

# The Iwasawa theory of totally real fields

## Ramanujan Mathematical Society Lectures Notes Series

- Vol. 1: Number theory
- Vol. 2: The Riemann zeta function and related themes
- Vol. 3: Formal language aspects of natural computing
- Vol. 4: Commutative algebra and combinatorics
- Vol. 5: Convexity in discrete structures
- Vol. 6: Number theory and discrete geometry
- Vol. 7: Discrete mathematics
- Vol. 8: Lectures on operator theory
- Vol. 9: Essays on geometric group theory
- Vol. 10: Techmüller theory and moduli problems
- Vol. 11: Perspectives in geometry and topology
- Vol. 12: The Iwasawa theory of totally real fields
- Vol. 13: Advances in discrete mathematics and applications

*Ramanujan Mathematical Society*

Lecture Notes Series

Volume 12

# The Iwasawa theory of totally real fields

From the workshop on the Iwasawa theory  
of totally real fields, held at the Indian  
Institute of Technology, Guwahati,  
September 2008

*Volume editors*

J. Coates  
C. S. Dalawat  
A. Saikia  
R. Sujatha

 International Press  
[www.intlpress.com](http://www.intlpress.com)

Ramanujan Mathematical Society  
Lecture Notes Series, Volume 12  
The Iwasawa theory of totally real fields

Volume editors:

J. Coates  
C. S. Dalawat  
A. Saikia  
R. Sujatha

Copyright © 2010 by the Ramanujan Mathematical Society, Mysore, India.

Published in 2011 by International Press, Somerville, Massachusetts, U.S.A. under license from the Ramanujan Mathematical Society.

This work is also published in the Republic of India, exclusively by the Ramanujan Mathematical Society.

All rights reserved. Individual readers of this publication, and non-profit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgement of the source is given. Reproduction, systematic copying, or mass reproduction of any material in this publication is permitted only under license from the Ramanujan Mathematical Society. Excluded from these provisions is material in articles to which the author holds the copyright. (If the author holds copyright, notice of this will be given with article.) In such cases, requests for permission to use or reprint should be addressed directly to the author.

ISBN: 978-1-57146-219-0

Printed in the United States of America.

15 14 13 12 11      1 2 3 4 5 6 7 8 9

# Foreword

A Workshop was held at the Indian Institute of Technology, Guwahati, from September 22 until September 30, 2008, with the goal of presenting as much as possible of the background and underlying ideas behind Wiles' celebrated proof of the "main conjecture" of cyclotomic Iwasawa theory over totally real base fields. This workshop was part of a series of schools and workshops organized in various parts of India each year, called "Advanced Training in Mathematics Schools", which are generously supported by the National Board for Higher Mathematics (NBHM). The Organizing Committee consisted of J. Coates, C. S. Dalawat, A. Saikia, and R. Sujatha, and lectures during the course of the Workshop were given by U. K. Anandavardhanan, D. Banerjee, J. Coates, C. S. Dalawat, Narasimha Kumar, E. Ghate, F. A. E. Nuccio, A. Saikia, T. Schmidt, A. C. Sharma, R. Sujatha and O. Venjakob.

We owe to Iwasawa the discovery of the first "main conjecture" in his revolutionary work on cyclotomic fields in the 1960's. He only considered this "main conjecture" for the Tate motive over the field obtained by adjoining all  $p$ -power roots of unity to the rational field  $\mathbb{Q}$ , where  $p$  is any prime number. However, it was quickly realized that similar ideas could be applied to elliptic curves, and provided a systematic framework for studying the conjecture of Birch and Swinnerton-Dyer, and other conjectural exact formulae in number theory. Today, we believe, although only fragmentary proofs are known so far, that these "main conjectures" hold in vast generality for all motives over  $p$ -adic Lie extensions of number fields. The great interest of the method discovered by Wiles for proving the "main conjecture" for Tate motives over any totally real number field is that it combines ideas from the theory of automorphic forms with purely arithmetic ideas in a way that suggests the beginning of some deeper general connections between these two vast citadels of modern number theory.

These notes present written versions of most of the material covered by the lectures in the Workshop, and it is hoped that they will be useful to others who wish to study these important questions. They are aimed very much at the beginner in the subject, and try to minimize wherever possible the background knowledge required.

In conclusion, we wish to warmly thank IIT Guwahati for providing such excellent facilities and such a delightful location for holding the Workshop.

**J. Coates**  
**C. S. Dalawat**  
**A. Saikia**  
**R. Sujatha**

# Members of the Advisory Board

R. Balasubramanian (IMSc, Chennai, India)  
R. B. Bapat (ISI, Delhi, India)  
Manjul Bhargava (Princeton U, NJ, USA)  
J. H. Coates (Cambridge, U, UK)  
W. Goldman (U of Maryland, Md, USA)  
G. Misra (IISC, Bangalore, India)  
V. Kumar Murty (U of Toronto, Canada)  
M. S. Narasimhan (IISC, Bangalore, India)  
Nitin Nitsure (TIFR, Mumbai, India)  
Gopal Prasad (U of Michigan, Michigan, USA)  
M. S. Raghunathan (TIFR, Mumbai, India)  
S. S. Sane (U of Mumbai, India)  
V. D. Sharma (IIT (Bombay), Mumbai, India)  
Alladi Sitaram (Formerly at ISI (Bangalore), India)  
V. Srinivas (TIFR, Mumbai, India)  
S. Thangavelu (IISC, Bangalore, India)  
V. S. Varadarajan (UCLA, California, USA)  
S. R. S. Varadhan (Courant Institute, New York, USA)  
S. T. Yau (Harvard U, Mass, USA)

# Contents

Foreword		iii
Members of the Advisory Board		iv
Classical Modular Forms and Galois Representations	<i>T. Schmidt</i>	1–14
$\Lambda$ -adic Forms and the Iwasawa Main Conjecture	<i>D. Banerjee, E. Ghate and Narasimha Kumar</i>	15–47
Ribet’s Construction of a Suitable Cusp Eigenform	<i>A. Saikia</i>	49–65
Ribet’s Modular Construction of Unramified $p$ -extensions of $\mathbb{Q}(\mu_p)$	<i>C. S. Dalawat</i>	67–81
Fitting Ideals	<i>F. A. E. Nuccio Mortarino Majno di Capriglio</i>	83–95
Deligne-Ribet’s work on $L$ -values	<i>O. Venjakob</i>	97–112
The Main Conjecture	<i>J. Coates and R. Sujatha</i>	113–140

