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Volume XXI

Advances in Geometric Analysis

edited by

Stanislaw Janeczko · Jun Li · Duong H. Phong

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Advances in Geometric Analysis

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Workshop on Geometry, Poland
in honour of Yau's 60th birthday

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Welcoming Speech (Extended) by Friedrich Hirzebruch

We just have finished a meeting in honor of the Institute of Mathematics of the Polish Academy of Sciences (IMPAN) on the occasion of its 60th anniversary. Now we are beginning a meeting in honor of Professor Shing-Tung Yau, also on the occasion of his 60th birthday.

IMPAN and Shing-Tung Yau are important for the development of mathematics throughout the world. Both are good friends of mine. I shall explain why.

A child of IMPAN is the Stefan Banach International Mathematical Center. It was founded in January 1972 by the Academies of Sciences of Bulgaria, Czechoslovakia, the German Democratic Republic, Poland, Romania, and the USSR for the development of international relations, also with the West, in particular by special semesters and meetings. The Banach Center gained international recognition very rapidly under the leadership of Professor Czesław Olech. The Center had a scientific council consisting of representatives of the founding academies. With the year 1992 the system ended. The European Mathematical Society (founded in 1990) was contacted. As president of the EMS, I had intensive discussions with Professor Olech (for the Banach Center, but also as vice president of the EMS) and with Professor Bogdan Bojarski as director of IMPAN. A new scientific council was formed, consisting of three representatives of the EMS, three representatives of the founding countries, and four representatives from Poland. I became a member for the EMS and the chairman of the council for nine years. My wife and I came to Poland frequently and made many good friends. The Banach Center is a branch of IMPAN. The cooperation with the director of IMPAN, Bogdan Bojarski, was very good, also with the scientific directors of the Center: Bronisław Jakubczyk, Łukasz Stettner, and Roman Dwilewicz. Of the members of the council I mention in particular Stanisław Janeczko, who became the director of IMPAN in 2002 and whom we are grateful for the excellent organization of the two conferences. The hospitality of the Polish colleagues and their wives during all our visits to Poland has always been overwhelming. My wife and I have enjoyed it a lot. Stanisław Janeczko has been a visitor to the Max Planck Institute for Mathematics in Bonn several times. We often have Polish visitors in Bonn. All this shows that I am a good friend of the first sexagenarian.

But now to the second sexagenarian. Shing-Tung Yau was born on April 4, 1949 and grew up in Hong Kong. He studied at The Chinese University of Hong

Kong where he received his B. A. in 1969. He went to Berkeley in the fall of 1969 where he studied under Shiing-Shen Chern and received his Ph. D. degree already in 1971. Then he was a member of the Institute for Advanced Study in Princeton, New Jersey. Two years at the State University of New York at Stony Brook followed. There I met him for the first time early in 1973. From Stony Brook Yau followed a call to Stanford. Since 1987, Yau is a professor at Harvard University.

Yau won one of the four Fields Medals for the Warsaw International Congress of Mathematicians 1982. Since the congress was postponed by one year, the medal was handed over in 1983. Louis Nirenberg reported about Yau's work. His introductory sentences are as follows:

"Yau has done extremely deep work in global differential geometry and elliptic partial differential equations, including applications in three-dimensional topology and in general relativity theory. He is an analytic geometer (or geometer's analyst) with remarkable technical power and insight. He has succeeded in solving problems on which progress had been stopped for years. Here are a few of his striking developments..."

Nirenberg describes six areas of Yau's work. The wide scope and depth of Yau's theorems become clear and this at a time when Yau was only 33 years old. I can enjoy Nirenberg's description, but I am not competent to report on it. One exception: There is Yau's work on the Calabi conjecture. It implies:

Let X be an algebraic surface (smooth and of general type), then the Chern numbers satisfy the inequality

$$c_1^2[X] \leq 3c_2[X]$$

(Miyaoka-Yau inequality) and equality holds if and only if the universal cover of X is the 2-dimensional complex ball. The "if"-part follows from the "proportionality principle" which I proved in 1955/1956. This is very easy compared to the other direction. The work of Yau involves partial differential equation (Monge-Ampère).

In the academic year 1981/1982, I gave a course at Bonn University and in 1984 a series of lectures at the Max Planck Institute for Mathematics constructing branched coverings of the complex projective plane assuming that the branch locus consists of lines, looking at cases that the covering is smooth, of general type and satisfies $c_1^2 = 3c_2$. Gottfried Barthel took notes and improved the presentation considerably. The excellent dissertation of Thomas Höfer with further results was incorporated, and the book *Geradenkonfigurationen und algebraische Flächen* by Barthel, Hirzebruch, Höfer, Vieweg 1987, took shape. The book also gives information on earlier work concerning the Miyaoka-Yau inequality.

In 1996, I gave a course at the ETH in Zürich with further developments of these topics. Paula Tretkoff took notes, a lot of joint work and work of Paula Tretkoff alone followed. She wrote a book which appears with Princeton University Press. I am glad that I got involved in a small portion of Yau's work.

More than 25 years are over since Yau received the Fields Medal. He has continued mathematical research with great energy and obtained pathbreaking results. The conference will highlight some of them. During the past ten years

Yau worked in applied mathematics, for example face recognition (with software for which patents were granted) and brain research.

Yau is a great communicator for mathematics. He had and has many brilliant students at Harvard University. Yau introduced the conferences which are sponsored by the *Journal of Differential Geometry* and take place at Harvard/MIT every three years. For 1999 he had the great idea to have such a JDG conference to honor the four mathematicians who founded Index Theory. I was, of course, very happy that Yau included me (Atiyah, Bott, Hirzebruch, Singer).

At Harvard/MIT Yau has also the annual Bourbaki style conference of two days *Current Development in Mathematics*.

Yau is one of the founders of the publishing house *International Press*.

Yau's activity in China is incredible. He founded three mathematical institutes[1]: the Institute of Mathematical Sciences at The Chinese University of Hong Kong, the Morningside Center of Mathematics in Beijing, and the Center of Mathematical Sciences at Zhejiang University in Hangzhou. Each institute serves different purposes. Yau visits them regularly. I heard that a huge Oberwolfach style institute is planned for the island Hainan in Southern China. Yau founded the *International Congress of Chinese Mathematicians* taking place every three years. The fifth one will be in Beijing in December 2010.

Yau intends to develop mathematics in China on all levels, also for high school and college students, he organized two meetings for them in 2004: *Why Math? Ask Masters* in Hangzhou and *The Wonder of Mathematics* in Hong Kong.

Finally, let me mention that Yau also tries to bring mathematics and mathematical physics closer to a general audience of people interested in the sciences. Jointly with Steve Nadis he wrote the book *The Shape of Inner Space: String Theory and the Geometry of the Universe's Hidden Dimensions* (Basic Books 2010).

I could only say a few words about Yau's work. Some of his achievements in mathematical research will be seen in the lectures of this conference. I am sorry that I have to leave for Bonn and am unable to attend.

We all wish Shing-Tung Yau health and further success for many years to come.

April 4, 2009

[1] Yau founded the fourth mathematical institutes in 2009: Mathematical Sciences Center at Tsinghua University in Beijing.

Speech of Thanks by Shing-Tung Yau

I am very grateful to Professor Stanisław Janeczko and other Polish colleagues who made this wonderful conference possible to celebrate my 60th birthday in Warsaw. Warsaw is a special city to me, and it is a city I have been always fond of. In 1983 I received the Fields Medal in the great Kongresowa hall in Warsaw. My wife and I came to the capital of Poland for this joyful moment of my life, and we are deeply indebted to the hospitality of our Polish friends. We also went to Krakow and witnessed the glory of the great old empire.

When Professor Janeczko invited me to visit Warsaw again, I could not resist the invitation which brought back indelible memories of the good old days. I enjoyed the conference and I am honored to have many distinguished mathematicians who came to this conference. I am especially delighted to have seen Professor Hirzebruch and his wife; I learnt a great deal from Professor Hirzebruch's classical book. In my heart, I always think of him as my teacher, and I am particularly flattered to see his welcoming speech in this book.

It is a memorable and important event in my career. I am really glad to see my old friends from Germany, the US and China. We worked together and contributed to the great subject of geometric analysis. To them I am grateful. Also, the staff and the students at the Banach Institute worked very hard and their hospitality will never be forgotten.

During this trip I also realized that there were and still are so many great mathematicians in Poland, just to mention the great works by Szegő and Kac which had a real, deep impact on the subject of analysis in the twentieth century. I do admire them.

Finally, I would like to thank the family of Professor Janeczko for their hospitality at their home.

February 14, 2011

Preface

This volume is dedicated to our friend and master Shing-Tung Yau. It contains invited expository and research papers from the conference, *The Conference on Geometry*, in honor of Shing-Tung Yau's 60th birthday, held on April 6–8, 2009, in the Stefan Banach International Mathematical Center, Warsaw, Poland. The conference was directly connected to the celebration of the 60th Anniversary of the Institute of Mathematics of the Polish Academy of Sciences. It was organized exactly 25 years after the International Congress of Mathematicians in Warsaw at which Professor Shing-Tung Yau received his Fields Medal. The whole event went very successfully with the participation of mathematicians and physicists from many countries, with the large number of Polish participants, as a testimony to his great influence on researchers in Poland. The conference brought together researchers from several different areas of contemporary research, including complex analysis, differential geometry, mathematical physics and algebraic geometry. The invited speakers consisted of Zbigniew Błocki, Huai-Dong Cao, Fabrizio Catanese, Piotr Chruściel, John H. Coates, Mihalis Dafermos, Gary W. Gibbons, Jürgen Jost, Albrecht O. Klemm, Sławomir Kołodziej, Nanchung Conan Leung, Mario Micallef, Duong Hong Phong, Mike Scherfner, Udo Simon, Mark A. Stern, Le Dung Trang, Mu-Tao Wang and Kang Zuo. An introduction to the widely recognized works of Shing-Tung Yau was provided by Friedrich Hirzebruch, and inspiring memories from the Yau Fields Medal Ceremony were recalled by Czesław Olech—the organizer of ICM Warszawa 1982. The scientific committee consisted of Zbigniew Błocki, Stanisław Janeczko, Jürgen Jost, Jun Li and Duong Hong Phong. The organizational tasks were undertaken by Stanisław Janeczko, Teresa Regińska and Janusz Grabowski. The papers in this volume are contributions which are considered suitable for publication, after a suitable refereeing process. They range from purely expository to technical papers, and as a whole represent a good survey of contemporary work in geometric analysis. They cover some of the most recent and important developments in geometry and theoretical physics today. Topics include Monge-Ampère equations, Kähler-Ricci flows, other fully non-linear elliptic and parabolic equations; canonical metrics in Kähler geometry; notions of quasi-local mass in general relativity, geometric properties of gauge theories; and new algebro-geometric and symplectic methods. The topics are all at the interface of several major branches of mathematics (geometry, analysis, and mathematical physics).

The contributions presented here are grouped into four Parts. The first part is dedicated to the complex Monge-Ampère equations and nonlinear partial differen-

tial equations. It consists of the papers by Z. Błocki, *On Geodesics in the Space of Kähler Metrics*, by U. Simon and R. Xu, *Geometric Modelling Techniques for the Solution of Certain Monge-Ampère Equations*, by O. Savin, *A Localization Property at the Boundary for Monge-Ampère Equation*, by S. Dinew and S. Kołodziej, *Pluripotential Estimates on Compact Hermitian Manifolds*, by D. H. Phong and J. Sturm, *On Pointwise Gradient Estimates for the Complex Monge-Ampère Equation*, and by L. Caffarelli, Y. Y. Li and L. Nirenberg, *Some Remarks on Singular Solutions of Nonlinear Elliptic Equations. II: Symmetry and Monotonicity via Moving Planes*.

The second part is devoted to canonical metrics in Kähler geometry. It includes the papers *Calabi-Yau Metrics on Kummer Surfaces as a Model Gluing Problem* by S. K. Donaldson, *The Kähler-Ricci Flow on Singular Calabi-Yau Varieties* by J. Song and Y. Yuan, and *The K-energy on Small Deformations of Constant Scalar Curvature Kähler Manifolds* by V. Tosatti.

The third part includes new results in general relativity and Yang-Mills theory and consists of the papers by D. X. Kong and K. F. Liu, *Time-periodic Solutions of the Einstein Field Equations*, by M. T. Wang, *Quantitative Properties of the new Quasilocal Mass*, by M. Scherfner, S. Weiss and S.-T. Yau, *A Review of the Chern Conjecture for Isoparametric Hypersurfaces in Spheres*, and by M. Stern, *Geometry of Stable Yang-Mills Connections*.

The last part includes algebraic and symplectic methods. It is represented by the papers by K. Chan, N. C. Leung, *Matrix Factorizations from SYZ Transformations*, by D. Joyce, *On Manifolds with Corners*, by G. Ishikawa and S. Janeczko, *Symplectic Invariants of Parametric Singularities*, by F. Catanese, *Irreducibility of the Space of Cyclic Covers of Algebraic Curves of Fixed Numerical Type and the Irreducible components of Sing* $(\overline{\mathfrak{M}_g})$, and by M. Amram, D. Garber, R. Shwartz and M. Teicher, *8-point — Regenerations and Applications*.

The editors would like to express their gratitude to the Stefan Banach International Mathematical Center for its support, to Jan Krzysztof Kowalski for his help in the preparation of the material, and to all those who have contributed in whatever ways to the successful event and final form of the book.

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Duong H. Phong
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April 20, 2011