Dirac Operators: Yesterday and Today

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Preface

The Summer School ”Dirac Operators: Yesterday and Today” was an ambitious endeavour to show to a public of advanced students a tableau of modern mathematics. The idea was to make available a topic, of considerable importance for mathematics itself because of its deep impact on many mathematical questions, but also for its interactions with one of mathematics historic partners, physics.

The Dirac operator was undoubtedly created by Paul Adrien Maurice Dirac to deal with an important physical question. That it now has a deep presence in mathematics is a long story, for which Michael F. Atiyah and Isadore M. Singer are the main players. Indeed, if the basic objects involved in the definition of the Dirac operator, namely the spinor fields, had been introduced for their own sake by Élie Cartan, and then firmly developed by Richard Brauer and Hermann Weyl, the remarkable insight that there is a very distinguished operator acting on them was Dirac's. Fifty years later Atiyah and Singer turned it into a mathematical cornerstone by one of the far-reaching developments of their monumental work on the index theorem. In some sense one can say that they rediscovered the Dirac operator in a Riemannian setting. As a consequence of their work, this operator now has a central role in modern mathematics, at the crossroads of many disciplines, and in particular as a unifying tool bringing together analysis and geometry under the umbrella of what is now called “global analysis.”

To understand problems connected with the Dirac operator, one needs to present its many facets. This set of notes covering courses and advanced seminars that accompanied the courses presents basic ingredients that constitute the foundations of the theory. It is exemplary of the recent evolution of mathematics, and of the strengthening of its historic unity which occurs in a very dynamic way. One needs to bring together many components of mathematics, often considered as separate: algebra, analysis and geometry in its broadest context. Indeed, at the basis of the study of Dirac operators one finds representation theory of the orthogonal groups, bundle theory and the theory of elliptic operators on manifolds. Last but not least, one of course cannot forget the link to theoretical physics, which has originally inspired most of the developments, and which recently broadened considerably the scope of the subject.

As it often is the case with multi-faceted objects, one never knows what should come first. This set of notes proposes an itinerary, going from the simplest notions to more advanced ones. There is always the risk that the reader loses patience from frustration of waiting too long before she or he can get a global view of the landscape. The discovery of this breath-taking view, when all pieces have been properly put in place, will be one of the high points of the journey, a very valuable pay-off. The hope of course is soon after to gain new viewpoints on classical questions. In the case at hand, some anticipate that parts of Riemannian Geometry, undoubtedly the classical backbone of the theory of Dirac operators, will soon be much enriched by a systematic spinorial approach. In a sense particle physicists have led the way in establishing long ago, almost as soon as Dirac came up with his model for relativistic electron motion, that particle physics must associate spinor fields and tensor fields. Throughout the twentieth century, it got richer and richer as a result of one of the most extraordinary endeavours of human history. In this context spinor fields are the wave functions of half-spin particles, while tensor fields are the wave functions of spinless particles or particles having an integral spin.

It took many years for mathematicians to really accept the vision that spinorial objects and tensorial objects have to be considered on an equal footing. In the late 70s and more systematically in the 80s, physicists went one step further. They started to focus their attention on theories that would associate, in an even more intimate way spinorial and tensorial fields, the so-called supersymmetric theories. Centering around the idea that special transformations actually exchanging spinorial and tensorial particles, the notion of supersymmetry would dramatically impact physical models in several parts of physics. It still has to prove realistic by some experimental evidence. When formulated in mathematical terms, this notion brings to the forefront special geometric structures in a very natural way. These, in turn, seem to have a physical bearing in string theory, something difficult to guess a priori. These notes do give a few hints of these important developments as one will read in some seminars.

Another way in which concepts at the heart of this lecture series find an exemplary domain of illustration is non-commutative geometry. Indeed, in order to introduce the notion of a metric in this context, Alain Connes proposed very successfully that this be done via the definition of an object with is the fundamental solution of a would-be "Dirac operator." This is a remarkable realisation of the principle this preface means to stress, namely that in modern Geometry the functional point of view, and very concretely the consideration of special operators related to classical geometric situations, is fundamentally fruitful. A very deep lesson this set of notes will hopefully help to digest.

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The Editors

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