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Advanced Lectures in Mathematics
Volume XVII

Geometry and Analysis No. 1

Editor: Lizhen Ji

International Press
www.intlpress.com

高 等 教 育 出 版 社
HIGHER EDUCATION PRESS

Advanced Lectures in Mathematics, Volume XVII
Geometry and Analysis, No. 1

Volume Editor: Lizhen Ji, University of Michigan, Ann Arbor

2010 Mathematics Subject Classification. 58-06.

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ISBN: 978-1-57146-224-4

Printed in the United States of America.

15 14 13 12 11 1 2 3 4 5 6 7 8 9

ADVANCED LECTURES IN MATHEMATICS

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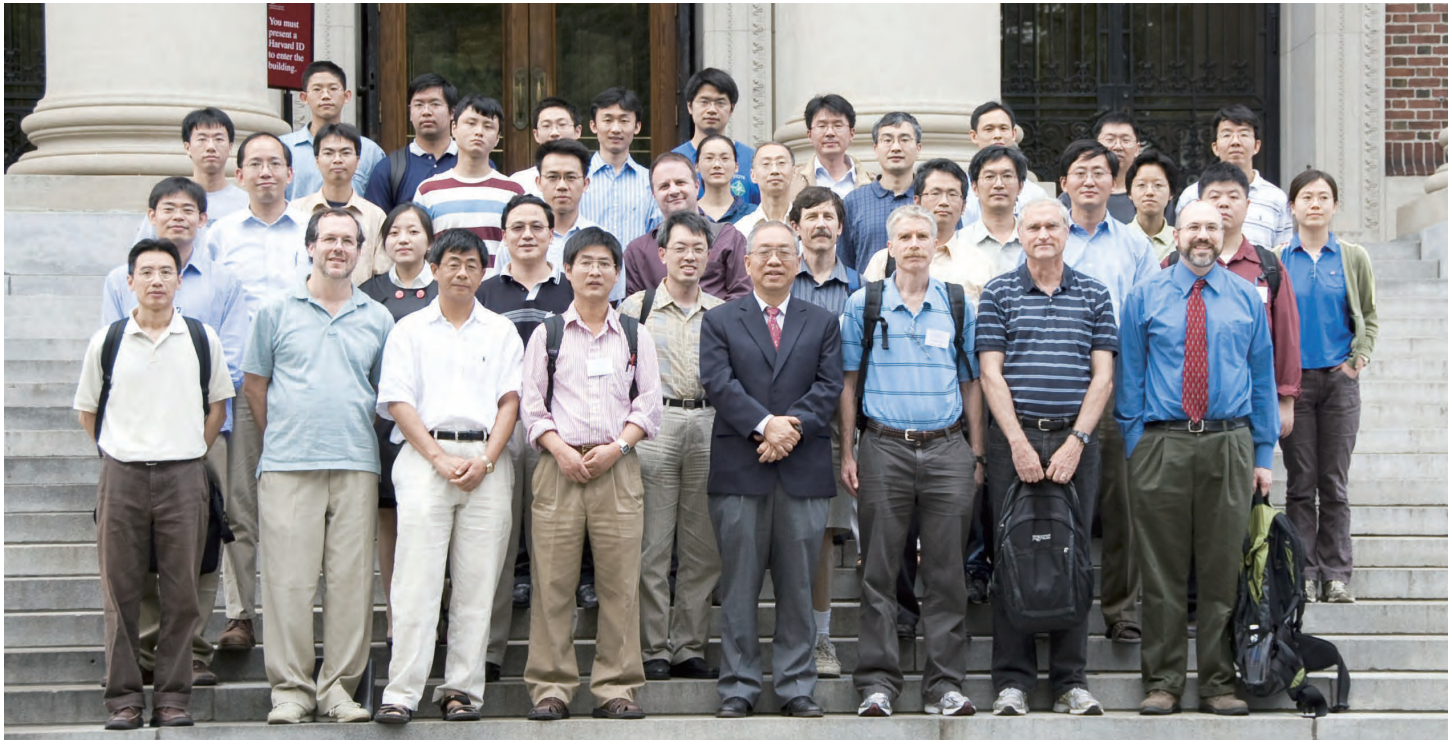
to Shing-Tung Yau
in honor of his sixtieth birthday

Group photo of the conference “ Geometric Analysis: Present and Future 2008”



Shing-Tung Yau and his students

Fields Medal 1982

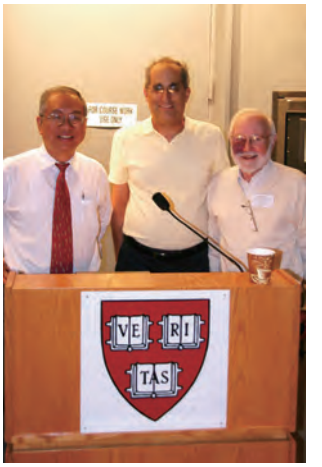
Crafoord Prize 1994

Wolf Prize 2010

Shing-Tung Yau and Dennis Sullivan

Shing-Tung Yau and Shimon Peres (the president of Israel)

Shing-Tung Yau and Eugenio Calabi



Shing-Tung Yau, Edward Witten and Isadore Singer

Richard Hamilton, Shing-Tung Yau and his wife

Shing-Tung Yau and Tristan Hubsch

Preface

To celebrate the 60th birthday of Professor Shing-Tung Yau, a conference titled *Geometric Analysis: Present and Future* was held at Harvard University from August 27 to September 1, 2008.

The purpose of this conference is to summarize what has been achieved in the fields around geometric analysis in the past, to discuss the most recent results and to map out directions for the future. The title *Geometric Analysis* was interpreted very broadly and reflected the wide range of interests of Yau. It was also reflected in the topics of many talks at the conference.

There were 47 distinguished speakers and they are:

1. Robert Bartnik, Monash University, Australia.
2. Robert Bryant, Mathematical Sciences Research Institute, Berkeley, USA.
3. Jean-Pierre Bourguignon, Insttlut Hauites Etudes Scientifiques, France.
4. Luis Caffarelli, University of Texas, Austin, USA.
5. Demetrios Christodoulou, ETH, Switzerland.
6. Fan Chung, University of California, San Diego, USA.
7. David Gieseker, University of California, Los Angeles, USA.
8. Brian Greene, Columbia University, USA.
9. Pengfei Guan, McGill University, Canada.
10. Victor Guillemin, Massachusetts Institute of Technology, USA.
11. Richard Hamilton, Columbia University, USA.
12. Nigel Hitchin, Oxford University, UK.
13. Jiaxing Hong, Fudan University, China.
14. Tristan Hubsch, Howard University, USA.
15. Gerhard Huisken, Albert-Einstein-Institut, Germany.
16. Blaine Lawson, State University of New York at Stony Brook, USA.
17. Jun Li, Stanford University, USA.
18. Peter Li, University of California, Irvine, USA.
19. Bong Lian, Brandeis University, USA.
20. Chang-Shou Lin, Taiwan University, China.

21. Fang-Hua Lin, New York University, USA.
22. Kefeng Liu, University of California, Los Angeles, USA.
23. Melissa Liu, Columbia University, USA.
24. Gregory Margulis, Yale University, USA.
25. Alina Marian, University of Illinois at Chicago, USA.
26. Williams Meeks, University of Massachusetts Amherst, USA.
27. Yoichi Miyaoka, University of Tokyo, Japan.
28. Duong Phong, Columbia University, USA.
29. Wilfried Schmid, Harvard University, USA.
30. Richard Schoen, Stanford University, USA.
31. Leon Simon, Stanford University, USA.
32. Isadore Singer, Massachusetts Institute of Technology, USA.
33. Yum-Tong Siu, Harvard University, USA.
34. Joel Smoller, University of Michigan, USA.
35. James Sparks, Oxford University, UK.
36. Andrew Strominger, Harvard University, USA.
37. Cliff Taubes, Harvard University, USA.
38. Chuu-Lian Terng, University of California, Irvine, USA.
39. Karen Uhlenbeck, University of Texas, Austin, USA.
40. Cumrun Vafa, Harvard University, USA.
41. Chin-Lung Wang, Taiwan University, China.
42. Xujia Wang, Australian National University, Australia.
43. Ben Weinkove, University of California, San Diego, USA.
44. Edward Witten, Institute for Advanced Study, Princeton, USA.
45. Zhouping Xin, The Chinese University of Hong Kong, China. USA.
46. Eric Zaslow, Northwestern University, USA.
47. Xi-Ping Zhu, Sun Yat-Sen University, China.

There was also an open problem session in the evening of August 27, and a lunch forum *Women in Mathematics* on August 28. The conference banquet was held at the American Academy of Arts and Sciences in the evening of August 29.

The conference was a great success. We would like to thank all the speakers for their excellent talks and active participation in the conference. We would also like to take this opportunity to thank the following sponsors of this conference:

1. The National Science Foundation, USA.
2. The Chinese University of Hong Kong, China.
3. Department of Mathematics, The Chinese University of Hong Kong, China.

4. Department of Mathematics, Harvard University, USA.
5. Morningside Center of Mathematics, Chinese Academy of Sciences, China.
6. Journal of Differential Geometry, Lehigh University, USA.
7. Pacific Institute for the Mathematical Sciences.
8. Department of Mathematics, Massachusetts Institute of Technology, USA.
9. Department of Mathematics, Brandeis University.
10. Pure and Applied Mathematics Quarterly, USA.
11. Center of Mathematical Sciences, Zhejiang University, China.
12. Peter Viem Kwok, CITIC Resources Holdings Limited, Hong Kong, China.

The organizers of this conference consisted of the following:

1. Lizhen Ji, University of Michigan, USA.
2. Ka-Sing Lau, The Chinese University of Hong Kong, China.
3. Peter Li, University of California, Irvine, USA.
4. Kefeng Liu, University of California, Los Angeles, USA.
5. Wilfried Schmid, Harvard University, USA.
6. Rick Schoen, Stanford University, USA.
7. I.M. Singer, Massachusetts Institute of Technology, USA.
8. Clifford Taubes, Harvard University, USA.
9. Cumrun Vafa, Harvard University, USA.
10. Zhouping Xin, The Chinese University of Hong Kong, China.

Certainly, on behalf of the organizers, we would also like to thank the staff members of the mathematics department, Harvard University, for their generous help with this big and intense conference.

The current set of two volumes *Geometry and Analysis* is the proceedings of the conference and mainly consists of contributions of the speakers. Together with the following three volumes of *Handbook of Geometric Analysis* prepared in conjunction with the conference:

1. *Handbook of geometric analysis. Volume 1.* Edited by Lizhen Ji, Peter Li, Richard Schoen and Leon Simon. Advanced Lectures in Mathematics (ALM), 7. International Press, Somerville, MA; Higher Education Press, Beijing, 2008. xii+676 pp.
2. *Handbook of geometric analysis. Volume 2.* Edited by Lizhen Ji, Peter Li, Richard Schoen and Leon Simon. Advanced Lectures in Mathematics (ALM), 13. International Press, Somerville, MA; Higher Education Press, Beijing, 2010. x+432 pp.
3. *Handbook of geometric analysis. Volume 3.* Edited by Lizhen Ji, Peter Li, Richard Schoen and Leon Simon. Advanced Lectures in Mathematics (ALM), 14. International Press, Somerville, MA; Higher Education Press, Beijing, 2010. x+472 pp.

We hope that they give a global perspective on the current status of geometric analysis and will be helpful in training the next generation of mathematicians. We would like to thank the authors of all these papers for their substantial contributions, and the referees for their generous help in the reviewing process.

These two volumes of *Geometry and Analysis* consist of 4 parts:

1. Summaries and commentaries on the work of Yau, which contains 20 short articles commenting on different aspects of Yau's work, lists of his papers and books, and a recent CV of him.
2. Differential geometry and differential equations, which contains 14 papers.
3. Mathematical physics, algebraic geometry and other topics, which also contains 14 papers.
4. Appendices, which contains a biography of Yau, and two survey papers by Yau on geometric analysis and Calabi-Yau manifolds.

Though geometric analysis has a long history, the decisive contributions of Yau since 1970s have made it an indispensable tool in many subjects such as differential geometry, topology, algebraic geometry, mathematical physics, etc, and hence have established it as one of the most important fields of modern mathematics. Yau's impacts are clearly visible in the papers of these two volumes, and we hope that these two volumes of *Geometry and Analysis* and the three volumes of the *Handbook of Geometric Analysis* will pay a proper tribute to him in a modest way.

According to the Chinese tradition, a person is one year old when he is born, and hence Yau turned 60 already in 2008. The number 60 and hence the age 60 is special in many cultures, especially in the Chinese culture. It is the smallest common multiple of 10 and 12, two important periods in the Chinese astronomy. Therefore, it is a new starting point (or a new cycle). A quick look at Yau's list of publications in Part 1 shows that Yau has not only maintained but increased his incredible output both in terms of quality and quantity.

On behalf of his friends, students and colleagues, we wish him continuing success and many productive, energetic years to come!

Lizhen Ji
March 2010

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