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Preface

Hyperkähler manifolds have appeared at first within the framework of differential geometry as an example of Riemannian manifolds with holonomy of a special restricted type. However, they have soon exhibited such diverse and unexpected links with various branches of mathematics that now hyperkähler geometry by itself forms a separate research subject. Among the traditional areas fused within this new subject are differential and algebraic geometry of complex manifolds, holomorphic symplectic geometry, geometric representation theory, Hodge theory and many others. The most recent addition to the list is the link between hyperkähler geometry and theoretical physics: it turns out that hyperkähler manifolds play a critical part in the modern version of the string theory, which is in itself the basis for the future unified field theory and quantum gravity.

Perhaps because the structure of a hyperkähler manifold is so rich, such manifolds are quite rare and hard to construct. Thus every new example or a class of examples of hyperkähler manifolds is of considerable interest. The main goal of this book is to describe two recent developments in this area, one dealing with compact hyperkähler manifolds, the other - with a rather general class of non-compact ones. In order to make the presentation as self-contained as possible, we have included much preliminary material and gave an exposition of most of the basic facts of the theory. We believe that this makes it possible to read the book without any prior knowledge of hyperkähler geometry and to use it as an introduction to the subject. On the other hand, it is our hope that the new examples of hyperkähler manifolds constructed here would be of interest to a specialist in the field.

For the detailed description of the new results proved in the book the reader is referred to the introductions to the individual chapters. In this general introduction we restrict ourselves to giving a brief historical overview of the theory of hyperkähler manifolds and indicating the place of our results in the general framework of hyperkähler geometry.

Historical overview

Recall that one can define a Kähler manifold as a Riemannian manifold M equipped with an almost complex structure parallel with respect to the Levi-Civita connection. It is well-known that such an almost complex structure is automatically integrable, thus every Kähler M is a complex manifold. Moreover, the Riemannian metric and the complex structure together define a non-degenerate closed 2-form ω on M, thus making M a symplectic manifold.

The notion of a hyperkähler manifold is obtained from this definition by replacing the field of complex numbers with the algebra of quaternions. A hyperkähler manifold is by definition a Riemannian manifold M equipped with two anticommuting almost complex structures parallel with respect to the Levi-Civita connection. These two almost complex structures generate an action of the quaternion algebra in the tangent bundle to M, which is also parallel. Every quaternion h with $h^2 = -1$ defines an almost complex struture on M. This almost complex structure is parallel, hence integrable. Thus every hyperkähler manifold is canonically complex, and in many different ways.

It is convenient to fix once and for all a quaternion I with $I^2 = -1$ and to consider a hyperkähler manifold M as complex by means of the corresponding complex structure. It is canonically Kähler. Moreover, one can combine the other complex structures on M with the Riemannian metric and obtain, apart from the Kähler 2-form ω, a canonical closed non-degenerate holomorphic 2-form Ω on M. Thus every hyperkähler manifold carries canonical Kähler and holomorphically symplectic structures.

A Riemannian manifold of dimension $4n$ is hyperkähler if and only if its holonomy group is contained in the symplectic group $Sp(n)$. As such, hyperkähler manifolds first appeared in the classification of all possible holonomy groups given by M. Berger [Ber]. The term "hyperkähler manifold" was introduced by E. Calabi in his paper [C], where he also constructed several non-trivial examples of hyperkähler metrics. All of Calabi's examples were non-compact. In fact, all these manifolds were total spaces of cotangent bundles to Kähler manifolds.

At the time of the original paper of Calabi's, it seemed that hyperkähler manifolds are a rather unusual phenomenon, not unlike, for example, sporadic simple groups. However, starting with the beginning of the eighties, there has been a wave of discoveries in the area, and we now know a lot of examples of hyperkähler metrics which occur "in the nature". These examples split naturally into two groups, depending on whether the underlying complex manifold is compact.

A powerful tool for constructing compact hyperkähler manifolds is the famous Calabi-Yau Theorem, which provides a Ricci-flat Kähler metric on every compact manifold of Kähler type with trivial canonial bundle. Its usefulness for the hyperkähler geometry lies in the fact that every hyperkähler manifold is canonically holomorphically symplectic. The converse statement is far from being true: a holomorphically symplectic Kähler manifold does not have to be hyperkähler. However, the converse is true if we require in addition that the hololomorphic symplectic form is parallel with respect to the Levi-Civita connection. It is easy to see that every Kähler manifold equipped with a parallel holomorphic symplectic form is hyperkähler.

In general it is very hard to check whether a holomorphic symplectic form on a compact Kähler manifold is parallel. However, there exists a theorem of S. Bochner's [Boch] which claims that this is always the case when the Kähler metric is Ricci-flat. Since the canonical bundle of a holomorphically symplectic manifold is trivial, indeed, trivialized by the power of the symplectic from, the Calabi-Yau Theorem shows that every compact holomorphically symplectic manifold of Kähler type carries a Ricci-flat Kähler metric. This metric must be hyperkähler by the Bochner Theorem. Thus every compact holomorphically symplectic manifold of Kähler type is hyperkähler.

Well-known examples of compact holomorphically symplectic manifolds of dimension 2 are abelian complex surfaces and K3 surfaces. In higher dimensions non-trivial examples of such manifolds have been given by A. Beauville [Beau], extending earlier results of A. Fujiki [F]. Beauville's examples are the Hilbert schemes of points on an abelian or a K3 surface. All these manifolds are of Kähler type, hence hyperkähler.

Given a compact hyperkähler manifold M, one can consider moduli spaces $\mathcal{M}$ of stable holomorphic vector bundles on M with fixed Chern classes. When M is 4-dimensional, hence either an abelian surface or a K3 surface, the moduli space $\mathcal{M}$ is known to be smooth and hyperkähler (see [Kob] for an excellent exposition of these results). When the Chern classes are such that $\mathcal{M}$ is compact, we obtain in this way a new compact hyperkähler manifold. This situation was studied in detail by S. Mukai in [M1], [M2]. It is now known that some of the compact moduli spaces of stable bundles on a K3 surface are deformationally equivalent to hyperkähler manifolds of the type constructed by Beauville. Conjecturally all these moduli spaces lie in the Beauville's deformation class. We refer the reader to [Huy] for an overview of this subject.

These results were partially generalized to higher dimensions in [V1].

The moduli space of stable bundles on a higher-dimensional hyperkähler manifold is no longer automatically smooth. However, it is a singular hyperkähler variety in the sense of [V2]. This implies, in particular, that it is hyperkähler near every smooth point. Moreover, a singular hyperkähler veriety can be canonically desingularized to a smooth hyperkähler manifold.

An outstanding problem in the theory of compact hyperkähler manifolds is to find an example of such a manifold which would be simply connected and not equivalent deformationally to a product of the ones constructed by Beauville. Several important results ([OGr]) on this subject have appeared recently, but the problem is still not completely closed. The first chapter of the present book describes a different approach to this subject. The results of Mukai and Verbitsky are extended and strengthened in a way that conjecturally leads to the hoped-for examples of compact hyperkähler manifolds belonging to a new deformation class.

Both the Calabi-Yau Theorem and the Bochner Theorem are results of a global nature and cannot be used to construct *non-compact* hyperkähler manifolds. Two general methods are known which can be used for this purpose. The first one uses the link between the theory of hyperkähler manifolds and the holomorphic geometry provided by the notion of the twistor space. The twistor space construction, introduced by R. Penrose, has a long and glorious history. The reader is referred to [HKLR] for a detailed exposition of this subject. Here we only mention that the twistor space X for a $4n$-dimensional hyperkähler manifold M is a holomorphic manifold of complex dimension $2n+2$ canonically associated to M, and that the holomorphic structure on the twistor space X embodies most of the differential-geometric properties of the hyperkähler manifold M.

There exists a theorem which allows one to reconstruct a hyperkähler manifold M from its twistor space X equipped with some additional structures. Therefore, if it is impossible to construct M explicitly, one can construct X instead. In this way one can, for example, construct an infinite-dimensional family of hyperkähler metrics on the vector space $\mathbb{C}^{2n}$ (see [HKLR]).

Another general method of constructing non-compact hyperkähler manifolds is the famous hyperkähler reduction technique introduced by Hitchin [Hi1], Hitchin et al. [HKLR]. It is this technique that led to recent discoveries of hyperkähler structures on many interesting manifolds. Fortunately, the hyperkähler reduction is well-covered in the literature (see, for example, [Hi3]). Therefore we will only list the most important examples of hyperkähler manifolds obtained by this method.

- One of the original examples of hyperkähler manifolds given by Calabi was the total space of the cotangent bundle to a complex projective space. This manifold admits an elementary construction via the hyperkähler reduction.

- Let $\Gamma \in SL(2, \mathbb{C})$ be a finite subgroup. The quotient $\mathbb{C}^2/\Gamma$ has an isolated singularity at 0 which can be blown up to a non-singular complex manifold. This manifold can be equipped with a hyperkähler metric by means of the hyperkähler reduction. The resulting metric is asymptotically locally Euclidean (ALE) at infinity. It was discovered by P.B. Kronheimer [Kr1] and generalized by Kronheimer and H. Nakajima [KN]. Nakajima [N] has recently generalized this example even further and obtained a whole family of hyperkähler manifolds called the quiver varieties.

- Let G be a compact Lie group, and let LG be the (infinite-dimensional) Lie group of maps from the unit cricle S^1 to G. A hyperkähler metric on the quotient LG/G has been constructed by S. Donaldson [D].

- Let S be a compact complex curve, and let $\mathcal{M}$ be the moduli space of bundles on S equipped with a flat connection. Hitchin [Hi2] has constructed a hyperkähler structure on the space $\mathcal{M}$. This construction has been recently generalized by C. Simpson [Sim] to the case when S is an arbitrary projective complex manifold.

A related group of examples is obtained by considering the solutions to a system of ordinary differential equations called the Nahm equations. These equations first appeared in the work of Schmid [Sch] on the variations of Hodge structures. They have been used by Kronheimer [Kr2],[Kr3] to obtain a hyperkähler metric on an orbit in the coadjoint representstion of an arbitrary semisimple complex ie group G. More recently Kronheimer's method was used in papers [BG],[DS] to obtain new examples of hyperkähler manifolds. This method is not directly related to the hyperkähler reduction but shares many features with it. Some of the metrics obtained by reduction can be also constructed via the Nahm equations, and vice versa.

Unfortunately, while hyperkähler reduction is a generous source of new hyperkähler metrics, it can only be pushed so far. One of the problems which seems to lie outside of the scope of this approach is that of constructing a hyperkähler metric on the total space of the cotangent bundle to a non-homogeneous Kähler manifold. The second chapter of the present book describes such a construction. This construction is local and works for an arbitrary Kähler manifold. The methods used are necessarily different from the ones already exploited in the literature and consist of explicit but cumbersome application of the deformation theory in the spirit of Kodaira and Spencer [Kod].

Bibliography

[Beau] A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Diff. Geom. **19** (1983), 755–782.

[Ber] M. Berger, *Sur les groupes d'holonomie des variétés à connexion affine et des variétés riemanniennes*, Bull. Soc. Math. France, **83** (1955), 279–330.

[BG] O. Biquard and P. Gauduchon, *Hyperkähler metrics on cotangent bundles of Hermitian symmetric spaces*, in "Proc. of the Special Session an Geometry and Physics held at Aarhus University and of the Summer School at Odense University, Denmark, 1995", Lecture Notes in Pure and Applied Mathematics, volume 184, Marcel Dekker Inc. 1996, 768 pages.

[Boch] S. Bochner, *Curvature in Hermitian metric*, Bull. Amer. Math. Soc. **53** (1947), 179–195.

[C] E. Calabi, *Métriques kähleriennes et fibrés holomorphes*, Ann. Ecol. Norm. Sup. **12** (1979), 269–294.

[DS] A. Dancer and A. Swann, *The structure of quaternionic kähler quotients*, in "Proc. of the Special Session an Geometry and Physics held at Aarhus University and of the Summer School at Odense University, Denmark, 1995", Lecture Notes in Pure and Applied Mathematics, volume 184, Marcel Dekker Inc. 1996, 768 pages.

[D] S.K. Donaldson, *Boundary value problems for Yang-Mills fields*, J. Geom. Phys. **8** (1992), 89–122.

[F] A. Fujiki, *On primitively symplectic compact Kähler V-manifolds of dimension four*, in Classification of algebraic and analytic manifolds, Progr. Math. **39** (1983), 71–250.

[Hi1] N.J. Hitchin, *Metrics on moduli spaces*, Proc. Lefschetz Centennial Conference (Mexico City 1984), Contemp. Math. **58**, Part I, AMS, Providence, RI, 1986.

[Hi2] N.J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. **55** (1987), 59–126.

[Hi3] N.J. Hitchin, *Hyperkähler manifolds*, Séminaire Bourbaki, Exposé 748, Astérisque, **206** (1992), 137–166.

[HKLR] N.J. Hitchin, A. Karlhede, U. Lindström, M. Roček, *Hyperkähler metrics and supersymmetry*, Comm. Math. Phys (1987).

[Huy] D. Huybrechts, *Birational symplectic manifolds and their deformations*, alg-geom/9601015.

[Kod] K. Kodaira and D. Spencer, *On deformations of complex analytic structures I, II*, Ann. Math. **67** (1958), 328–466.

[Kob] S. Kobayashi, *Differential geometry of complex vector bundles*, Princeton University Press, 1987.

[Kr1] P.B. Kronheimer, *The construction of ALE spaces as hyper-Kaehler quotients*, J. Diff. Geom. **29** (1989), 665–683.

[Kr2] P.B. Kronheimer, *A hyper-Kählerian structure on coadjoint orbits of a semisimple complex group*, J. of LMS, **42** (1990), 193–208.

[Kr3] P.B. Kronheimer, *Instantons and the nilpotent variety*, J. Diff. Geom. **32** (1990), 473–490.

[KN] P.B. Kronheimer and H. Nakajima, *Yang-Mills instantons on ALE gravitational instantons*, Math. Ann. **288** (1990), 263–307.

[M1] S. Mukai, *Symplectic structure of the moduli space of sheaves on abelian or K3 surfaces*, Invent. Math. **77** (1984), 101–116.

[M2] S. Mukai, *Moduli of vector bundles on K3 surfaces, and symplectic manifolds*, Sugaku Expositions **1** (1988), 138–174.

[N] H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. **76** (1994), 365–416.

[OGr] Kieran G. O'Grady, *Desingularized moduli spaces of sheaves on a K3*, alg-geom/9708009.

[Sim] C. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety*, I: Publ. Math. I.H.E.S. **79** (1994), 47–129; II: Publ. Math. I.H.E.S. **80** (1994), 5–79.

[Sch] W. Schmid, *Variations of Hodge structure: the singularities of the period mapping*, Invent. Math. **22** (1973), 211–319.

[V1] M. Verbitsky, *Hyperholomorphic bundles over a hyperkähler manifold*, alg-geom/9307008 (1993), also published in: Jour. of Alg. Geom. **5** (1996), 633–669.

[V2] M. Verbitsky, *Hypercomplex Varieties*, alg-geom/9703016 (1997) (to appear in Comm. in Anal. and Geom.)

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