Lectures on Quantum Groups

by

Pavel Etingof and Olivier Schiffmann

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to our parents

Contents

Introduction

Quantum groups is a new exciting area of mathematics, which originated from mathematical physics (field theory, statistical mechanics), and developed greatly over the last 15 years. It is connected with many other, old and new, parts of mathematics, and remains an area of active, fruitful research today.

This book arose from a graduate course on quantum groups given by the first author at Harvard in the Spring of 1997, when it was written down in an extended and improved form by the second author.

The purpose of this book is to give an elementary introduction to the aspect of the theory of quantum groups which has to do with the notion of quantization. It is written for a general mathematical audience: we tried to do everything from scratch, assuming only the basic algebra and geometry.

The first seven lectures are devoted to the theory of quasiclassical objects which are relevant in the theory of quantum groups: Poisson manifolds (algebras), Poisson-Lie groups, Lie bialgebras, the classical Yang-Baxter equation and its solutions (classical r-matrices). The material here is largely standard. At the end of this part we consider in detail the classification of classical r-matrices for simple Lie algebras, given by Belavin and Drinfeld. Our exposition in Lectures 1-7 is similar to that of Chari and Pressley [CP].

In Lectures 8-12, we discuss the definition and properties of the main characters in our story – bialgebras and Hopf algebras. Here we discuss quantum R-matrices, the double construction, and the notion of quantization of Lie bialgebras. We formulate the results about existence of quantization, anticipated by Drinfeld [Dr1] and proved recently in [EK1].

In Lectures 13-14 we discuss monoidal categories. This material is standard, and contained in the book of Maclane [Mac], as well as in several textbooks on quantum groups. We give it in a form suitable for subsequent exposition. In particular, we stress the importance of non-symmetric and non-strict monoidal categories.

In Lectures 15-16 we discuss quasi-bialgebras and quasi-Hopf algebras, which are algebraic counterparts of non-strict monoidal categories, in the same sense as bialgebras and Hopf algebras are algebraic counterparts of strict monoidal categories. We consider the main properties of quasi-bialgebras, and the simplest examples of them. Then we study quasitriangular quasi-Hopf algebras, the Knizhnik-Zamilodchikov equation and the corresponding quasi-Hopf algebra, define and study equivalence by a twist, and cite Drinfeld's classification result. From this result, we deduce the Drinfeld-Kohno theorem about the monodromy of the Knizhik-Zamolodchikov equations.

In Lecture 17 we introduce Lie associators and the Grothendieck-Teichmuller group, give their main properties, and define, following Drinfeld, the free, transitive action of the Grothendieck-Teichmuller group on the space of Lie associators.

In Lecture 18 we discuss the Tannaka-Krein philosophy for tensor categories, which allows one to get a better understanding of the notion of a bialgebra and a quasitriangular bialgebra.

In Lectures 19-22 we describe the method of quantization of Lie bialgebras developed recently in [EK1, EK2]. This part is the culmination point of the book, where many methods and notions of the previous chapters come together. In this part, we prove that any Lie bialgebra can be quantized, and that this quantization is given by a universal, functorial construction.

Finally, in the Appendix we give some applications of the material of the book to number theory (counting independent values of zeta-functions). The contents of the Appendix is discussed somewhat differently in [Kass].

Now a few words about the nature of this book. It is written in the spirit of lecture notes rather than that of a serious monograph. Our goal was not to cover the maximal amount of material, nor to present it in the most complete form, but to expose a number of deep and interesting results in a reader-friendly way. In view of this, we did not discuss many important parts of the theory of quantum groups (representation theory, quantum groups at roots of unity, knot invariants, Drinfeld new realizations, relations to q-special functions, etc.), and did not give many basic references. Luckily, there exist many textbooks on quantum groups [Kass, CP, ShSt, Jos, J, Lu, Maj], where this missing information can be readily found.

Two unusual features of this book, compared to other textbooks, are extensive use of pictorial language for writing and checking algebraic relations, and over fifty problems and exercises (with solutions).
We hope that these features will facilitate active reading of the book, and make it accessible to a wide audience.

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