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Variational Principles for Discrete Surfaces

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Preface

This book consists of mathematical and algorithmic studies of geometry of polyhedral surfaces based on the variations principles. The part of mathematics is based on a lecture series given by Feng Luo at the Center of Mathematical sciences at Zhejiang University, China, in June and July 2006. The algorithmic theory and applications to computer graphic are based on the work of Xianfeng Gu and are written by him. The task of writing the part of mathematics of the note was done by Junfei Dai who prepared them with great care and made a number of improvements in the exposition.

The aim of this book is to introduce to the students and researchers an emerging field of polyhedral surface geometry and computer graphics based on variation principles. These variational principles are derived from the derivatives of the cosine law for triangles. From mathematical point of view, one of the most fascinating identity in low-dimensional polyhedral geometry is the Schlaefli formula. It relates in a simple and elegant to way the volume, edge lengths and dihedral angles of tetrahedra in the spheres and hyperbolic spaces in dimension 3. The formula can be considered as a foundation of 3-dimensional variational principles for triangulated objects. For a long time, mathematicians have been considering the Gauss-Bonnet formula as the 2-dimensional counterpart of Schlaefli. The recent breakthrough in this area was due to the work of Colin de Verdiere in 1995 who found the first 2-dimensional identity relating edge lengths and inner angles similar to the Schlaefli identity. The mathematical work produced in this book can be considered as establishing all 2-dimensional counterparts of Schaefli formula. It turns out there are continuous families of Schlaefli type identities in dimension 2. These identities produce many interesting variational principles for polyhedral surfaces. In the part of mathematics of the book, we are focusing on a study of the rigidity phenomena on polyhedral surfaces. Some moduli space problems are also discussed in the book.

In the part of algorithm of the book, we introduce discrete curvature flow from both theoretical and practical points of view. Discrete curvature flow is a powerful tool for designing metrics by prescribed curvatures. The algorithm maps general surfaces with arbitrary topologies to three canonical spaces. Therefore, all geometric problems of surfaces in 3D space are converted to 2D ones. This greatly improves the efficiency and accuracy for engineering applications. The discrete Ricci flow algorithm, and Ricci energy optimization algorithm are rigorous, robust, flexible and efficient. They have been applied for surface matching, registration, shape classification, shape analysis and many fundamental applications in practice.

This book is written for senior undergraduate students and graduate students majoring in mathematics or computer science. The mathematical requirements to follow the proofs in the book are some basic knowledge of differential geometry and elementary surface topology. We have not stated and proved theorems in the book in the most general form to avoid the technical details. For computer science majors with basic knowledge in data structure and algorithm, all the algorithms in the book can be implemented straightforwardly by following the pseudo codes and the software system can be built step by step from scratch. Some data sets and source code are also available upon request.

This book should be valuable for researchers in surface geometry, computer graphics, computer vision, geometric modelling, visualization, medical imaging and scientific computation fields. The computational algorithms are also useful for geometric modelers, industrial products designer, digital artists, animators, game developers and anyone who needs digital geometry processing tools.

We owe much to many colleagues and friends with whom we have discussed the subject matter over the years. The first author would like to thank Ben Chow who introduced him the wonderful world of discrete curvature flows and the Ricci flow. The second author is very grateful to all professors in the *Center of Visual Computing* at Stony Brook: Arie Kaufman, Hong Qin, Dimitris Samaras, Klaus Mueller and all faculty members in the Computer Science department in Stony Brook University. The second author deeply appreciate the encouragements and valuable advices from Professor Shing-Tung Yau.

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Last but not least, we also want to thank our families, without their supports, this book could not be accomplished.

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