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Volume XX

Surveys in Geometric Analysis and Relativity

edited by

Hubert L. Bray · William P. Minicozzi II

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Volume Editors:

Hubert L. Bray, Duke University

William P. Minicozzi II, Johns Hopkins University

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to Richard Schoen
in honor of his sixtieth birthday

Preface

This volume of 23 survey articles is dedicated to Richard M. Schoen on the occasion of his 60th birthday in recognition of his many important contributions as a leading researcher in geometric analysis and general relativity. We also thank him for the equally important roles he has played as mentor, colleague, collaborator, and friend.

Rick Schoen was born on October 23, 1950 in Celina, Ohio. In 1972 he graduated *summa cum laude* from the University of Dayton and received an NSF Graduate Fellowship. In March 1977, Rick received his Ph.D. from Stanford University under the direction of Leon Simon and Shing-Tung Yau, and soon after received a Sloan Postdoctoral Fellowship. His early work was on minimal surfaces and harmonic maps. By the time that Rick received his Ph.D., he had already proven major results, including his 1975 curvature estimates paper with Simon and Yau.

In the late 1970's, Schoen and Yau developed new tools to study the topological implications of positive scalar curvature. This work grew out of their study of stable minimal surfaces, eventually leading to their proof of the positive mass theorem in 1979. All together, their work was impressive for the way it connected neighboring fields, first using analysis to understand geometry, and then using geometry to understand physics.

In the early 1980's, Rick published a number of fundamental papers on minimal surfaces and harmonic maps. His work on minimal surfaces includes an influential Bernstein theorem for stable minimal surfaces with Doris Fischer-Colbrie. Rick met his future wife Doris in Berkeley, where Doris received her Ph.D. in 1978. They have two children, Alan and Lucy, seen in the photographs in this book, both of whom graduated from Stanford.

Other works from the early 1980's include an extremely useful curvature estimate for stable surfaces, a uniqueness theorem for the catenoid, and a partial regularity theory for stable hypersurfaces in high dimensions with Leon Simon. In 1982, Rick and Karen Uhlenbeck proved the partial regularity of energy minimizing harmonic maps. In 1983, Rick was awarded the very prestigious MacArthur Prize Fellowship.

Rick is also very well known for his celebrated solution to the remaining cases of the Yamabe problem in 1984, this time using a theorem from physics, namely the positive mass theorem, to solve a famous problem in geometry. The resulting fundamental theorem in geometry, that every smooth Riemannian metric on a closed manifold admits a conformal metric of constant scalar curvature, had been

Preface

open since the 60's. This work was cited in 1989 when Rick received the Bocher prize of the American Mathematical Society. His work on scalar curvature at this time set the direction for the field for the next 25 years.

Rick was elected to the American Academy of Arts and Sciences in 1988 and the National Academy of Sciences in 1991. He has been a Fellow of the American Association for the Advancement of Science since 1995 and won a Guggenheim Fellowship in 1996.

Starting around 1990, Rick began two major programs. The first was to develop a theory of harmonic maps with singular targets, starting with a joint paper with Mikhail Gromov where they used harmonic maps to establish p -adic superrigidity for lattices in groups of rank one. In a series of papers, Rick and Nick Korevaar laid the foundations for a general theory of mappings to NPC spaces, established the basic existence and regularity results, and applied their theory to settle problems in a number of areas of mathematics. The second big program was a variational theory of Lagrangian submanifolds, including the existence and regularity theory, done in a series of papers with Jon Wolfson.

Over the last decade, Rick has continued to make major contributions to geometric analysis and general relativity. Among other results in general relativity, Rick has made fundamental contributions to the constraint equations (with Corvino and others) which dictate the range of possible initial conditions for a spacetime and proved theorems on the topology of higher dimensional black holes (with Galloway). In geometric analysis, he has several important results with Simon Brendle on Ricci flow, including the proof of the differentiable sphere theorem, as well as a compactness theorem for the Yamabe equation with Marcus Khuri and Fernando Marques.

Rick has written 2 books and roughly 80 papers and has solved an impressively wide variety of major problems and conjectures. He has supervised 35 students and counting, and he has hosted many postdocs. Even with his great success, Rick is still one of the hardest working people in mathematics, giving us all the distinct impression that he must love it. His impact on mathematics, both in terms of his ideas and the example he sets, continues to be tremendous.

We would like to thank all of the authors for their contributions, the publishers Lizhen Ji and Liping Wang for their help, as well as Jaigyoung Choe, Michael Eichmair, John Rawnsley, Peter Topping, and Doris Fischer-Colbrie for contributing photographs. We hope you enjoy reading the beautiful survey articles included in this volume as much as we have enjoyed helping to put it all together.

Hubert L. Bray and William P. Minicozzi II
April 20, 2011

Contents

On the Positive Mass, Penrose, and ZAS Inequalities in General Dimension	
<i>Hubert L. Bray</i>	1
1 Dedication	1
2 Introduction	2
3 A Trio of Inequalities	15
References	25
Recent Progress on the Yamabe Problem	
<i>Simon Brendle, Fernando C. Marques</i>	29
1 The Yamabe Problem	29
2 The Compactness Conjecture	31
3 Non-compactness Results in Dimension $n \geq 25$	34
4 A Compactness Result in Dimension $n \leq 24$	37
5 The Parabolic Yamabe Flow	40
References	45
Some Recent Progress on Mean Curvature Flow for Entire Lagrangian Graphs	
<i>Jingyi Chen</i>	49
1 Introduction	49
2 Longtime Existence With Lipschitz Continuous Initial Data	50
3 Uniqueness and Viscosity Solutions.....	52
4 Self-similar Solutions.....	53
References	57
Radial Viewpoint on Minimal Surfaces	
<i>Jaigyoung Choe</i>	59
1 Introduction	59
2 Cone.....	60
3 Horizon	62
4 Non-Euclidean Space.....	63
5 Ray preserving Metric.....	65
6 Varying Curvature	67
7 Embeddedness	70

References	72
Minimal Surfaces and Mean Curvature Flow	
<i>Tobias H. Colding, William P. Minicozzi II</i>	73
1 Introduction	73
2 Harmonic Functions and the Heat Equation	74
3 Energy of a Curve	78
4 Birkhoff: A Closed Geodesic on a Two Sphere	80
5 Curve Shortening Flow	84
6 Minimal Surfaces	88
7 Classification of Embedded Minimal Surfaces	96
8 Mean Curvature Flow	112
9 Width and mean curvature flow	117
10 Singularities for MCF	119
11 Smooth Compactness Theorem for Self-shrinkers	124
12 The Entropy	126
13 An Application	131
14 Non-compact self-shrinkers	132
References	135
Scalar Curvature and the Einstein Constraint Equations	
<i>Justin Corvino, Daniel Pollack</i>	145
1 Introduction	145
2 The Constraint Equations	147
3 A Tour of Asymptotically Flat Solutions	149
4 The Conformal Method	167
5 Gluing Constructions	172
References	182
On the Intrinsic Differentiability Theorem of Gromov-Schoen	
<i>Georgios Daskalopoulos, Chikako Mese</i>	189
1 Introduction	189
2 Definitions	190
3 Main Theorem	192
References	203
Minimal Surface Techniques in Riemannian Geometry	
<i>Ailana Fraser</i>	205
1 Introduction	205
2 Brief Overview of Some Geodesic Methods	206
3 Existence of Minimal Surfaces	208
4 Second Variation Theory for Minimal Surfaces and Applications ..	211

References.....	217
Stability and Rigidity of Extremal Surfaces in Riemannian Geometry and General Relativity	
<i>Gregory J. Galloway</i>	221
1 Minimal Hypersurfaces in Manifolds of Nonnegative Scalar Curvature.....	221
2 Marginally Outer Trapped Surfaces.....	226
3 Positivity of Mass for Asymptotically Hyperbolic Manifolds.....	232
References.....	237
Convex Hypersurfaces of Constant Curvature in Hyperbolic Space	
<i>Bo Guan, Joel Spruck</i>	241
1 Introduction.....	241
2 Formulas on Hypersurfaces.....	246
3 The Asymptotic Angle Maximum Principle and Gradient Estimates.....	250
4 Curvature Estimates.....	251
5 Uniqueness and Foliations.....	254
References.....	257
Ricci Flow in Two Dimensions	
<i>James Isenberg, Rafe Mazzeo, Natasa Sesum</i>	259
1 Introduction.....	259
2 General Considerations.....	261
3 Compact Surfaces.....	262
4 Open Surfaces.....	267
5 Flows on Incomplete Surfaces.....	274
References.....	278
Doubling and Desingularization Constructions for Minimal Surfaces	
<i>Nikolaos Kapouleas</i>	281
1 Introduction.....	281
2 Doubling Constructions.....	288
3 Desingularization Constructions.....	296
4 Minimal Surfaces in the Round Three-Sphere.....	299
5 The Building Blocks for the Desingularization Construction.....	304
6 An Initial Surface for the Desingularization Construction.....	309
7 The Family of Initial Surfaces for the Desingularization Construction.....	312
8 Main Estimates and Outline of the Proof.....	317
References.....	322

The Metric Properties of Lagrangians	
<i>Yng-Ing Lee</i>	327
1 Introduction	327
2 A Short Survey	328
3 Definitions and Properties	332
4 Singularities and Geometric Measure Theory	334
5 Gluing and Singular Perturbation	336
References	338
Structure of Complete Manifolds with Positive Spectrum	
<i>Peter Li</i>	343
1 Introduction	343
2 Riemannian Case	344
3 Kähler Case	348
4 Quaternionic Kähler Manifolds, Cayley Manifolds, and Locally Symmetric Spaces	351
5 Manifolds of Finite Volume	354
6 Further Generalizations	356
References	360
Topology of Sobolev Mappings and Associated Variational Problems	
<i>Fang Hua Lin</i>	363
Introduction	363
1 Analytical and Topological Properties of Sobolev Maps	364
2 Singularity of Energy Minimizing Maps	373
3 Limits of Singular Sets of p -Energy Minimizing Maps	380
References	391
A Survey of Research on Boundary Behavior of Compact Manifolds via the Positive Mass Theorem	
<i>Pengzi Miao</i>	395
1 Introduction	395
2 Statement of the Positive Mass Theorem	395
3 On compact Manifolds with Nonnegative Scalar Curvature	397
4 On Compact Manifolds with Negative Scalar Curvature	406
References	410
Recent Progress on Singularities of Lagrangian Mean Curvature Flow	
<i>André Neves</i>	413
1 Introduction	413
2 Preliminaries	415

3	Basic Techniques.....	416
4	Applications I: Blow-ups.....	424
5	Applications II: Self-Expanders.....	428
6	Application III: Stability of Singularities.....	430
7	Open Questions.....	434
	References.....	436
Geometric Structures of Collapsing Riemannian Manifolds I		
	<i>Aaron Naber, Gang Tian</i>	439
1	Introduction.....	439
2	Structure of Collapsed Spaces.....	443
3	Geometry of Toric Quotients.....	448
4	Geometry of Toric Quotients II.....	454
5	Proof of Theorems 1.1 and 1.2.....	458
6	Proof of Theorem 1.3.....	457
A	Geometry of Quotients.....	457
B	Orbifolds.....	460
	References.....	465
Deformation of Kähler-Einstein Metrics		
	<i>Xiaofeng Sun, Shing-Tung Yau</i>	467
1	Introduction.....	467
2	Complex Structures of Kähler-Einstein Manifolds.....	468
3	Deformation of Kähler-Einstein Metrics.....	473
4	Local Trivialization of Polarization Bundles and Deformation of Sections.....	476
5	Curvature of L^2 Metrics on Direct Image Sheaves.....	483
6	Appendix.....	486
	References.....	489
Reverse Bubbling in Geometric Flows		
	<i>Peter M. Topping</i>	491
1	Introduction.....	491
2	The Harmonic map Flow.....	495
3	Ricci Flow.....	499
4	Addendum — Mean Curvature Flow.....	505
	References.....	506
Review on Harmonic Diffeomorphisms Between Complete Noncompact Surfaces		
	<i>Tom Y. H. Wan</i>	509
1	Introduction.....	509
2	Harmonic Map Theory of Universal Teichmüller Space.....	510

3	Asymptotic Behavior of Open Harmonic Embedding From the Complex Plane Into Hyperbolic Plane.....	512
	References.....	515
Compactifications of Complete Riemannian Manifolds and Their Applications		
	<i>Xiaodong Wang</i>	517
1	Introduction.....	517
2	The Geometric Compactification.....	518
3	The Martin Compactification.....	519
4	The Busemann Boundary.....	520
5	A Comparison Theorem.....	524
	References.....	528
Some Aspects of Weil-Petersson Geometry of Teichmüller Spaces		
	<i>Sumio Yamada</i>	531
1	Introduction.....	531
2	Harmonic Maps into $\overline{\mathcal{T}}$ and an Application.....	533
3	Finite Rank Properties of $\overline{\mathcal{T}}$	537
4	Coxeter-Tits Construction.....	539
5	Weil-Petersson Geodesic Completeness.....	541
6	Weil-Petersson Geometry of the Universal Teichmüller Space.....	541
7	Embeddings of the Coxeter Complex into \mathcal{UT}	543
8	Summary and Open Problems.....	544
	References.....	545