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Handbook of Geometric Analysis, No. 2

Editors: Lizhen Ji, Peter Li, Richard Schoen, and Leon Simon

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Advanced Lectures in Mathematics, Volume XIII
Handbook of Geometric Analysis, No. 2

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Preface

The marriage of geometry and analysis, in particular non-linear differential equations, has been very fruitful. An early deep application of geometric analysis is the celebrated solution by Shing-Tung Yau of the Calabi conjecture in 1976. In fact, Yau together with many of his collaborators developed important techniques in geometric analysis in order to solve the Calabi conjecture. Besides solving many open problems in algebraic geometry such as the Severi conjecture, the characterization of complex projective varieties, and characterization of certain Shimura varieties, the Calabi-Yau manifolds also provide the basic building blocks in the superstring theory model of the universe. Geometric analysis has also been crucial in solving many outstanding problems in low dimensional topology, for example, the Smith conjecture, and the positive mass conjecture in general relativity.

Geometric analysis has been intensively studied and highly developed since 1970s, and it is becoming an indispensable tool for understanding many parts of mathematics. Its success also brings with it the difficulty for the uninitiated to appreciate its breadth and depth. In order to introduce both beginners and non-experts to this fascinating subject, we have decided to edit this handbook of geometric analysis. Each article is written by a leading expert in the field and will serve as both an introduction to and a survey of the topics under discussion. The handbook of geometric analysis is divided into several parts, and this volume is the second part.

Shing-Tung Yau has been crucial to many stages of the development of geometric analysis. Indeed, his work has played an important role in bringing the well-deserved global recognition by the whole mathematical sciences community to the field of geometric analysis. In view of this, we would like to dedicate this handbook of geometric analysis to Shing-Tung Yau on the occasion of his sixtieth birthday.

Summarizing the main mathematical contributions of Yau will take many pages and is probably beyond the capability of the editors. Instead, we quote several award citations on the work of Yau.

The citation of the Veblen Prize for Yau in 1981 says: "*We have rarely had the opportunity to witness the spectacle of the work of one mathematician affecting, in a short span of years, the direction of whole areas of research.... Few mathematicians can match Yau's achievements in depth, in impact, and in the diversity of methods and applications.*"

In 1983, when Yau was awarded a Fields medal, L. Nirenberg described Yau's work up to that point:

"*Yau has done extremely deep work in global geometry and elliptic partial differential equations, including applications in three-dimensional topology and in general relativity theory. He is an analyst's geometer (or geometer's analyst) with remarkable technical power and insight. He has succeeded in solving problems on which progress had been stopped for years.*"

More than ten years later, Yau was awarded the Carfoord prize in 1994, and the citation of the award says:

"The Prize is awarded to ... Shing-Tung Yau, Harvard University, Cambridge, MA, USA, for his development of non-linear techniques in differential geometry leading to the solution of several outstanding problems.

Thanks to Shing-Tung Yau's work over the past twenty years, the role and understanding of the basic partial differential equations in geometry has changed and expanded enormously within the field of mathematics. His work has had an impact on areas of mathematics and physics as diverse as topology, algebraic geometry, representation theory, and general relativity as well as differential geometry and partial differential equations. Yau is a student of legendary Chinese mathematician Shiing-Shen Chern, for whom he studied at Berkeley. As a teacher he is very generous with his ideas and he has had many students and also collaborated with many mathematicians."

In 2010, Yau was awarded the Wolf Prize for his lifetime achievements in geometric analysis and mathematical physics, and the award citation probably gives one of the best summaries of his major works up to 2010:

"Shing-Tung Yau (born 1949, China) has linked partial differential equations, geometry, and mathematical physics in a fundamentally new way, decisively shaping the field of geometric analysis. He has developed new analytical tools to solve several difficult nonlinear partial differential equations, particularly those of the Monge-Ampere type, critical to progress in Riemannian, Kahler and algebraic geometry and in algebraic topology, that radically transformed these fields. The Calabi-Yau manifolds, as these are known, a particular class of Kahler manifolds, have become a cornerstone of string theory aimed at understanding how the action of physical forces in a high-dimensional space might ultimately lead to our four-dimensional world of space and time. Prof. Yau's work on T-duality is an important ingredient for mirror symmetry, a fundamental problem at the interface of string theory and algebraic and symplectic geometry. While settling the positive mass and energy conjectures in general relativity, he also created powerful analytical tools, which have broad applications in the investigation of the global geometry of space-time.

Prof. Yau's eigenvalue and heat kernel estimates on Riemannian manifolds count among the most profound achievements of analysis on manifolds. He studied minimal surfaces, solving several classical problems, and then used his results, to create a novel approach to geometric topology. Prof. Yau has been exceptionally productive over several decades, with results radiating onto many areas of *pure and applied mathematics and theoretical physics. In addition to his diverse and fundamental mathematical achievements, which have inspired generations of mathematicians, Prof. Yau has also had an enormous impact, worldwide, on mathematical research, through training an extraordinary number of graduate students and establishing several active mathematical research centers.*"

Indeed, he has already trained more than 60 Ph.D. students.

We wish Yau a happy sixtieth birthday and continuing success in many years to come!

The Editors:

Lizhen Ji
Peter Li
Richard Schoen
Leon Simon

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