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Geometry of special holonomy and related topics

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Preface

Riemannian geometry is a very rich subject in itself, having close relationships with many different branches of mathematics, and with the sciences in general. In physics, Einstein theory of general relativity taught us the importance of Einstein metrics. However, it is very difficult to find Einstein metrics on general manifolds.

The most important existence result on the existence of Einstein metrics on a compact Riemannian manifold is the theorem of the second editor which says that if a manifold is Kähler, with zero first Chern class, then it admits Kähler metrics with zero Ricci curvature. Such a manifold is called a *Calabi-Yau manifold*. The Kählerian condition means that there is a complex structure which is compatible with the Riemannian metric up to first order. This extra structure reduces the holonomy group of the Levi-Civita connection of the Riemannian metric from the orthogonal group $O\,(2n)$ to the unitary group $U\,(n)$. On a Calabi-Yau manifold, the holonomy group further reduces to the special unitary group $SU\,(n)$, due to the existence of a parallel holomorphic volume form. When the Riemannian manifolds has more parallel tensors, their holonomy groups will be even smaller, for instance $Sp\,(n)$, G_2 or Riemannian symmetric spaces. All possible Riemannian holonomy groups were classified by Berger in the 1950s. In the article "Geometric structures on Riemannian manifolds," included in this volume, the author describes various holonomy groups and their corresponding geometries. All of them can be described in uniform manners in terms of normed division algebras and orientability.

In the 1980s, physicists studying string theory found that our spacetime should be ten–dimensional. Besides the usual spacetime $\mathbb{R}^{3,1}$, the remaining dimensions are warped in a tiny Calabi-Yau threefold. Furthermore a physical duality in string theory can be translated into a duality between the complex geometry and the symplectic geometry on different Calabi-Yau manifolds. Kontsevich has proposed an interpretation of the concept of Mirror symmetry, which was based on super-conformal algebra, by linking the derived category of one Calabi-Yau manifold with the Fukaya category of the mirror Calabi-Yau manifold. Kontsevich gave a talk on this subject in 2010 at the Fifth International Congress of Chinese Mathematicians

(ICCM). His proposed correspondence is now called homological mirror symmetry.

Strominger, Yau, and Zaslow argued that physically mirror symmetry is a T-duality. Namely, that mirror Calabi-Yau manifolds ought to admit dual special Lagrangian torus fibrations, at least in the large complex structure limits. The base of any (singular) Lagrangian fibration has a natural (singular) affine structure. In the large complex structure limits, this affine structure is expected to break into simple pieces. For instance, a large complex structure limit of the quartic K3 surfaces is the degeneration to the union of coordinate hyperplanes in the projective three space. The corresponding limiting affine structure becomes four triangles forming the boundary of a tetrahedron. In the article "An invitation to toric degenerations," Gross and Siebert describe a canonical construction of degenerations into a union of toric varieties. These include large complex structure limits for Calabi-Yau complete intersections in toric varieties. More importantly, in their construction, the complex geometry of the family can be read off from the tropical datum in the limit, at least in principle. This is an important step into proving the SYZ mirror symmetry conjecture.

In string theory, the Calabi-Yau manifold should also be equipped with B-field. Roughly speaking, it is a harmonic two–form on the manifold. The B-field can be used to transformed the *generalized* geometry of the manifold. In generalized geometry, the tangent bundle T is replaced by the direct sum $T \oplus T^*$. That is to say, going from $GL(2n, \mathbb{R})$ to $O(2n, 2n)$ as $T \oplus T^*$ has a canonical split-definite inner product. Similarly, (linear) generalized complex structure is a $U(n, n)$ structure on $T \oplus T^*$. Amazingly this notion includes both the complex structures and the symplectic structures as special cases. Generalized geometry has also played important roles in mirror symmetry. In the article "Lectures on generalized geometry," Hitchin give a wonderful lecture on generalized geometry. He also gives a proof of Goto's existence theorem for generalized Kähler structures using deformation theory.

In string theory, Calabi-Yau manifolds of complex dimension three are special, as they are internal manifolds in our ten–dimensional spacetimes. Recall that in low dimensional topology, for real three dimension oriented manifolds, Casson invariants count the number of flat bundles over these manifolds. These invariants can be refined to define a homology theory, called the Chern-Simons Floer homology groups, such that Casson invariants are their Euler characteristics. Donaldson and Thomas, in an earlier paper "Gauge theory in higher dimensions," define a complex version for Casson invariants, called the *Donaldson-Thomas invariants*, which count the number of stable holomorphic bundles over Calabi-Yau threefolds. In the paper "Gauge theory in higher dimensions, II" within this volume, Donaldson and Segal explain how we should generalize the Chern-Simons Floer theory to the complex setting, namely a holomorphic vector bundle over the moduli space of Calabi-Yau threefolds whose rank is the Donaldson-Thomas invariants.

Their construction uses the G_2-geometry of real seven dimensional manifolds. The natural embedding $SU(3) \subset G_2$ also explains why Calabi-Yau manifolds of complex dimension three is of particular interest.

Joyce and Song have developed a complete theory for Donaldson-Thomas invariants for coherent sheaves on Calabi-Yau threefolds and studied their wall-crossing properties. In "Generalized Donaldson-Thomas invariants," Joyce has here summarized their important work.

Odd-dimensional analogs of Calabi-Yau manifolds are Sasaki-Einstein manifolds. Namely, a link of a Calabi-Yau metric cone is a Sasaki-Einstein manifolds. They also play important roles in string theory and duality in physics. In "Sasaki-Einstein manifolds," Sparks gives an exposition of the Sasaki-Einstein geometry and describes various constructions and obstructions of these metrics.

We have seen that special geometry is a very rich and fascinating subject. It has an intimate relationship with physics which benefits both subjects enormously.

The Editors

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