is the line bundle over Λ defined in §7.8 (the tensor product of the Maslov bundle and the half-density bundle) there is a canonical leading symbol map
$$\sigma : I^k(X;\Lambda)/I^{k+1}(X;\Lambda)\to\mathcal{C}^\infty(\mathbb{L}_\Lambda)\,. \tag{20}$$

3. To apply these results to canonical relations. In more detail, if $\Gamma : T^*X \twoheadrightarrow T^*Y$ is a canonical relation and $\Gamma^\sharp$ is, as in §3, the associated Lagrangian submanifold of $T^*(X\times Y)$, then, given an element, μ, of $I^{k-n/2}(X\times Y,\Gamma^\sharp)$, $n=\dim Y$, we can define an operator
$$F_\mu : \mathcal{C}_0^\infty(X)\to\mathcal{C}^\infty(Y) \tag{21}$$
by the recipe
$$F_\mu f(y) = \int f(x)\mu(x,y,\hbar)\,dx\,; \tag{22}$$
and we will call this operator a *semi-classical Fourier integral operator* of order k. We will also define its *symbol* to be the leading symbol of μ and we will denote the space of these operators by $\mathcal{F}^k(\Gamma)$. One of our main goals in Chapter 8 will be to show that the assignment
$$\Gamma\to\mathcal{F}^k(\Gamma) \tag{23}$$
is a functor, i.e. to show that if $\Gamma_i : T^*X_i \twoheadrightarrow T^*X_{i+1}$, $i=1,2$, are canonical relations and Γ_1 and Γ_2 are transversally composable, then for $F_i\in\mathcal{F}^{k_i}(\Gamma_i)$, F_2F_1 is in $\mathcal{F}^{k_1+k_2}$ and the leading symbol of F_2F_1 can be computed from the

[3]Strictly speaking to define π_* one needs to equip X and Z with densities, dx and dz, so as to make sense of the pairing
$$\int \pi_*\mu\nu\,dx = \int \mu\pi^*\nu\,dz\,.$$
However in §8 we will give a slightly different definition of π_* that avoids these choices: We will let Γ_π be an enhanced canonical relation in the sense of §4.7, i.e. equipped with a $\frac{1}{2}$-density symbol, and let μ and ν be $\frac{1}{2}$-densities. Thus in this approach $I^k(X;\Lambda)$ becomes a space of *half-densities.*

leading symbols of F_2F_1 by the composition law for symbols that we defined in Chapter 7. (We will also prove an analogous result for cleanly composable canonical relations.)

4. To apply these results to the identity map of T^*X onto T^*X. If Γ is this identity map then $\Gamma \circ \Gamma = \Gamma$ and this composition is a transversal composition, so the space of Fourier integral operators, $\mathcal{F}(\Gamma)$, is a filtered ring. This ring is the *ring* of *semi-classical pseudodifferential* operators and we will henceforth denote it by $\Psi(X)$. We will show that the symbol calculus for this ring is much simpler than the symbol calculus for arbitrary Γ; namely, we will show that $\mathbb{L}_\Gamma \cong \mathbb{C}$ and hence that the leading symbol of an element of Ψ^k/Ψ^{k+1} is just a $\mathcal{C}^\infty$ function on T^*X.

5. To observe that $I(X, \Lambda)$ is a *module* over $\Psi(X)$: More explicitly if $\Lambda : \text{pt.} \twoheadrightarrow T^*X$ is a Lagrangian manifold and Γ is the identity map of T^*X onto itself, then $\Gamma \cdot \Lambda = \Lambda$, and this composition is transversal. Hence, for $\mu \in I^k(X; \Lambda)$ and $P \in \Psi^\ell(X)$, $P\mu \in I^{k+\ell}(X; \Lambda)$. We will make use of this module structure to deal with some problems in PDE theory that we were unable to resolve in Chapter 1, in particular, to construct solutions $\bmod O(\hbar^\infty)$ of the Schrödinger equation and other semi-classical differential equations in the presence of caustics.

6. To give a concrete description of the algebra of semi-classical pseudodifferential operators for $X = \mathbb{R}^n$, in particular to show that locally on $\mathbb{R}^n$ these operators are of the form

$$\hbar^{-k} Pf(x) = (2\pi\hbar)^{-\frac{n}{2}} \int a(x, \xi, \hbar) e^{i\frac{x \cdot \xi}{\hbar}} F_\hbar f(\xi)\, d\xi \tag{24}$$

where $F_\hbar$ is the semi-classical Fourier transform (17).

Finally one last (very important) goal of Chapter 8 will be to describe the role of "microlocality" in semi-classical analysis. If P is the pseudodifferential operator (24) and (x, ξ) a point in $T^*\mathbb{R}^n$ we will say that P *vanishes* on an open neighborhood, U, of (x, ξ) if the function $a(x, \xi, \hbar)$ vanishes to infinite order in $\hbar$ on this open neighborhood. We will show that this definition is coordinate independent and hence that one can make sense of the notion "$P = 0$ on U" for X an arbitrary manifold, P an element of $\Psi(X)$ and U an open subset of T^*X. Moreover, from this notion one gets a number of useful related notions. For instance, for an open set, U, in T^*X one can define the ring of pseudodifferential operators, $\Psi(U)$, to be the quotient of $\Psi(X)$ by the ideal of operators which vanish on U, and one can define the *microsupport* of an operator, $P \in \Psi(X)$ by decreeing that $(x, \xi) \notin \text{Supp}(P)$ if P vanishes on a neighborhood of (x, ξ). Moreover, owing to the fact that $I(X, \Lambda)$ is a module over $\Psi(X)$ one can define analogous notions for this module. (We refer to §6 of Chapter 8 for details.) In particular these "microlocalizations" of the basic objects in semi-classical analysis convert this into a subject which essentially lives on T^*X rather than X.

One last word about microlocality: In definition (19) we have been a bit sloppy in not specifying conditions on the support of $a(z, \hbar)$. For this expression to be well-defined we clearly have to assume that for every $p \in X$, $a(z, \hbar)$ is compactly supported on the fiber above p, or at least, in lieu of this, impose some decay-at-infinity conditions on the restriction of a to these fibers. However sometimes one can get around such assumptions using microlocal cutoffs, i.e. define generalized elements, μ of $I^k(X;\Lambda)$ by requiring that such an element satisfy $P\mu \in I^k(X;\Lambda)$ for every compactly supported cutoff "function", $P \in \Psi(X)$. In Chapter 9 we will apply this idea to the ring of pseudodifferential operators itself. First, however, as an illustration of this idea, we will show that the algebra of *classical* pseudodifferential operators: operators with polyhomogeneous symbols (but with no $\hbar$ dependence) has such a characterization. Namely let $\Psi_0(X)$ be the ring of semi-classical pseudodifferential operators having compact micro-support and let $\Psi_{00}(X)$ be the elements of this ring for which the micro-support does not intersect the zero section. We will prove

Theorem 1. *A linear operator, $A : \mathcal{C}_0^\infty(X) \to \mathcal{C}^{-\infty}(X)$, with distributional kernel is a classical pseudodifferential operator with polyhomogeneous symbol if and only if $AP \in \Psi_{00}(X)$ for all $P \in \Psi_{00}(X)$, and is a differential operator if $AP \in \Psi_0(X)$ for all $P \in \Psi_0(X)$.*

We will then generalize this to the semi-classical setting by showing that semi-classical pseudodifferential operators with polyhomogeneous symbols are characterized by the properties:

(i) $A_\hbar$ depends smoothly on $\hbar$.

(ii) For fixed $\hbar$, $A_\hbar$ is polyhomogeneous.

(iii) $A_\hbar P \in \Psi_0(X)$ for all $P \in \Psi_0(X)$.

The second half of Chapter 9 will be devoted to discussing the symbol calculus for this class of operators, for the most part focusing on operators on $\mathbb{R}^n$ of the form (24).[4]

If $a(x,\xi,\hbar)$ is polyhomogeneous of degree less than n in ξ then the Schwartz kernel of P can be written in the form

$$\hbar^k(2\pi\hbar)^{-n}\int a(x,\xi,\hbar)e^{i\frac{(x-y)\cdot\xi}{\hbar}}\,d\xi\,; \tag{25}$$

however, we will show that these are several alternative expressions for (25): $a(x,\xi,\hbar)$ can be replaced by a function of the form $a(y,\xi,\hbar)$, a function of the form $a(\frac{x+y}{2},\xi,\hbar)$ or a function of the form $a(x,y,\xi,\hbar)$ and we will show how all these symbols are related and derive formulas for the symbols of products of these operators. Then in the last section of Chapter 9 we will show that there is a local description in coordinates for the space $I(X;\Lambda)$ similar to (25) and give a concrete description in coordinates of the module structure of $I(X;\Lambda)$ as a module over $\Psi(X)$.

In Chapter 10 we will study the functional calculus associated with polyhomogeneous semi-classical pseudodifferential operators. We recall that if $\mathcal{H}$ is a Hilbert

[4]We will, however, show that our results are valid under change of variables and hence make sense on manifolds.

space and A a densely defined self-adjoint operator on $\mathcal{H}$ then by Stone's theorem A generates a one-parameter group of unitary operators

$$U(t) = e^{itA}$$

and one can make use of this fact to define functions of A by the recipe

$$f(A) = \frac{1}{2\pi} \int \hat{f}(t) e^{itA}\, dt$$

for f a compactly supported continuous function and $\hat{f}$ its Fourier transform. We will give an account of these results in Chapter 13 and also describe an adaptation of this theory to the setting of semi-classical pseudodifferential operators by Dimassi–Sjostrand. In Chapter 10, however, we will mainly be concerned with the "mod $O(\hbar^{-\infty})$" version of this functional calculus. More explicitly we will show that if $P : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)$ is a self-adjoint elliptic pseudodifferential operator of order zero with leading symbol $P_0(x, \xi)$ and $f : \mathbb{R} \to \mathbb{R}$ a compactly supported $\mathcal{C}^\infty$ function then $f(P)$ is a semi-classical pseudodifferential operator with Schwartz kernel

$$(2\pi\hbar)^{-n} \int b_f(x, \xi, \hbar) e^{i\frac{(x-y)\cdot\xi}{\hbar}}\, d\xi \tag{26}$$

where $b_f(x, \xi, \hbar)$ admits an asymptotic expansion

$$\sum \hbar^k \sum_{\ell \le 2k} b_{k,\ell}(x, \xi) \left(\frac{1}{i}\frac{d}{ds}\right)^\ell f(P_0(x, \xi)) \tag{27}$$

in which the $b_{k,\ell}$'s are explicitly computable, and from this we will deduce the following generalization of the Weyl law that we described in Section 2 above.

Theorem 2. *Suppose that for some interval, $[a, b]$, the set $P_0^{-1}([a, b])$ is compact. Then the spectrum of P intersected with (a, b) consists of a finite number of discrete eigenvalues, $\lambda_k(\hbar)$, $q \le k \le N(\hbar)$ where*

$$N(\hbar) \sim (2\pi\hbar)^{-n} \quad \textit{volume} \quad (P_0^{-1}([a, b]))\,. \tag{28}$$

We will in fact derive this result from a much sharper result. Namely the formula (27) gives us for $f \in \mathcal{C}_0^\infty(\mathbb{R})$ an asymptotic expansion for

$$\text{Trace } f(P) = \int b_f(x, \xi, \hbar\, dx\, d\xi \tag{29}$$

in powers of $\hbar$ and hence an asymptotic expansion of the sum

$$\sum f(\lambda_k(\hbar)) \qquad 1 \le k \le N(\hbar)\,. \tag{30}$$

The second half of Chapter 10 will basically be concerned with applications of this result. For P the Schrödinger operator we will compute the first few terms in this expansion in terms of the Schrödinger potential, V, and will prove in dimension one an inverse result of Colin de Verdiere which shows that modulo weak asymmetry assumptions on V, V is spectrally determined. We will also show in dimension one that there is a simple formula linking the spectral measure $\mu(f) = \text{trace } f(P)$ and the quantum Birkhoff canonical form of P.

The results above are concerned with semi-classical pseudodifferential operators on $\mathbb{R}^n$; however we will show at the end of Chapter 10 that they can easily be generalized to manifolds and will show that these generalizations are closely related to classical heat trace results for elliptic differential operators on manifolds.

In Chapter 11 we will discuss results similar to these for Fourier integral operators. A succinct table of contents for Chapter 11 (which we won't bother to reproduce here) can be found at the very beginning of the chapter. However, in fifty words or less the main goal of the chapter will be to compute the trace of a Fourier integral operator $F : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$ whose canonical relation is the graph of a symplectomorphism, $f : T^*X \to T^*X$, and to apply this result to the wave trace

$$\text{trace } \exp i\frac{tP}{\hbar} \tag{31}$$

where P is an elliptic zeroth order semi-classical pseudodifferential operator.

The last chapter in this semi-classical segment of the monograph, Chapter 12, has to do with a topic that, as fas as we know, has not been much investigated in the mathematical literature (at least not from the semi-classical perspective). Up to this point our objects of study have been *exact* Lagrangian manifolds and *exact* canonical relations, but these belong to a slightly larger class of Lagrangian manifolds and canonical relations: If (M, α) is an exact symplectic manifold and $\Lambda \subseteq M$ a Lagrangian submanifold we will say that Λ is *integral* if there exists a function $f : \Lambda \to S^1$ such that

$$\iota_\Lambda^* \alpha = \frac{1}{\sqrt{-1}} \frac{df}{f} \,. \tag{32}$$

To quantize Lagrangian manifolds of this type we will be forced to impose a quantization condition on $\hbar$ itself: to require that $\hbar^{-1}$ tend to infinity in $\mathbb{Z}^+$ rather than in $\mathbb{R}^+$. An example which illustrates why this constraint is needed is the Lagrangian manifold, $\Lambda_\varphi = \text{graph } d\varphi$ in the cotangent bundle of the n-torus, $\mathbb{R}^n/2\pi\mathbb{Z}^n$ where $\varphi(x) = \sum k_i x_i$, $k \in \mathbb{Z}^n$. As a function on the torus this function is multi-valued, but $d\varphi$ and Λ_φ are well-defined, and

$$\iota_\Lambda^* \alpha = \pi_\Lambda^* \frac{df}{f}$$

where π_Λ is the projection of Λ onto the torus and $f = e^{i\varphi}$, so Λ_φ is integral.

Suppose now that we quantize Λ_φ by the recipe (18), i.e. by associating to it oscillatory functions of the form

$$a(x, \hbar) e^{i\frac{\varphi(x)}{\hbar}} \,. \tag{33}$$

It's clear that for these expressions to be well-defined we have to impose the constraint, $\hbar^{-1} \in \mathbb{Z}_+$ on $\hbar$.

In Chapter 12 we will discuss a number of interesting results having to do with quantization in this integral category. The most interesting perhaps is some "observational mathematics" concerning the classical character formulas of Weyl, Kirillov and Gross–Kostant–Ramond–Sternberg for representations of Lie groups: Let G be a compact simply-connected semi-simple Lie group and γ_α the irreducible representation of G with highest weight, α. By semi-classical techniques adapted to this integral symplectic category, one can compute symbolically the leading order asymptotics of the character, $\chi_n = \text{trace}\,\gamma_{n\alpha}$ as n tends to infinity. However, somewhat surprisingly, the asymptotic answer is, in fact, the exact answer (and in particular valid for $n = 1$).

The plan of attack, part 3

The last four chapters of this monograph are basically appendices and have to do with results that were cited (but not proved or not explained in detail) in the earlier chapters. Most of these results are fairly standard and are well-exposed in other texts, so we haven't, in all instances, supplied detailed proofs. (In the instances where we've failed to do so, however, we've attempted to give some sense of how the proofs go.) We've also, to provide some perspective on these results, discussed a number of their applications besides those specifically alluded to in the text.

1. Chapter 13:

   Here we gather various facts from functional analysis that we use, or which motivate our constructions in Chapter 10. All the material we present here is standard, and is available in excellent modern texts such as Davies, Reed-Simon, Hislop-Sigal, Schecter, and in the classical text by Yosida. Our problem is that the results we gather here are scattered among these texts. So we had to steer a course between giving a complete and self-contained presentation of this material (which would involve writing a whole book) and giving a bareboned listing of the results.

   We also present some of the results relating semi-classical analysis to functional analysis on L_2 which allow us to provide the background material for the results of Chapters 9–11. Once again the material is standard and can be found in the texts by Dimassi-Sjöstrand, Evans-Zworski, and Martinez. And once again we steer a course between giving a complete and self-contained presentation of this material giving a bareboned listing of the results.

2. Chapter 14: The purpose of this chapter is to give a rapid review of the basics of calculus of differential forms on manifolds. We will give two proofs of Cartan's formula for the Lie derivative of a differential form: the first of an algebraic nature and then a more general geometric formulation with a "functorial" proof that we learned from Bott. We then apply this formula to the "Moser trick" and give several applications of this method. (This Moser trick is, incidentally, the basic ingredient in the proof of the main results of Chapter 5.)

   (In earlier versions and in some publications we have referred to "Cartan's formula" as "Weil's formula". But it has been pointed out to us that this formula appears on page 84 (equation (5)) of Élie Cartan's 1922 book *Leçons sur les invariants intégraux.*)

3. Chapter 15: The topic of this chapter is the lemma of stationary phase. This lemma played a key role in the proofs of two of the main results of this monograph: It was used in Chapter 8 to show that the quantization functor that associates F.I.O's to canonical relations is well-defined and in chapter 11 to compute the traces of these operators. In this chapter we will prove the standard version of stationary phase (for oscillatory integrals whose phase functions are just quadratic forms) and also "manifold" versions for oscillatory integrals whose phase functions are Morse or Bott-Morse. In addition we've included, for edificational purposes, a couple corollaries of stationary phase that are not explicitly used earlier on: the Van der Corput theorem (for estimating the number of lattice points contained in a convex region of n-space) and the Fresnel version in geometric optics of Huygens's principle.

4. In Chapter 15 we come back to the Weyl calculus of semi-classical pseudodifferential operators that we developed in Chapter 9 and describe another way of looking at it (also due to Hermann Weyl.) This approach involves the representation theory of the Heisenberg group and is based upon the following fundamental result in the representation theory of locally compact topological groups: If one is given a unitary representation of a group of this type, this representation extends to a representation of the convolution algebra of compactly supported continuous functions on the group. Applying this observation to the Heisenberg group and the irreducible representation, $\rho_\hbar$, with "Planck's constant $\hbar$", one gets an algebra of operators on $L^2(R^n)$ which is canonically isomorphic to the Weyl algebra of Chapter 9, and we show that this way of looking at the Weyl algebra makes a lot of its properties much more transparent.

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