# Index Theory

## with Applications to Mathematics and Physics

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*To David*

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| Aims | Logical Order by Boxed Chapter Numbers |
|---|---|
| Index Theorem and Topolog. $K$-Theory | $\boxed{\text{App. B}} \to \boxed{\text{1.1-1.3, 2.1,2.2, 3.1-3.8, Thm. 5.11}} \to \boxed{\text{6-7}} \to \boxed{\text{8.5,9}} \to \boxed{\text{10-12.2}} \to \boxed{\text{13.1-13.5, 13.10,13.11}} \to \boxed{\text{18.1}}$ |
| Index Theorem via Heat Equation | $\boxed{\text{App. A}} \to \boxed{\text{1.2,3.3}} \to \boxed{\text{5.2, 6-7, 8.5}} \to \boxed{\text{8.5,9}} \to \boxed{\text{12.3}} \to \boxed{\text{15}} \to \boxed{\text{17}}$ |
| Gauge-Theoretic Physics | $\boxed{\text{14}} \to \boxed{\text{5-6}} \to \boxed{\text{9.2}} \to \boxed{\text{12.3}} \to \boxed{\text{13.8,13.11}} \to \boxed{\text{15}} \to \boxed{\text{16}} \to \boxed{\text{19}}$ |
| Spectral Geometry | $\boxed{\text{1-4}} \to \boxed{\text{6,15}} \to \boxed{\text{12.3}} \to \boxed{\text{13.4-13.11}} \to \boxed{\text{17.4-17.6}} \to \boxed{\text{18}}$ |
| Global and Micro-Local Analysis | $\boxed{\text{App. A,B}} \to \boxed{\text{1, 2, 3.1-3.5}} \to \boxed{\text{3.10,4}} \to \boxed{\text{5-6}} \to \boxed{\text{14-15}} \to \boxed{\text{7-9}} \to \boxed{\text{10.1, 10.2}} \to \boxed{\text{12.3,13}} \to \boxed{\text{16,17.6,18}}$ |

# Preface

**Target Audience and Prerequisites**. The mathematical philosophy of index theory and all its basic concepts, technicalities and applications are explained in Parts I-III. Those are the easy parts. They are written for upper undergraduate students or graduate students to bridge the gap between rule-based learning and first steps towards independent research. They are also recommended as general orientation to mathematics teachers and other senior mathematicians with different background. All interested can pick up a single chapter as bedside reading.

In order to enjoy reading or even work through Parts I-III, we expect the reader to be familiar with the concept of a smooth function and a complex separable Hilbert space. Nothing more — but a will to acquire specialized topics in functional analysis, algebraic topology, elliptic operator theory, global analysis, Riemannian geometry, complex variables, and some other subjects. Catching so many different concepts and fields can make the first three Parts a bit sophisticated for a busy reader. Instead of ascending systematically from simple concepts to complex ones in the classical Bourbaki style, we present a patch-work of definitions and results when needed. In each chapter we present a couple of fully comprehensible, important, deep mathematical *stories.* That, we hope, is sufficient to catch our four messages:

(1) Index theory is about *regularization*, more precisely, the index quantifies the defect of an equation, an operator, or a geometric configuration from being regular.
(2) Index theory is also about *perturbation invariance*, i.e., the index is a meaningful quantity stable under certain deformations and apt to store certain topological or geometric information.
(3) Most important for many mathematicians, the index *interlinks quite diverse mathematical fields*, each with its own very distinct research tradition.
(4) Index theory *trains the student* to recognize all the elementary topics of linear algebra in finite dimensions in the sophisticated topics of infinite-dimensional and nonlinear analysis and geometry.

Part IV is different. It is also self-contained. Choosing one or two chapters of this Part IV of the book would make a suitable text for a graduate course in selected topics of global analysis. All concepts will be explained fully and rigorously, but much shorter than in the first Parts. This last Part is written for graduate students, PhD students and other experienced learners, interested in low-dimensional topology and gauge-theoretic particle physics. We try to explain the very place of index theory in *geometry* and for revisiting *quantum field theory.* There are thousands of other calculations, observations and experiments. But there is something special about the actual and potential contributions of index theory. Index theory is about chirality (asymmetry) of zero modes in the spectrum and classifies connections (back ground fields) and a variety of other intrinsic properties in geometry and physics. It is not just about some more calculations, some more numbers and relations.

**Outline of History**. When first considering infinite-dimensional linear spaces, there is the immediate realization that there are injective and surjective linear endomorphisms which are not isomorphisms, and more generally the dimension of the kernel minus that of the cokernel (i.e., the index) could be any integer. However, in the classical theory of Fredholm integral operators which goes back at least to the early 1900s (see [**144**]), one is dealing with compact perturbations of the identity and the index is zero. FRITZ NOETHER (in his study [**321**] of singular integral operators and the oblique boundary problem for harmonic functions, published in 1920), was the first to encounter the phenomenon of a nonzero index for operators naturally arising in analysis *and* to give a formula for the index in terms of a winding number constructed from data defining the operator. Over some decades, this result was expanded in various directions by G. HELLWIG, I.N. VEKUA and others (see [**422**]), contrary to R. COURANT's and D. HILBERT's expectation in [**113**] that "linear problems of mathematical physics which are correctly posed behave like a system of $N$ linear algebraic equations in $N$ unknowns", i.e., they should satisfy the *Fredholm alternative* and always yield vanishing index. Meanwhile, many working mainly in abstract functional analysis were producing results, such as the stability of the index of a Fredholm operator under perturbations by compact operators or bounded operators of sufficiently small operator norm (e.g., first J.A. DIEUDONNÉ [**117**], followed by F.V. ATKINSON [**49**], B. YOOD [**446**], I.Z. GOHBERG and M.G. KREIN [**175**], etc.).

Around 1960, the time was ripe for I.M. GELFAND [**157**] to propose that the index of an elliptic differential operator (with suitable boundary conditions in the presence of a boundary) should be expressible in terms of the coefficients of highest order part (i.e., the principal symbol) of the operator, since the lower order parts provide only compact perturbations which do not change the index. Indeed, a continuous, ellipticity-preserving deformation of the symbol should not affect the index, and so GELFAND noted that the index should only depend on a suitably defined homotopy class of the principal symbol. The hope was that the index of an elliptic operator could be computed by means of a formula involving only the topology of the underlying domain (the manifold), the bundles involved, and the symbol of the operator. In early 1962, M.F. ATIYAH and I.M. SINGER discovered the (elliptic) Dirac operator in the context of Riemannian geometry and were busy working at Oxford on a proof that the $\widehat{A}$-genus of a spin manifold is the index of this Dirac operator. At that time, S. SMALE happened to pass through Oxford and turned their attention to GELFAND's general program described in [**157**]. Drawing on the foundational and case work of analysts (e.g., M.S. AGRANOVICH, A.S. DYNIN, L. NIRENBERG, R.T. SEELEY and A.I. VOLPERT), particularly that involving pseudo-differential operators, ATIYAH and SINGER could generalize HIRZEBRUCH's proof of the Hirzebruch-Riemann-Roch theorem of 1954 (see [**204**]) and discovered and proved the desired index formula at Harvard in the Fall of 1962. Moreover, the Riemannian Dirac operator played a major role in establishing the general case. The details of this original proof involving cobordism actually first appeared in [**325**]. A $K$-theoretic embedding proof was given in [**44**], the first in a series of five papers.

This proof was more direct and susceptible to generalizations (to $G$-equivariant elliptic operators in [**42**] and families of elliptic operators in [**47**]).

The proof of the Index Theorem in [**44**] was inspired by GROTHENDIECK's proof and thorough generalization of the Hirzebruch-Riemann-Roch Theorem, explained in [**83**]. We shall present the approach in detail in Chapters 10-12 of this book. The invariance of the index under homotopy implies that the index (say, the *analytic index*) of an elliptic operator is stable under rather dramatic, but continuous, changes of its principal symbol while maintaining ellipticity. Using this fact, one finds (after considerable effort) that the analytical index of an elliptic operator transforms predictably under various global operations such as embedding and extension. Using $K$-theory and Bott periodicity, a topological invariant (say, the *topological index*) with the same transformation properties under these global operations is constructed from the symbol of the elliptic operator. One then verifies that a general index function having these properties is unique, subject to normalization. To deduce the Atiyah–Singer Index Theorem (i.e., *analytic index = topological index*), it then suffices to check that the two indices are the same in the trivial case where the base manifold is just a single point. A particularly nice exposition of this approach for twisted Dirac operators over even-dimensional manifolds (avoiding many complications of the general case) is found in E. GUENTNER's article [**191**] following an argument of P. BAUM.

Not long after the $K$-theoretical embedding proof (and its variants), there emerged a fundamentally different means of proving the Atiyah–Singer Index Theorem, namely the *heat kernel method*. This is worked out here (see Chapter 17 in the important case of the chiral half $\mathcal{D}^+$ of a twisted Dirac operator $\mathcal{D}$. In the index theory of closed manifolds, one usually studies the index of a chiral half $\mathcal{D}^+$ instead of the total Dirac operator $\mathcal{D}$, since $\mathcal{D}$ is symmetric for compatible connections and then index $\mathcal{D} = 0$.) The heat kernel method had its origins in the late 1960s (e.g., in [**288**], inspired by [**299**] of 1949) and was pioneered in the works [**328**], [**163**], [**33**]. In the final analysis, it is debatable as to whether this method is really much shorter or better. This depends on the background and taste of the beholder. Geometers and analysts (as opposed to topologists) are likely to find the heat kernel method appealing. The method not only applies to geometric operators which are expressible in terms of twisted Dirac operators, but also largely for more general elliptic pseudo-differential operators, as R.B. MELROSE has done in [**289**]. Moreover, the heat method gives the index of a "geometric" elliptic differential operator naturally as the integral of a characteristic form (a polynomial of curvature forms) which is expressed solely in terms of the geometry of the operator itself (e.g., curvatures of metric tensors and connections). One does not destroy the geometry of the operator by using ellipticity-preserving deformations. Rather, in the heat kernel approach, the invariance of the index under changes in the geometry of the operator is a consequence of the index formula itself more than a means of proof. However, considerable analysis and effort are needed to obtain the heat kernel for $e^{-t\mathcal{D}^2}$ and to establish its asymptotic expansion as $t \to 0^+$. Also, it can be argued that in some respects the K-theoretical embedding/cobordism methods are more forceful and direct. Moreover, in [**270**], we are cautioned that the index theorem for families (in its strong form) generally involves torsion elements in K-theory that are not detectable by cohomological means, and hence are not computable in terms of local densities produced by heat asymptotics. Nevertheless, when this difficulty does not arise, the K-theoretical expression for the topological index may be less appealing than the integral of a characteristic form, particularly for those who already understand and appreciate the geometrical formulation of characteristic classes. More importantly, the heat kernel approach exhibits the index as just one of a whole sequence of spectral invariants appearing as coefficients of terms of the asymptotic expansion (as $t \to 0^+$) of the trace of the relevant heat kernel. (On p. 118, we guide the reader to the literature about these particular spectral invariants and their meaning in modern physics. The required mathematics for that will be developed in Section 17.4.) All disputes aside, the student who learns *both* approaches and formulations to the index formula will be more accomplished (and probably a good deal older).

**Further Reading**. What the coverage of topics in this book is concerned, we hope our table of contents needs no elaboration, except to say that space limitations prevented the inclusion of some important topics (e.g., the index theorem for families; index theory for manifolds with boundary, other than the Atiyah-Patodi-Singer Theorem; $L^2$-index theory and coarse geometry of noncompact manifolds; R. Nest's and B. Tsygan's algebraic and operator theoretic index theory of [**314, 315**]; P. Kronheimer's and T. Mrowka's visionary work on knot homology groups from instantons; lists of all calculated spectral invariants; aspects of analytic number theory). However, we now provide some guidance for further study. A fairly complete exposition, by Atiyah himself, of the history of index theory from 1963 to 1984 is found in Volume 3 of [**25**] and duplicated in Volume 4. Volumes 3, 4 and 5 contain many unsurpassed articles written by Atiyah and collaborators on index theory and its applications to gauge theory. In the informative — and charming [**445**], S.-T. Yau collected *The founders of index theory: reminiscences of and about Sir Michael Atiyah, Raoul Bott, Friedrich Hirzebruch, and I. M. Singer.* N. Hitchin's short text [**213**] on the 2004 Abel Prize Laureates describes the index theorem, where it came from, its different manifestations and a collection of applications. It indicates how one can use the theorem as a tool in a concrete fashion without necessarily retreating into the details of the proof. We all owe a debt of gratitude to H. Schröder for the definitive guide to the literature on index theory (and its roots and offshoots) through 1994 in Chapter 5 of the excellent book [**164**] of P.B. Gilkey. We have benefited greatly not only from this book, but also from the marvelous work [**270**] by H.B. Lawson and M.L. Michelsohn. In that book, there are proofs of index formulas in various contexts, and numerous beautiful applications illustrating the power of Dirac operators, Clifford algebras and spinors in the geometrical analysis of manifolds, immersions, vector fields, and much more. The classical book [**386**] of P. Shanahan is also a masterful, elegant exposition of not only the standard index theorem, but also the $G$-index theorem and its numerous applications. A fundamental source on index theory for certain open manifolds and manifolds with boundary is the authoritative book [**289**] of R.B. Melrose. In [**363**], Th. Schick reviews coarse index theory, in particular, for complete partitioned manifolds. It has been introduced by J. Roe and provides a theory to use tools from $C^*$-algebras to get information about the geometry of non-compact manifolds via index theory of Dirac type operators. [**201**] of N. Higson and J. Roe gives a well-written presentation of the underlying ideas of analytic K-homology and develops some of its applications. For a concrete calculation see also the concise [**306**, Section 7.4.2] and the Notes (forthcoming) [**202**]. See also [**134, 135, 136**] of J. Eichhorn for heat kernel asymptotics on non-compact manifolds and [**313**] of B.-W. Schulze and collaborators for index theory on singular spaces. [**454**] of W. Zhang gives an excellent introduction to various aspects of Atiyah-Singer index theory via the Bismut/Witten-type deformations of elliptic operators. Very close to our own view upon index theory is the plea [**154**] of M. Furuta for reconsidering the index theorem, with emphasis on the localization theorem. In the case of boundary-value problems for Dirac operators, we put quite some care in the writing of our [**80**] jointly with K.P. Wojciechowski. The recent book [**152**] of D. Fursaev and D. Vassilevich contains a detailed description of main spectral functions and methods of their calculation with emphasis on heat kernel asymptotics and their application in various branches of modern physics. Following up on the classic [**210**] of F. Hirzebruch and D. Zagier on interrelations between the index theorem and elementary number theory, the comprehensive [**373**] of S. Scott covers the theory of traces and determinants on Banach algebras of operators on vector bundles over closed manifolds, with emphasis on various algebras of pseudo-differential operators. He gives a series of calculations that give the flavor of the subject in tractable cases, and relates these calculations to Poisson and Selberg trace formulas. There is an impressive nonstandard proof of the local Atiyah-Singer index theorem, using resolvent expansions in place of the usual heat equation techniques. A wealth of radically new ideas of (partly yet unproven) geometric use of instantons are given in [**266**] of P.B. Kronheimer and T.S. Mrowka. Very inspiring is [**223**] of E.P. Hsu on stochastic analysis on manifolds. It gives a reformulation of the heat equation proof of the index theorem in terms of Wiener process asymptotics. Basically, that is what we should have after A. Einstein's famous 1905-discovery of the basis of heat conduction in diffusion. The details are interesting, though, in particular because they open a window to discrete analysis. A taste of the recent revival of D-branes and other exotic instantons in string theory can be gained from [**161**] of H. Ghorbani, D. Musso and A. Lerda. Indications can be found in the review [**358**] of F. Sannino about, how strongly coupled theories of gauge theoretic physics result in perceiving a composite universe and other new physics awaiting to be discovered. In the mathematically rigorous and richly illustrated [**352**], N. Reshetikhin explains why and how topological invariants by necessity appear in various quantizations of gauge theories.

**The Question of Originality: Seeking a Balance between Mathematical Heritage and Innovation**. Parts I-III and the two appendices teach what mathematicians today consider general knowledge about the index theorem as one of the great achievements of 20th century mathematics. But, actually, there are two novelties included which even not all experts may be aware of: The *first novelty* appears when rounding up our comprehensive presentation of the topology of the space of Fredholm operators: we do not halt with the Atiyah-Jänich Theorem and the construction of the index bundle, but also confront the student with a thorough presentation of the various definitions of determinant line bundles. This is to remind the student that index theory is not a more or less closed collection of results but a philosophy of regularization, of deformation invariance and of visionary cross connections within mathematics and between its various branches.

A *second novelty* in the first three Parts is the emphasis on global constructions, e.g., in introducing and using the concept of pseudo-differential operators.

Apart from these two innovations, the student can feel protected in the first three Parts against any originality.

Basically, Part IV follows the same line. Happily we could also avoid excessive originality in the chapters dealing with instantons and the Donaldson-Kronheimer-Seiberg-Witten results about the geometry of moduli spaces of connections. There we also summarize, refer, define, explain great lines and details like in the first three Parts, though emphasizing variational aspects based on [**59**].

However, the *core* of Part IV is different. It consists of an original, full, quite lengthy (in parts almost unbearably meticulous) proof of the *Local Index Theorem for twisted Dirac operators* in Chapter 17 and its applications to standard geometric operators. That long Chapter is thought as a new contribution to the ongoing search for a deeper understanding of the index theorem and the "best" approach to it.

Clearly, a student looking for the most general formulation of the index theorem and a proof apt for wide generalizations should concentrate on our Part III, the so-called Embedding (or K-theoretic) Proof. However, a student wanting to trace the germs of index calculations back in the geometry of the considered standard operators (all arising from various decompositions of the algebra of exterior differential forms) should consult Section 17.5 with a full proof of the *Local Index Formula* for twisted Dirac operators on spin manifolds (all terms will be explained) and Section 17.6, where we derive the *Index Theorem for Standard Geometric Operators.* These geometric index theorems are by far less general than Part III's embedding proof, but they are more geometric, and we hold, also more geometric than the *usual* heat equation proofs of the index theorem. Not striving for greatest generality, we obtain index formulas for the standard elliptic geometric operators and their twists. The standard elliptic geometric operators include the signature operator $d+\delta\colon\ (1+*)\,\Omega^*\,(M)=\Omega^+(M)\to\Omega^-(M)=(1-*)\,\Omega^*\,(M)$, the Euler-Dirac operator $d+\delta\colon\,\Omega^{\mathrm{ev}}\,(M)\to\Omega^{\mathrm{odd}}\,(M)$, and the Dolbeault-Dirac operator $\sqrt{2}\,(\bar\partial+\bar\partial^*)\,:\,\Omega^{-,\mathrm{ev}}\,(M)\to\Omega^{+,\mathrm{odd}}\,(M)$ (all symbols will be defined). The index formula obtained for the above operators yields the Hirzebruch Signature Theorem, the Chern-Gauss-Bonnet Theorem, and the Hirzebruch-Riemann-Roch Theorem, respectively. While these operators generally are not globally twisted Dirac operators, locally they are expressible in terms of chiral halves of twisted Dirac operators. That applies also to the Yang-Mills operator. Thus, even if the underlying Riemannian manifold $M$ (assumed to be oriented and of even dimension) does not admit a spin structure, we may still use the Local Index Theorem for twisted Dirac operators to compute the index density and hence the index of these operators. While it is possible to carry this out separately for each of the geometric operators, basically all of these theorems are consequences of one single index theorem for generalized Dirac operators on Clifford module bundles (all to be defined). Using the Local Index Theorem for twisted Dirac operators, we prove this index theorem first (our Theorem 17.59), and then we apply it to obtain the geometric index theorems, yielding the general Atiyah-Singer Index Theorem for practically all geometrically defined operators.

**Style and Notations**. To present the rich world of index theory, we have chosen two different styles. We write all definitions, theorems, and proofs as concise as possible to free the reader from dispensable side information. Where possible, we begin the introduction of a new concept with a simple but generic example or a review of the local theory, immediately followed by the corresponding global or general concept. That is one half of the book, so to speak the odd numbered pages. The other half of the book consists of exercises (often with extended hints) and historical reviews, motivations, perspectives, examples. We wrote those sections in a more open web-like style. Important definitions, notions, concepts are in bold face. Background information is in small between the signs ▶ and ◀. In remarks and notes, leading terms are in italics.

The reader will notice our bias towards elder literature when more recent references would not add substantially more value. This is due not so much to the age of the authors (both born before the middle of the last century) but rather to the common pride of mathematicians belonging to a community where bibliographic impact factors and research indices should rather be calculated in citations after some decades of years than in numbers of recently appeared, cited and soon forgotten publications.

There is also a distinction, due to HARALD BOHR and disseminated by BØRGE JESSEN, between *expansive* and *consolidating* periods of each individual science. While physics and biology had consolidating periods in the first half of the last century and suffer now of the rapid change of ever new single and dispersed results, mathematics has had and still has good decades of consolidation and of long-time validity of key results. To the present authors, there is no reason to hide our indifference to changing fashions.

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Both authors agreed to dedicate this book to their teachers, to the memory of S.-S. CHERN (thesis adviser of DB) on the occasion of his centenary in October 2011 and to the memory of F. HIRZEBRUCH (thesis adviser of BBB) who appreciated the announced dedication intended for his 85th birthday in October 2012, which he did not live to celebrate. I take the liberty to change the dedication. This book is dedicated to DAVID.

*Bernhelm Booß-Bavnbek, Roskilde (Denmark), December 2012*

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# Index of Notation

# Index of Names/Authors

Cursive numbers refer to the bibliography.

Most given names and personal data are from *Indexes of Biographies*
of selected mathematicians at
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# Subject Index

Terms and topics are not in alphabetical order but grouped according to teaching goals. → refers, or auto-refers, to a separate main entry.

# About the Authors

**David D. Bleecker, Ph.D.** (University of California at Berkeley under S.-S. Chern) taught on the faculty of the Department of Mathematics at the University of Hawaii, Manoa, for more than thirty years, and is now retired. He is known for his work in differential geometry and gauge-theoretic physics. He is also known to a larger audience as a textbook author of unrivaled care for details. Among his student textbooks are the widely-read *Basic Partial Differential Equations* (International Press, 1996) with George Csordas; the more advanced *Gauge Theory and Variational Principles;* and this book's predecessor with Bernhelm Booß-Bavnbek, *Topology and Analysis: The Atiyah-Singer Index Formula and Gauge-Theoretic Physics.*

**Bernhelm Booß-Bavnbek, Ph.D.** (Universität Bonn under F. Hirzebruch) is senior lecturer in mathematics and mathematical modeling with the Mathematics-Physics Group IMFUFA at Roskilde University (Denmark), where he has been for more than thirty years. He is known for his work in global analysis and spectral geometry. He is also known to a larger audience as a textbook author and editor with lively interest in a broad sweep of mathematics, applications of mathematics, and communication about mathematics. Among his research monographs are the now-classic treatise *Elliptic boundary problems for Dirac operators,* co-written with Krzysztof Wojciechowski; the volume *Mathematics and War*, with Jens Høyrup as co-editor; the monograph entitled *BetaSys: Systems Biology of Regulated Exocytosis in Pancreatic $\beta$-cells,* as main editor; and this book's first German-language predecessor, *Topologie und Analysis – Eine Einführung in die Atiyah-Singer-Indexformel.*

## Praise for *Index Theory with Applications to Mathematics and Physics:*

This book ... describes in impressive detail one of the greatest achievements of 20th-century mathematics, the Atiyah-Singer index theorem. As a bonus, the author includes an extensive and detailed discussion of some of the applications of the theorem and more recent developments in the direction of gauge theoretic physical models. For the student, the gold lining in the book is that all the background is there—starting at a level that is accessible to anyone with a reasonable mathematical education. There is also a bridge to the point of view from the physics side. The student can dip into the book at many points and use it to learn individual topics. The discussion covers a very broad sweep of mathematics which will be of interest to anyone aiming to understand geometric analysis, whether or not gauge theories are their primary focus. Two proofs of the Atiyah Singer theorem are given, one based on K-theory and the other on the heat kernel approach... The authors have included many discussion sections that illuminate the thinking behind the more general theory... There is much valuable historical context recorded here. Moreover, it captures the flavour of contemporary mathematics, with the interaction of geometric, topological, analytic, and physical perspectives and methods being in the forefront of the discussion.

Alan Carey
Mathematical Sciences Institute at Australian National University, Canberra

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Those wishing to start their study of the theory of elliptic operators will appreciate the book's use of rigorous exposition intermingled with intuition-building casual and historical discussion. Experts will find many features of the book useful and convenient.

Paul Kirk
Indiana University Bloomington

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The real strength of the book is its detailedness, which makes it particularly attractive for the learning (graduate) student. But the senior researcher, likewise, will treasure this somewhat unconventional textbook as a very valuable source of information.

Matthias Lesch
Universität Bonn, Germany

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Two or three famous Index Formulas discovered and proved in the course of middle decades of the last century are some of the highest peaks in a mountain country rising from the vast plains of functional analysis, theory of smooth manifolds, and homotopical topology.

This treatise, written with ambition, wit and (mathematical) eloquence, strives to combine the qualities of a guide-book, historical chronicles, and a hiking manual for enthusiastic travellers and budding future explorers of this vast territory.

Any reader possessing will and enthusiasm can profit from studying (parts of) this book and enjoy finding his or her own path through this land.

Yuri I. Manin
Max Planck Institute for Mathematics, Bonn, Germany

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My struggles with this theorem are vivid in my mind. I still remember the sunny autumn afternoon when I picked from the library the first edition of this book. From the first moment I opened it I thought it spoke to me, the beginning graduate student with a limited mathematical experience. The proofs had the level of detail I needed at that stage in my life. What awed me most was the wealth of varied examples from exotic worlds I did not even suspect existed. Reading those examples and the carefully crafted proofs I began to get a glimpse at the wonderful edifice behind the index theorem.

The present edition has the same effect on the more mature me. And it does quite a bit more. Old examples are refined, and new ones added. Facts whose proofs were only sketched in the old edition are now given complete, or almost complete proofs. The chapter on gauge theory is completely and massively rewritten and it incorporates some of the spectacular developments in this area that took place in the intervening time. As importantly, throughout the book, the exposition is sprinkled with many mathematical “anectodes” which give the reader a glimpse into the minds of the pioneers of the subject.

I believe that any youngster who will pick this new edition will be as grateful as I am to the authors for the care and their concern for the reader. The teacher of this subject, as well, will have many things for which to be grateful: this is one of the few places containing such a wealth of examples and care for detail.

Liviu Nicolaescu
University of Notre Dame, Indiana

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The present book is a landmark, being a complete introduction to practically all aspects of the Index Theorem, complete with basic examples and exercises, covering topological ideas: homotopy invariance and K-theory; analysis: elliptic operators and heat equations; geometry: principal bundles and curvature; physics: gauge theory and Seiberg-Witten theory. The first English version appeared in 1985, but the present book is a largely reworked and much expanded treatise; it is a highly informative presentation, based on the authors' expert knowledge and pedagogical interests. Indeed, it seems to be presently the most complete single book on the many approaches to and applications of the Atiyah-Singer Index Theorem. Experts as well will find inspiration in this book.

Bent Ørsted
Aarhus University, Denmark

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The student who wants to explore the whole shape of this huge and complex territory—as well as the usual climbing routes to the summit of the Index Theorem itself—can hardly do better than to enlist the genial and enthusiastic guide service of Bleecker and Booss-Bavnbek.

John Roe
Pennsylvania State University

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Professors Bleecker and Booß-Bavnbek have followed ... developments in index theory from the beginning, and made original contributions of their own... Assuming only basic analysis and algebra, [this book] gives detailed constructions and proofs for all the necessary concepts, along with illuminating digressions on the various paths through the rich territory of index theory.

Robert Seeley
Professor Emeritus, University of Massachusetts, Boston

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The very appreciable feature of the book is its discussion of the applications of the theorem in low-dimensional topology and in gauge theories. For these reasons, the book will be a valuable reference for theoretical and mathematical physicists, especially those interested in the foundations of quantum gauge theories, at the basis of the standard model of elementary particle physics. Mathematically oriented graduate students will greatly profit from reading this excellent book.

Franco Strocchi
Scuola Normale Superiore, Pisa, Italy

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Students of mathematics and physics will find this book to be an excellent resource for study of this vast subject. In particular, attention is given to exposition of the subject from different angles, which is very helpful as a bridge between physics and mathematics.

Cumrun Vafa
Donner Professor of Science, Physics Department, Harvard University

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Readers from a wide range of backgrounds will find much to learn here.

Edward Witten
Institute for Advanced Studies, Princeton, New Jersey

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