Lectures on Differential Geometry

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Preface for the English Translation

In the Spring of 1984, The authors gave a series of lectures in the Institute for advanced study in Princeton. Professor K.H. Zhong of the Academic Sinica in Peking took the lecture notes. These lectures become the first four chapters of this book. Then in the the academic year 1984-1985, we continued to give the lectures in San Diego where Professor Wei-Yue Ding of the Academic Sinica and Professor Kung-Ching Chang of Peking University took the notes. Parts of these lecture notes become the last two chapter of this book. Since all the notes were written in Chinese, they were initially published in Chinese. While they were widely circulated in China, it was clear that it will be useful for the book to appear in English. Professor Ding and Professor S.Y. Cheng were kind enough to translate them into English. We are very grateful to all of these mathematicians for their tremendous help, without which the book would never appear. This book has been widely circulated in Chinese since 1986. We hope the translation will be useful for the English speaking audience.

Since much of this material was presented in lectures given almost ten years ago, many of the discussions and conjectures are out of date. While we have done some updating of these discussions together with references, we have not done this systematically because we felt it better not to delay publication.

In order to provide a helpful overview of the subject, we also included three articles that were written by the second author, published previously in other publications. These are Chapters VII, VIII, and IX. We thank the respective publishers for allowing us to reproduce these manuscript here.

Translation of Original Preface
translated by Kaising Tso

Among all branches of mathematics, mathematicians have had high respect to the study of geometry since ancient times. The reason is that in geometry one studies certain forms of natural phenomena, and the vivid perception provided by natural phenomena has always been an important source of inspiration for mathematicians. As a result, geometry has a very close kinship with other branches of mathematics. On the other hand, apparently it is advancing along with progress in natural sciences. Einstein's general relativity proposed in the early part of the century and Yang-Mills' gauge field theory studied in the past two decades are a perfect illustration of how geometry and physics meet.

A significant part of geometry is differential geometry. In modern differential geometry one studies the analytic structure of a manifold and various geometric properties derived from it. Its origin may be traced back to Gauss and Riemann. Soon after Riemann proposed the geometry which later bore his name, the study of local geometry prospered quickly. Tensor analysis was invented. At the same time, Klein made known to the public his celebrated Erlangen Programme where invariants of the transformation group of the space were studied from a group-theoretic point of view. Many different geometries were introduced. In addition, the study of Riemann surfaces was promoted by the uniformization theory in complex function theory. All these developments together with the classical theory of surfaces laid the foundation for differential geometry of the twentieth century.

Differential geometry progressed rapidly in this century. The progress may be described in the following four categories.

First, classification of Lie groups and Riemannian symmetric spaces was carried out by Cartan and Weyl. Cartan generalized the concept of a connection by combining Klein's theory and Riemannian geometry. Moreover, he further developed Cartan-Kahler theory by introducing exterior differentiation. All these efforts pushed forward the local theory of differential geometry in a big step.

Second, mathematicians including de Rham, Hodge, Kodaira, Hopf, Lefschetz, Whitney, Weil and Chern established a close relationship between differential geometry and topology and algebraic geometry, which were blossoming during this period. Global differential geometry started taking its form.

Third, along the vein of classical differential geometry, geometry of convex surfaces, synthetic and integral geometry advanced greatly under the leadership of Alexandroff, Cohn-Vossen, Pogorelov, Busemann, Rauch, and Santalo.

Finally, by the maturity of the theory of differential equations, geometers started using analytical methods to tackle problems in geometry. In the reverse direction, people discovered a large number of significant differential equations from differential geometry. New approaches or methods are very often required to solve these equations. Thus analysts also watched progress in geometry closely. Leaders in this aspect include Hadamard, Morse, Lewy, Morrey, Bochner, Nash, Moser, Nirenberg, and Efimov. Their works have formed the cornerstone for the application of nonlinear partial differential equations in geometry in the past twenty years.

We attempt to describe major achievements in all these aspects in this book. The reader will find that differential geometry is a whole subject where all these aspects interlace in a natural way. In this first volume we shall study differential equations on a manifold and derive results relating its curvature to its topology. Only a single equation will be treated. Systems of differential equations will be postponed to the forthcoming second and third volumes, where we shall discuss subjects including Hodge theory, minimal submanifolds, harmonic maps, gauge fields, Kahler manifolds, and Mong-Ampere equations. Relationship between geometry and topology, algebraic geometry, general relativity and high energy physics will also be discussed.

The present volume has six chapters. In the first chapter we treat the Laplace operator, the most important differential operator in differential geometry. Its importance lies in the fact that the linearization of many important nonlinear operators gives rise to Laplace operators of some Riemannian metrics. In specific places we need to approximate nonlinear operators by linear ones. In this aspect estimations independent of the Riemannian metric are sought for. The function space we shall use either consists of bounded functions or square summable functions. As we know, the object of study in classical harmonic analysis is the class of harmonic functions in the Euclidean space or its bounded domains. The proper generalization in geometry is a complete Riemannian manifold, where non-negative Ricci curvature tensor corresponds to the Euclidean space and negative sectional curvature corresponds to bounded Euclidean domains. In principle, all major results in classical harmonic analysis have their counterparts on manifolds. In the first two chapters of this volume we discuss some of these important generalizations. It is worthwhile to point out that very often new methods must be introduced to achieve this goal. We have also found that many geometric problems can be solved by these analytic methods. When the Laplace operator acts on the space of square summable functions, spectral analysis is the key issue. The spectrum is discrete on a compact manifold. We shall study eigenfunctions and eigenvalues in Chapter 3. In Chapter 4 we study the spectrum via the heat kernel. It is also desirable to look at the wave kernel for the obvious reason that it links the spectrum of a manifold to its geodesics. Very little is known about the spectrum of a noncompact manifold, especially when it is continuous. We hope to discuss it in a later occasion.

We study a nonlinear partial differential equation arising from conformal deformation in Chapter 5 and Chapter 6. Research in this direction has never ceased since Poincaré. First we discuss the Yamabe problem, restricting to the compact case. The noncompact case has not yet been solved completely. We wish to return to it in due course. Chapter 6 is devoted to conformally flat Riemannian manifolds. The reader will find from this chapter that a rather good understanding for these manifolds can be obtained when they have positive scalar curvature. However, the problem becomes formidable in the case of negative scalar curvature.

We have not been able to find a suitable text book on differential geometry in the large, especially a systematic one which is based on topology and algebraic geometry and uses analysis as its major tool. Our book may be regarded as an attempt in this direction.

This volume is based on a series of the author's lectures. The first four chapters came from lectures given at Princeton in 1983. The notes were taken and written up by Jiaqing Zhong. The last two chapter came from lectures given at San Diego in 1984 and 1985, respectively. Chapter 5 was written by Yichao Xu and Weiyue Ding. Chapter 6 contains results mainly obtained by the authors during this period; and it was written by K.C. Chang. All these colleagues involved in writing up the manuscript are outstanding mathematicians in their own fields. Their valuable opinions incorporated in the text surely improve the book a lot. Additionally, the first author's students Gang Tian, Huaidong Cao, and Jun Li proofread the manuscript. We would like to take this opportunity to thank all of them for their assistance.

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UC San Diego, Feb 1, 1986