Integral Evaluations Using the Gamma and Beta Functions and Elliptic Integrals in Engineering

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A Self-Study Approach

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FOREWORD

This book is intended to provide to the students of engineering and science faculties and to the engineers and scientists an alternative tool for integral evaluation. Using only the classical methods, many times, the work is difficult, long and because of this, prone to errors. In addition, many integrals can not be evaluated in terms of the elementary functions, but can be evaluated in terms of gamma function and elliptic integrals. This book will fill a gap that has existed for many years in engineering and science fields.

For many modern advanced technology areas, the evaluation of the integrals using the gamma and beta functions is of prime importance. It is easy to apply this method, to generalize the solutions and at the same time to obtain the convergence conditions of given integrals. The complete elliptic integrals are also important in engineering.

All chapters of this book are very lucidly treated and are important to the author's goal of making engineers and scientists effective and competitive in their fields. There are many worked examples and generalized solutions. The book is very well coordinated and I am confident that it will be very well received by a large number of engineers, scientists and students worldwide.

The book will also encourage the educators to include gamma and beta functions and complete elliptic integrals in the engineering and science curricula at universities and colleges.

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LIST OF SYMBOLS, NOTATIONS AND FORMULAS

$a \in A$ a is an element of the set A, or the point a belongs to the set A.

$a \notin A$ a is not an element of the set A.

$N = \{1, 2, 3, \ldots\}$ is the set of natural numbers (positive integers).

$Z = \{0, \pm 1, \pm 2, \ldots\}$ is the set of integers.

Q is the set of rational numbers and they are real numbers of the form m/n where m and n are integers and $n \neq 0$.

– The irrational numbers are real numbers that can not be expressed in the form m/n where m and n are integers and $n \neq 0$.

R is the set of real numbers $= R^+ \cup R^- \cup \{0\}$.

$R^+ = \{x \in R \mid x > 0\}$ is the set of positive real numbers.

$R^- = \{x \in R \mid -x \in R^+\}$ is the set of negative real numbers.

$(a, b) = \{x \in R \mid a < x < b\}$ is an open interval.

$[a, b) = \{x \in R \mid a \leq x < b\}$ is an interval closed at left and open at right.

$(a, b] = \{x \in R \mid a < x \leq b\}$ is an interval open at left and closed at right.

$[a, b] = \{x \in R \mid a \leq x \leq b\}$ is a closed interval.

R^n-space, $n \in N$, is a real n-dimensional Euclidean Space.

$n! = 1 \cdot 2 \cdot 3 \cdots n, \quad 0! = 1.$

$$(2n+1)!! = 1 \cdot 3 \cdots (2n+1) = \frac{(2n+1)!}{n! 2^n}$$

$(2n)!! = 2 \cdot 4 \cdots (2n) = n! 2^n$

$(n-1)! 2^{n-1} = (2n-2)!!$

$\Gamma(p)$ is the complete gamma function

$B(p, q)$ is the complete beta function

$\gamma(p, x)$, $\Gamma(p, x)$ are the incomplete gamma functions.

$B_x(p, q)$ is the incomplete beta function

$\psi(x)$ is the digamma function

$\beta(x)$, see Eq.(1.171).

$P(x^2/v)$ is the probability integral of x^2-distribution function.

$Q(x^2/v)$ is the percentage points of x^2-distribution function.

$K(k)$ is the complete elliptic integral of the first kind.

$E(k)$ is the complete elliptic integral of the second kind.

a. *The order of the function* $f(x)$ with respect to another function $g(x)$. Suppose that the point x approaches a point x_0. We say that $f(x)$ is of the order of $g(x)$, if there exists a number $M \geq 0$ such that

$$|f(x)| \leq M |g(x)|$$

in some sufficiently small neighbourhood of the point x_0. We write

$$f(x) = \underset{x=x_0}{O}\,[g(x)]\,.$$

In particular, if

$$\lim_{x=x_0} \left| \frac{f(x)}{g(x)} \right|$$

exists and is finite, then $f(x)$ is of the order of $g(x)$ in the neighbourhood of x_0.

b. A function $f(x)$ integrable over the interval $(-a, a)$ and satisfying the relation $f(-x) = f(x)$ on that interval is called an even function and then

$$\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx, \qquad a > 0.$$

c. A function $f(x)$ integrable over the integral $(-a, a)$ and satisfying the relation $f(-x) = -f(x)$ on that interval is called an odd function and then

$$\int_{-a}^{a} f(x)\, dx = 0, \qquad a > 0.$$

d. Integrating by parts, with u and v functions of x, we have

$$\int_{a}^{b} uv'\, dx = uv\Big|_{a}^{b} - \int_{a}^{b} vu'\, dx$$

or

$$\int_{a}^{b} u\frac{dv}{dx}\, dx = uv\Big|_{a}^{b} - \int_{a}^{b} v\frac{du}{dx}\, dx$$

or

$$\int_{a}^{b} u\, dv = uv\Big|_{a}^{b} - \int_{a}^{b} v\, du.$$

For applications see Secs.1.9.2, 4.2.7 and 4.2.15.

e. For other formulas, see Appendix A6.

INTRODUCTION

For the evaluation of many integrals, the Euler's gamma and beta functions and the complete elliptic integrals are among the most useful functions in engineering, physics and probability. The gamma and beta functions are also used for the generalization of many integrals and in the definition of other special functions, e.g., the Bessel, Legendre, hypergeometric functions, and so on.

Using the properties of these functions, readers will find in the book how to use them for integral evaluations. In addition, for many integrals, using these functions we will try to eliminate the long and tedious traditional methods of integral evaluations. Note that there are many integrals that cannot be evaluated in terms of the elementary functions, but can be evaluated in terms of the gamma function or in terms of the complete elliptic integrals; e.g.,

$$\int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{1}{8\sqrt{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2 = \frac{1}{2}K\left(\frac{1}{\sqrt{2}}\right), \qquad \text{Secs.2.2.4a and 4.2.19}$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx = \frac{4\pi\sqrt{2\pi}}{\left[\Gamma\left(\frac{1}{4}\right)\right]^2} = \sqrt{2}\left[2E\left(\frac{1}{\sqrt{2}}\right) - K\left(\frac{1}{\sqrt{2}}\right)\right]. \qquad \text{Sec.4.3.5}$$

This book can be used as a self-study book, textbook, supplemental textbook or reference book. The prerequisite could be any classical course of integral calculus given in college and university. In a step-by-step procedure, this book will show how to use the gamma and beta functions and complete elliptic integrals to evaluate, easily and with a high degree of accuracy, many integrals used in engineering work.

The gamma, beta, digamma functions and the function $\beta(x)$ may be used to simplify the evaluation of many definite integrals, line integrals, double, multiple and surface integrals. Using only the traditional methods, the evaluation of many integrals in engineering is very tedious. Such long calculations are prone to errors. Examples of the integrals evaluated by different methods illustrate that many times it is simpler to use the gamma, beta, digamma functions and the function $\beta(x)$. Examples

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^4} = \frac{1}{2}B\left(\frac{3}{4},\frac{1}{4}\right) = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{\pi}{\sqrt{2}}, \qquad \text{Sec.2.3.14}$$

$$\int_{0}^{\infty} \frac{dx}{1+x^n} = \frac{1}{n}B\left(1-\frac{1}{n},\frac{1}{n}\right) = \frac{\pi}{n\sin\dfrac{\pi}{n}}, \quad n>1, \qquad \text{Sec.2.3.16}$$

$$\int_{0}^{1} \frac{dx}{1+x^3} = \frac{1}{3}\beta\left(\frac{1}{3}\right). \qquad \text{Sec.2.8.4}$$

Moreover, there are integrals which cannot be expressed in terms of the elementary functions, but can be in terms of the gamma function or elliptic integrals. Examples

$$\begin{aligned}\int_{0}^{1} (1-x^2)^p\, dx &= \frac{p}{1+2p}B\left(\frac{1}{2},p\right)\\ &= \frac{p\sqrt{\pi}}{1+2p}\frac{\Gamma(p)}{\Gamma\left(\dfrac{1}{2}+p\right)}, \quad p>0, \qquad \text{Sec.2.3.8}\end{aligned}$$

$$\int_{0}^{\phi} \frac{d\theta}{\sqrt{\cos\theta-\cos\phi}} = \sqrt{2}K\left(\sin\frac{\phi}{2}\right), \quad \phi\in[0,\pi). \qquad \text{Sec.4.2.11}$$

There are applications where the double and triple integrals are expressed in terms of the elementary functions, but when they are generalized in the R^n-space, the integrals cannot be expressed in terms of the elementary functions, but can be in terms of the gamma and beta functions (see the surface area of a sphere in the R^3-space and then of a sphere in the R^n-space, Sec.2.5.3), the digamma function, function $\beta(p)$ and elliptic integrals.

Also, for many improper generalized integrals (integrals depending on one or more parameters), the convergence conditions of the integrals expressed in terms of the gamma and beta functions result at the end of the evaluation from the properties of the gamma and beta functions. Example

$$\int_0^\infty \frac{x^{p-1}\,dx}{(1+x^q)^2} = \frac{1}{q}B\left(2-\frac{p}{q},\frac{p}{q}\right), \quad 0<p<2q. \qquad \text{Sec.2.3.25}$$

Many existing tables of integrals list only the indefinite and definite integrals; they do not list the double, triple, multiple, line and surface integrals which are used in engineering.

The existing tables give the gamma function for $p \in [1,2]$, but we also need tables for $p \in (0,1]$. Our tables are for $p \in (0,2]$, and in addition we give $[\Gamma(p)]^2$ and the graphic of $1/\Gamma(p)$, which are important in applications. Also, the tables of complete elliptic integrals, $E(k)$ and $K(k)$, are given in terms of k^2 and the angle psi, $k = \sin\psi$. The graphs of these functions are given with enough accuracy to be usable in engineering work. The computer programs are in Microsoft QuickBASIC. With little modification, the programs can be used for other computer languages.

The book contains a valuable range of practical applications and examples. The applications are selected in such a way as to be important to any engineering branch. At the end of some applications we ask the reader to evaluate the integrals using other methods. Doing these exercises, the reader can compare different methods of evaluation for the same integral to convince himself that the gamma and beta functions method is a simple and important method of integral evaluation. This book illustrates the fact that the methods of integral evaluations using the gamma, beta and elliptic integral functions are very powerful and useful for solving practical problems in different branches of engineering and science.

In this book, the theory has been limited to only the engineering needs. Thus, the book has several limitations:

(a) The gamma function $\Gamma(z)$, where z is a complex number, is not studied. For this theory, one could refer to those books that study the functions of a complex variable.
(b) Stirling's asymptotic series and formula are given without proof; these are used in statistical mechanics and probability theory, which deal with large factorials.
(c) Hölder's theorem is not discussed.

I predict that one day $\Gamma(p)$, $\psi(x)$, $K(k)$ and $E(k)$ will be regarded as elementary functions and will be included in the scientific calculator. This publication may encourage colleges and engineering faculties to include these functions and their applications as an undergraduate or graduate term course. This book should be used as a tool for solving practical problems. Taking into consideration the many exercises and applications evaluated by a step-by-step approach, the book may be recommended to students and engineers as a self-study book.

For easy reference, a list of the most important formulas with the gamma function, beta function and complete elliptic integrals can be found in Appendix A6.

I am greatly indebted to the late Dr. H.H.E. Leipholz, former Dean of Engineering, University of Waterloo, Ontario, Canada, and Dr. K.D. Srivastava, Professor of Electrical Engineering, University of British Columbia, Vancouver, for the encouragement given to me to write this book.

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