

Calculus

A computer algebra approach

Second Edition

Calculus

A computer algebra approach

Second Edition

by Iris Anshel and Dorian Goldfeld
Columbia University, New York



International Press

www.intlpress.com

Calculus: A computer algebra approach, 2nd edition
by Iris Anshel and Dorian Goldfeld (Columbia University, New York)

Copyright © 1996, 2011 by International Press
Somerville, Massachusetts, U.S.A.

All rights reserved. Individual readers of this publication, and non-profit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgement of the source is given. Republication, systematic copying, or mass reproduction of any material in this publication is permitted only under license from International Press.

ISBN 13: 978-1-57146-222-0

ISBN 10: 1-57146-222-8

Paperback re-issue 2011

This title was previously published in 1996 under ISBN 1-57146-038-1 (casebound).

Printed in the United States of America.

DEDICATION

The authors would like to dedicate this book to their children, Ada and Dahlia, who played quietly while their parents worked on the manuscript.

Acknowledgement

The authors would like to thank Professor Hervé Jacquet, of the Columbia University Mathematics Department, for his motivating idea that we initiate calculus reform at Columbia and for suggesting several elegant and succinct proofs. Professors, Hervé Jacquet, Duong H. Phong, Ramuald Dabrowsky, Patrick X. Gallagher and Cormac O'Sullivan provided indispensable editorial assistance during the preparation of the manuscript, and we thank International Press for making the text available to students at very low cost. The additional exercise appearing at the end of each chapter were contributed by Maia Berkane and Kevin Oden, the authors wish to express their gratitude their appreciation for their efforts. We are deeply grateful to Professor Morton Friedman, of the Columbia University School of Engineering, for creating a computer laboratory which has allowed us to present an experimental course based on this material. His constant support and encouragement, as director of the Gateway Project, have been invaluable: without his efforts this project would still be on the drawing board. Finally, we wish to thank the Gateway Project for financial support.

Contents

0 What is Calculus?	10
0.1 The Real Numbers	10
0.2 What is Calculus?	11
I Functions and their Graphs	15
1.1 Functions	15
1.2 The Domain and Range of a Function	18
1.3 The Graph of a Function.	21
1.4 The Trigonometric and Exponential Functions	24
II The Algebra of Functions	31
2.1 An Informal Introduction to Algebras	31
2.2 The Algebra of Functions	32
2.3 The Identity and the Inverse Function	35
III Lines, Circles, and Curves: a Review	41
3.1 Lines	41
3.2 Circles	47
IV Limits and Continuity	53
4.1 Limit of a Sequence	53
4.2 The Limit of a Function	54
4.3 Continuous Functions	58
4.4 The Algebra of Limits	62
V The Derivative	66
5.1 Tangent Lines	66
5.2 The Derivative of a Function	69
5.3 Computing the Derivative using Limits	70
5.4 Finding the Equation of the Tangent Line	74
5.5 Higher Derivatives	76
VI Basic Applications of the Derivative	81
6.1 Velocity	81
6.2 Newton's Method	85
VII The Rules of Calculus	89
7.1 The Primary Rules	89
7.2 The Product and the Quotient Rules	91

7.3 The Chain Rule	94
7.4 Derivatives of Trigonometric Functions	97
VIII Implicitly Defined Functions and their Derivatives	102
8.1 Implicit Functions	102
8.2 Implicit Differentiation	103
8.3 The Exponential, the Natural Logarithm, and the Hyperbolic Functions	105
8.4 The Derivative of the Inverse Function	111
IX The Maxima and Minima of Functions	115
9.1 Maximum and Minimum Values	115
9.2 The First Derivative Test	121
9.3 The Second Derivative Test	128
X Classical Optimization Theory	134
10.1 A Three Step Method for Finding Maxima and Minima	134
10.2 Mathematical Modeling	136
10.3 Surface Area and Volume Problems	141
10.4 A Simple Mathematical Model in Economics	146
XI Graphing Functions	152
11.1 Graphing with the First and Second Derivative Test	152
11.2 Graphing with Cusps	155
11.3 Concavity	157
XII Asymptotes	164
12.1 Generalities on Asymptotes	164
12.2 Vertical Asymptotes	166
12.3 Horizontal Asymptotes	170
XIII The Integral as Area	178
13.1 Intuitive Definition of the Integral as an Area	178
13.2 The Integral of an Arbitrary Function	182
13.3 The Integral as a Limit of a Sum	184
13.4 Properties of Integrals	193
XIV Sums, Induction, and Computation of Integrals	198
14.1 Sums	198
14.2 Induction	200
14.3 Computation of Integrals	204
14.4 Approximate Computation of Integrals	206

XV The Integral as an Antiderivative	213
15.1 The Fundamental Theorem of Calculus	213
15.2 The Indefinite Integral	218
15.3 Integration by the Method of Substitution	219
15.4 Integration by Parts	224
15.5 Basics on Differential Equations	229
15.6 Exponential Growth and Decay	232
XVI Basic Applications of the Integral	239
16.1 The Average Value of a Function	239
16.2 Computing Area	241
16.3 Computing Arc Length	248
16.4 Volume as Summation of Cross-Sectional Area	252
16.5 Volumes of Solids of Revolution	257
XVII Further Topics on Integration	266
17.1 Integral Representation of the log Function	266
17.2 Integral Representation of Inverse Trigonometric Functions	269
17.3 Integrating Rational Functions	272
17.4 Further Substitutions	277
17.5 Improper Integrals	282
XVIII Infinite Series	294
18.1 Geometric Series	294
18.2 General Infinite Series	298
18.3 The Integral Test	300
18.4 Other Tests for Convergence	303
18.5 Infinite Series with Positive and Negative Terms	310
18.6 Power Series	314
XIX Taylor Series	322
19.1 The Tangent Line Approximation	322
19.2 Approximation of Functions by Taylor Polynomials	324
19.3 Maclaurin Series	327
19.4 Binomial Series	330
19.5 Estimates of Errors in the Taylor Approximation of Functions	333
19.6 The General Taylor Expansion	337
19.7 Complex Taylor Series and Euler's Formula	339
19.8 L'Hospital's Rule	342
19.9 Solving Differential Equations with Taylor Series	344

XX Vectors in Two and Three Dimensions	351
20.1 Introduction to Vectors	351
20.2 The Algebra of Vectors	355
20.3 Basis Vectors in Two and Three Dimensions	359
20.4 The Dot Product	360
20.5 The Cross Product	366
20.6 Some Basic Properties of the Cross Product	370
20.7 Applications of the Cross Product	373
XXI Two and Three Dimensional Graphics	380
21.1 Lines in Space	380
21.2 Planes: Their Equations and Properties	383
21.3 Space Curves	389
21.4 Polar and Cylindrical Coordinates	393
21.5 Converting Polar Functions to Vector Valued Functions	399
XXII Calculus of Vector Valued Functions	406
22.1 Derivatives of Vector Valued Functions	406
22.2 Integration and Arclength	411
22.3 Arclength and Area in Polar Coordinates	415
22.4 Direction and Curvature	419
22.5 Velocity and Acceleration	425
XXIII Functions of Several Variables	431
23.1 Functions of Several Variables	431
23.2 Graphical Display	433
23.3 Partial Derivatives and the Gradient	437
23.4 The Total Derivative	442
23.5 The Chain Rule	446
23.6 Tangent Planes	450
XXIV Multidimensional Optimization	455
24.1 The Method of Steepest Descent	455
24.2 The Method of Critical Points	459
24.3 Taylor Series and the Classification of Critical Points	466
24.4 The Method of Lagrange Multipliers	471
XXV Double Integrals	480
25.1 Review of One Variable Integration	480
25.2 Double Integrals	481
25.3 Evaluation of Double Integrals	487
25.4 Double Integrals in Polar Coordinates	494
25.5 Evaluation of Double Integrals in Polar Coordinates	499

25.6 Computing Areas and Volumes with Double Integrals	502
25.7 Changing Variables in Double Integrals	505
XXVI Triple Integrals	514
26.1 Triple Integrals and the Fourth Dimension	514
26.2 Evaluation of Triple Integrals	519
26.3 Changing Coordinates in Triple Integrals	524
26.4 Cylindrical and Spherical Coordinates	527
XXVII Vector Fields and Line Integrals	535
27.1 Vector Fields	535
27.2 Line Integrals	546
27.3 Independence of Path	551
27.4 Green's Theorem in the Plane	559
XXVIII Surface Integrals	568
28.1 Surface Integrals	568
28.2 Surface Integrals for Open Surfaces	573
28.3 Surface Integrals for Closed Surfaces	582
28.4 The Divergence Theorem	587
28.5 Curl and Spin	590
28.6 Stoke's Theorem	595
XXIX Differential Forms: An Overview	602
29.1 Differential Forms and the Wedge Product	602
29.2 The d -Operator	607
29.3 The Generalized Stoke's Theorem	609
XXX Fourier Series	614
30.1 Periodic Functions	614
30.2 Fourier Expansions of Periodic Functions	618
30.3 Examples	622
XXXI APPENDIX	631
Symbols	636
Index	637

INTRODUCTION

The evolution of the teaching of calculus is at a critical juncture. For some time there has been an emphasis on the computational aspects of calculus in conjunction with the various applications of the method. It is the advent of highly accessible computer algebra systems (CAS) and various sophisticated calculators which has driven us to reevaluate how calculus should be presented. The use of this book does *not* require a computer laboratory, or even access to an elaborate CAS. A graphing calculator suffices in that the emphasis is not on mass computation or programming, but rather on the understanding of the underlying concepts.

A CAS is a computing device with the following capabilities:

- (1) *It is a calculator*, i.e., it can perform arithmetic and compute values of standard functions,
- (2) *It has 2–dimensional graphics capabilities*, i.e., it can produce graphic displays of functions of a single variable,
- (3) *It can do calculus*, i.e., it can compute derivatives and integrals of functions,
- (4) *It can do algebra*, i.e., it can expand and simplify algebraic expressions,
- (5) *It can create functions*,
- (6) *It has 3–dimensional graphics capabilities*, i.e., it can produce graphic displays of functions of several variables.

In light of the immense capabilities of the CAS we are led us to asking the following questions: (1) what approach should we take to this discipline now that the mechanics of computation have been automated, and (2) how can we incorporate CAS with the classical teaching methods?

It is the authors opinion that the new technology is a moment of opportunity. No one would consider learning, for example, Physics or Biology without working in a laboratory; computation for anyone without a CAS is very limited. Calculus is both an intellectual breakthrough and a powerful tool in research and development (in a variety of fields). To understand it as such (and to apply it as a problem solving tool in modern settings) one must obtain both an understanding of the vital concepts in their abstract form and in a wide range of examples. Our philosophy is to focus on the meaning of the most important definitions and theorems and to *experiment* with them on the CAS.

It should be noted that this text is not a “how–to” manual for a particular CAS. Our aim is to teach the concepts of calculus without getting lost in the quagmire of

programming. Almost all the exercises and examples can be done at the level of using the CAS as a calculator. Beyond this, even on the most sophisticated CAS, only a few commands need be learned.

Many students are familiar with introductory calculus upon entering college. With this in mind, we have opted to quickly review the foundational material (such as real numbers, functions and their graphs, lines, and circles). Although the introductory material is covered in relatively little space, almost nothing is assumed on the part of the reader (that is to say, the text is substantially self-contained). When the material becomes less familiar and more complex the sections and chapters become more substantial. Throughout the text we focus on core concepts which are presented from first principles. Once a given concept is introduced we utilize the CAS to rapidly view it in many different contexts. This approach encourages the student to develop *hands-on* experience with Calculus. We have found that such a CAS experience is superior to reading hundreds of worked out examples in a book. Evidence of the benefits of this method were apparent when, in 1993–94, an experimental computer laboratory was set up at Columbia University (in conjunction with a course based on this book).

It is the authors hope that this book is intuitively rigorous and indicates how mathematics is thought about. There is little purpose to memorizing massive numbers of formulae and algorithms (all of which the CAS knows). It is a natural outcome of this pedagogical perspective that solutions of problems are presented analytically, and whenever possible and appropriate, algorithmically. The emphasis is consistently on how to derive the formulae, why the algorithms work, and how the CAS can solve Calculus problems.

This edition begins by covering both differential and integral calculus for functions of one variable, mathematical modeling and optimization, basics of ordinary differential equations, and then moves on to differential calculus for vector valued functions and functions of several variables. Much time is spent on vector geometry, coordinate systems, and two and three dimensional graphical display using the CAS. The latter part of the text includes multiple integration, vector fields and line integrals, surface integrals Stoke's theorem, an overview via differential forms, as well as an introduction to Fourier series including a proof of the Fourier expansion theorem.

This book assumes that students have access to a basic CAS with capabilities (1), (2), and (3) above. Among the many exercises the majority can be solved either by hand or with a basic CAS.