BASIC PARTIAL DIFFERENTIAL EQUATIONS

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We dedicate this book to

Marie and Beverly

who, much more than ourselves,
steadfastly longed for its completion.

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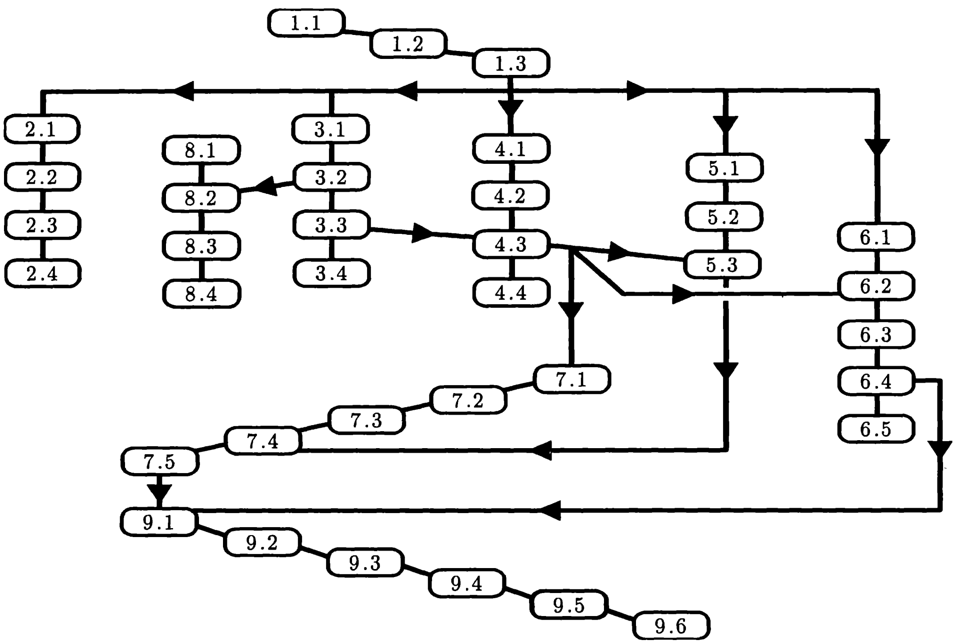
Dependence of Sections



Preface

Quantities which depend on space and/or time variables are often governed by differential equations which are based on underlying physical principles. Partial differential equations (PDEs) not only accurately express these principles, but also help to predict the behavior of a system from an initial state of the system and from given external influences. Thus, it is hard to overestimate the relevance of PDEs in all forms of science and engineering, or any endeavor which involves reasonably smooth, predictable changes of measurable quantities.

Having taught from the material in this book for fifteen years with much feedback from students, we have found that the book serves as a very readable introduction to the subject for undergraduates with a year and a half of calculus, but not necessarily any more. In particular, one need *not* have had a linear algebra course or even a course in ordinary differential equations to understand the material. As the title suggests, we have concentrated only on what we feel are the absolutely essential aspects of the subject, and there are some crucial topics such as systems of PDEs which we only touch on. Yet the book certainly contains more material than can be covered in a single semester, even with an exceptional class. Given the broad relevance of the subject, we suspect that a demand for a second semester surely exists, but has been largely unmet, partly due to the lack of books which take the time and space to be readable by sophomores. A glance at the table of contents or the index reveals some subjects which are regarded as rather advanced (e.g., maximum principles, Fourier transforms, quasi-linear PDEs, spherical harmonics, PDEs on manifolds, complex variable theory, conditions under which Fourier series are uniformly convergent). However, despite general impressions given (perhaps unwittingly) by mathematical gurus, *any* valid mathematical result or concept, regardless of how "advanced" it is, can be broken down into elementary, trivial pieces which are easily understood by all who desire to do so. With regard to the so-called "advanced" topics in this book, we feel that we have accomplished this to a degree which surprised even us. For us it was a constant and worthwhile challenge to make the book completely self-contained for those who have only been through the typical freshman/sophomore calculus sequence, even if they forgot most of it. We have successfully taught students who did not recall how to solve $y'(x) = y(x)$ with $y(0) = 1$ at the beginning of the semester, as was the case with over half of our students according to initial survey tests. However, before the semester's end, these same students could prove and understand the Maximum Principle for the heat equation and could easily deduce the continuous dependence of solutions on initial and boundary data. In essence, "advanced topics" are rarely difficult per se, but they may seem so, if (for the sake of elegance) too little time is spent explaining and motivating them.

We have avoided the temptation to first prove unmotivated results in great generality and then use them to deduce an abundance of particular cases. By and large, we have introduced results and techniques inductively through many solved examples. By the time students have seen enough examples, they can often anticipate, as well as understand, the argument for the general case. In particular, we have found that, in spite of the fact that the Sturm-Liouville Theory provides a uniform approach to boundary-value problems, it is not so wise to teach it first to students who are barely familiar with sines and cosines, and then cover the elementary boundary-value problems as special cases. We have proceeded in the opposite manner. After we have handled a variety of simple boundary conditions for the heat equation and treated Fourier series, the student is prepared to study and appreciate Sturm-Liouville Theory as a natural continuation of what has been learned without it. Proceeding from examples to theorems may result in a book which is physically longer, but students learn more rapidly and effectively this way. In short, it is easier to build from the ground up than from the roof down. In the process, we may have sacrificed some degree of elegance, but we have not sacrificed rigor. Nearly every basic result is proved rigorously at some stage, or at least we give a reference (e.g., for the convergence of eigenfunction expansions on manifolds). We certainly do not recommend proving everything in class, since this would severely limit the range of the material covered, but instead the interested student may be directed to the many detailed, thoroughly digestible proofs in the text. On the point of rigor, we mention that many solutions of PDEs are expressible in terms of integrals of Green's functions against boundary and/or initial data. In most PDE texts, such integral formulas are derived (if at all) under the assumption that solutions of the PDEs actually exist. To be honest, one should have the tools to check that the functions defined by such integral formulas actually solve the given problem. This necessarily entails the use of Leibniz's rule for differentiation under an integral, in particular when the interval of integration is unbounded. One feature of this book, which appears to be absent in other introductory texts, is that there is a complete proof of Leibniz's rule (cf. Appendix A.3). In place of Lebesgue Dominated Convergence Theorem, this proof uses an elementary version of dominated convergence for Riemann integrals based on ideas originating in [Lewin, 1986, 1987]. Thus, the notion of Lebesgue measure and integration is avoided.

Solving problems is the major part of learning in any mathematical subject. This book contains many problems which range from the purely routine to those which will challenge even the most brilliant student. Sometimes one finds that although some students can arrive at a solution to a problem through mimicking procedures, they still may not be able to interpret or use the solution or even understand why the expression they have found is actually a solution of the problem. We have tried to counter this tragedy by including many exercises which require the student to think, draw some conclusions, and interpret the results, instead of simply implementing purely computational procedures. Since some students will do anything to get the answer in the back of the book, we have been sparing with the answers. However, a solutions manual (with complete solutions to all but the most trivial problems) is available to instructors only. We personally worked out each of the problems.

Since the whole book cannot be covered in a single semester, instructors who are limited to a single semester must decide which sections or chapters to cover. Given the demand, instructors might consider the introduction of a second semester of PDEs. Below, we summarize the material covered in the chapters and sections. Following this, some suggestions are given on what sections must, should or could be included in a one-semester or two-quarter course.

Acknowledgements. It is our pleasure to acknowledge the comments and suggestions of our colleagues and students. In particular, we thank Hans Broderson, Philip Doyle, Karl Heinz Dovermann, Christopher Mawata, Ken Rogers, Mi-Soo Smith, Wayne Smith, David Stegenga, Joel Weiner, George Wilkens, and Les Wilson, who have adopted the notes in their courses. We also acknowledge Paolo Agliano and Paul Kohs who helped us with the typing of the preliminary version of the book. In addition, a warm *mahalo* is due to the secretarial staff of the Department of Mathematics at the University of Hawaii. A special *mahalo nui loa* is due to Pat Goldstein who cheerfully helped us with some of the clerical work. Last, but not least, we wish to thank our families for their patience and support during the preparation of this work.

Honolulu, 1996 David Bleecker & George Csordas

Chapter-by-chapter synopsis and suggestions for the instructor

Chapter 1 (Review and Introduction): If the students have had a course in ODEs, then Section 1.1 can be skipped, or assigned as reading. Some coverage of Sections 1.2 and 1.3 is necessary for a general overview of PDEs and their applications, and for an introduction to certain topics, such as separation of variables and the superposition principle. These concepts are used often in the sequel.

Chapter 2 (First-Order PDEs): For instructors who regard first-order PDEs as devoid of any real application, we urge them to read the introduction to Chapter 2, before deciding to skip Chapter 2 entirely. Not only are there wide applications to birth and death processes (e.g., the evolution of population densities), continuum mechanics and the development of shocks in traffic flow, but also the student sees how a change of variables can greatly simplify a PDE. Incidentally, we elected not to include examples and drill exercises for putting second-order, linear PDEs (with constant coefficients) into the standard normal forms (e.g, by rotation of axes, etc.), for the simple reason that second-order PDEs which arise in applications are already in a standard form. However, a complete statement of the Classification Theorem is given in Section 1.2, and a complete proof is given in the Appendix A.1. To compensate for lack of practice in change of variables drill for second-order PDEs, there are plenty of change-of-variable problems for first-order PDEs in Chapter 2. First-order PDEs which arise in applications are seldom in the standard form of a parametrized ODE. While Chapters 3-9 do not depend on Chapter 2, instructors should seriously consider doing at least Section 2.1 in which $au_x + bu_y + cu = f(x, y)$ is solved, when a, b and c are constants. The case of variable coefficients is covered in Section 2.2, and the quasi-linear case is covered in Section 2.3. The fully nonlinear case is covered in the optional Section 2.4.

Chapter 3 (The Heat Equation): Section 3.1 begins with a derivation of the heat equation. The simplest initial/boundary-value problems are solved *without* first introducing Fourier series. Here, we use separation of variables to find product solutions of the heat equation which meet the homogeneous boundary conditions B.C. and then find a linear combination which meets the initial condition. In Chapter 3, initial temperatures are chosen so that they are expressible (via trigonometric identities) as finite linear combinations of sines or cosines of the appropriate form. Students then naturally ask what can be done if this is not the case. In other words, they are naturally motivated for the introduction of Fourier series which is the topic of Chapter 4. In Section 3.2, uniqueness of solutions of various initial/boundary-value problems for the heat equation is proved, by showing that for homogeneous B.C. of the first or second kind, the mean-square of the temperature is nonincreasing. The Maximum Principle provides a second approach. We first illustrate the Maximum Principle through a number of examples and we show that it easily leads to continuous (uniform) dependence of solutions on initial/boundary data. The proof of the Maximum Principle is then given at the end of Section 3.2. Section 3.3 deals with the case of various simple B.C. which are time-independent, but possibly inhomogeneous. In Section 3.4, the case of time-dependent B.C. and heat sources are handled by means of Duhamel's principle. Section 3.4 can be skipped or covered later if time permits, and Section 3.3 can be covered quickly and lightly. However, Section 3.1 is certainly part of any first PDE course, and we strongly recommend that Section 3.2 be covered in some detail.

Chapter 4 (Fourier Series and Sturm-Liouville Theory): Students see the need for Fourier series in Chapter 3. In Section 4.1, we introduce the notion of orthogonality of functions, and the definition of Fourier series of a function as a formal expression which may or may not converge to the function. Many examples are computed, and the question of convergence is motivated. An estimate for the number of terms needed to uniformly approximate a C^2 function is stated (but the proof is deferred until Section 4.2). We provide a technique for obtaining much sharper estimates by means of integral estimates of the tail of a Fourier series. Section 4.2 contains detailed proofs of the convergence of Fourier series under various assumptions. We gently introduce the difference between pointwise convergence and uniform convergence. Pointwise convergence is proved for piecewise C^1 functions and uniform convergence for *continuous* piecewise C^1 functions. Without the luxury of time, we recommend that the lengthier proofs be skipped or assigned for reading. However, certainly one should get across the general idea that the smoother a function is on a circle, the more rapid is the convergence of its Fourier series. In Section 4.3, we introduce Fourier sine and cosine series which are used to handle (at least formally) the case (left dangling in Chapter 3) that the initial temperature was not a *finite* linear combination of the appropriate form. It is emphasized that infinite sums of C^2 functions need not be C^2, and hence the formal solutions obtained need not be strict solutions. However, by truncating the series at a sufficiently large number of terms one can often meet the I.C. within any positive error, which is all that is needed for applications. The validity of formal solutions under certain assumptions is deferred to Chapter 7. Sturm-Liouville Theory is covered in Section 4.4. At this point the student is ready to savor this subject which extends what is known already to the case of inhomogeneous rods and boundary conditions of the third kind. We provide a convincing sketch of a proof of the infinitude of the eigenvalues for Sturm-Liouville problems, by means of the Sturm Comparison Theorem. Practically none of the rest of the book depends on Section 4.4, except the statement found in Chapter 9 (Section 9.5) that Bessel functions have infinitely many zeros. Thus, in the face of time pressures, it is possible to omit Section 4.4 entirely, although one should at least tell students what it is about. We have found that Section 4.3 can and should be covered rapidly, and that one should stress the statements of the theorems in Section 4.2, but not necessarily the details of the proofs. Section 4.1 should be covered in detail, as it is frequently used later.

Chapter 5 (The Wave Equation): In Section 5.1, the wave equation for a transversely vibrating string is derived from Newton's equation. Some care is taken to explain why the assumption of transverse vibrations actually *implies* a linear wave equation instead of an approximately linear equation. The dubious assumption of "small" vibrations is thus eliminated. The simplest initial/boundary-value problems for a finite string are solved. Uniqueness of solutions of these problems is also proved in Section 5.1, using the energy-integral method. In Section 5.2, we cover D'Alembert's solution of wave problems on the infinite string. Consequences of D'Alembert's solution, such as finite propagation speed are covered, and the method of images for semi-infinite strings is explained. For finite strings, the method of images provides an alternative to the Fourier series approach. The continuous dependence of solutions for the finite string on initial conditions is also an easy consequence of D'Alembert's formula and the method of images. In Section 5.3 a variety of boundary conditions for the string are handled. Also, the inhomogeneous wave equation (i.e., with forcing term) is treated via both Duhamel's principle and the Fourier series approach. Section 5.1 should be covered in some detail, with the complete derivation possibly assigned as reading. Section 5.2 is equally crucial, but if time is running short Section 5.3 can simply be summarized, so that students are aware of what is covered in case they need it.

Chapter 6 (Laplace's Equation): In Section 6.1, Laplace's equation is motivated and it is shown that solutions may be interpreted as steady-state temperature distributions. The Dirichlet and Neumann problems are introduced. Section 6.2 concerns the solution of these problems on a rectangle. Since students are familiar with separation of variables and superposition, this material can be done quickly. Uniqueness and the Maximum Principle are motivated and utilized, but proofs are deferred until Section 6.4. In Section 6.3, we solve Dirichlet and Neumann problems on annuli and disks using polar coordinates. The Mean-Value Theorem and Poisson's Integral Formula are carefully proved, and the regularity of harmonic functions is demonstrated. In Section 6.4, the Maximum Principle for harmonic functions on bounded domains is proved along with continuous dependence of solutions of the Dirichlet problem on boundary data. The importance of these results has been amply demonstrated to students in the previous sections. Section 6.5 is on the application of complex variable theory to Laplace's equation. We assume *no* knowledge of complex-variables. We do not cover Cauchy's theorem, contour integration, or residue theory, for the simple reason that we do not need it here. However, the intimate connection between complex analytic functions and harmonic functions is brought out and exploited. Moreover, the concept and use of conformal mapping to solve problems in steady-state temperatures, fluid flow and electrostatics are handled without any difficulty. All of the material in Chapter 6 is important, and if too much time is spent on material in previous chapters, it may not be possible to cover all of Chapter 6. For a class of mostly engineers, it may be wiser to cover Section 6.5 instead of Section 6.4, if a choice must be made, whereas for mathematics majors the reverse choice is desirable.

Chapter 7 (Fourier Transforms): It will take an exceptional class to reach Chapter 7 in one semester, without skipping all but the most essential material in the previous chapters. However, if students are likely to take a full complex variable course in the future, many concepts in Chapter 6 will be treated in that course. Then, skipping much of Chapter 6 and proceeding with Chapter 7 becomes an attractive possibility. Of course, the possibility of introducing a second semester (or more quarters) of PDEs should be contemplated. The demand is there. In Section 7.1, we introduce complex Fourier series and define the Fourier transform. Many examples are computed. In Section 7.2, we develop the basic properties of Fourier transforms which make them a useful tool for finding solutions of PDEs (i.e., differentiation is carried to a multiplication operator, and multiplication of transforms corresponds to convolution). The idea that the regularity of a function increases the rate of decay of its Fourier transform (and vice versa), is brought out. Although, this is typically regarded as an advanced topic, we treat it in an elementary way, and it is a close relative of the idea (covered in Section 3.2) that the smoothness of a function on a circle increases the rate of decay of its Fourier coefficients. Section 7.3 covers the Inversion Theorem, inverse Fourier transforms and Parseval's equality. But the proof of the Inversion Theorem is deferred to a supplement at the end of Chapter 7. In Section 7.4, Fourier transforms are applied to solving PDEs. One may wish to cover Sections 7.1 to 7.3 quickly and concentrate on Section 7.4. Here, we solve the heat problem on the infinite rod, and the Dirichlet problem for the half plane. We felt that it was a good idea to emphasize the fact that Fourier transform methods not only presume that a solution of a problem exists, but also that it has certain decay properties. Thus, integral formulas for solutions obtained in this fashion should be checked independently through a careful application of Leibniz's rule for differentiating under the integral. For a class of mostly engineers, this point can be made, without going through the details of the verification. Although a derivation of D'Alembert's formula for the wave equation is given in Chapter 5, we also show how to get it by Fourier transform techniques and the Dirac delta distribution is discussed. In Section 7.5, heat problems for semi-infinite and finite rods are solved via the method of images. The validity of formal infinite-sum solutions, found in Chapter 4, is now handled with ease. Also, Fourier sine and cosine transformations are introduced and applied.

Chapter 8 (Numerical Solutions of PDEs–An Introduction): While the solution of PDEs by numerical methods could constitute a whole course, we offer an introduction to the subject in Chapter 8. Our aim is not to present, without proof or motivation, a huge number of algorithms. Instead, we have concentrated on the foundations of the numerical approach, and we work mostly with the familiar heat equation to illustrate the nature and possible pitfalls of the numerical approach. To broaden the horizons, we do provide an optional overview of other numerical methods for other PDEs for the interested reader in Section 8.4. In Section 8.1, the "big O" notation is introduced. There we focus on Taylor's Theorem which plays a fundamental role in the approximation of partial derivatives by finite differences. This leads us to the approximation of PDE problems by a finite system of equations for the values of the unknown function at grid points. For the heat equation, these systems are easily solved by the explicit method in Section 8.2. Moreover, in the case of the heat equation, the discretization error (i.e., the difference of the numerical solution from the actual solution) is proved to approach zero as the grid point separation goes to zero, at least in the absence of round-off errors. In Section 8.3, we obtain exact solutions for a finite grid by means of the theory of difference equations. We then examine how systematic round-off errors lead to the conclusion that the best results are not always obtained by taking the grid size as small as possible. Continuing with the simple case of the heat equation, we obtain theoretical estimates for optimal grid sizes, which are born out to be correct in concrete examples. We believe that it is better to discuss in some depth a number of crucial issues for a single equation, than only briefly comment on a lot of PDEs and techniques. Again, Section 8.4 provides some overview and plenty of references for further study.

Chapter 9 (PDEs in Higher Dimensions): The fundamental ideas in Chapters 3 though 7 are extended in Section 9.1 in a straightforward manner to the case of several cartesian spatial coordinates. We solve heat problems on rectangles and cubes, and consider Laplace's equation on a solid rectangle. Double Fourier transforms and series are easily motivated and introduced. In Section 9.2, it is made clear that the primary objects from which solutions of the heat, wave and potential problems are constructed are the eigenfunctions of the Laplace operator which meet the B.C.. This basic fact is often hidden behind the process of separation of variable and the plethora of special functions which thereby arise in various coordinate systems. A great variety of series expansions for functions all fall into the category of eigenfunction expansions. In Section 9.2, we also prove a uniform convergence result for double Fourier series, and discuss simple properties of double Fourier transforms. In Section 9.3, we begin our study of the standard PDEs in terms of spherical coordinates. The spherical harmonics are defined as eigenfunctions of the Laplace operator on a sphere. They arise as the angular part of eigenfunctions of the Laplace operator on space and can be expressed through associated Legendre functions. We solve a number of heat and wave problems with spherical symmetry. The three-dimensional version of D'Alembert's formula is derived and Huygen's principle is discussed. The determination of all eigenvalues and spherical harmonics, dimensions of eigenspaces, etc. is covered in Section 9.4. There is a complete proof of the uniform convergence of the Laplace series for C^2 functions on a sphere. Moreover, a number of problems with angular dependence (e.g., heat flow in a ball) are solved through the use of spherical harmonics and spherical Bessel functions. In Section 9.5, we consider PDEs in cylindrical coordinate systems and some more PDEs in spherical coordinates, but with nontrivial potentials, such as Schrödinger's equation. The special functions which arise in the process are discussed. While spherical Bessel functions can be expressed in terms of sines and cosines, the cylindrical Bessel functions (of integer order) cannot, which is why we did not handle cylindrical coordinates before spherical ones. We consider a number of applications, ranging from the vibrating circular drum, to the determination of the energy levels and wave functions for the (nonrelativistic) hydrogen atom and the degeneracy of the energy levels which forms the basis for the periodic table. Section 9.6 deals with the standard heat, wave and potential problems on compact submanifolds with boundary in $\mathbf{R}^n$. Laplace operators are defined on these objects in an easily understood way. Although, we do not prove the existence theory for eigenfunctions and eigenvalues in this general setting, some of the more readable references are cited. Admittedly, the eigenfunctions are difficult to concretely compute or approximate, but once the eigenfunctions are given, the solution of the standard heat, wave and potential problems on manifolds proceeds in a way which is quite analogous to the many special cases covered in the rest of the book. This last section essentially unifies and consolidates these special cases into a single framework. Moreover, there is some discussion of Weyl's asymptotic formula for the eigenvalues of the Laplace operator, and the geometric information about the manifold which can be "heard" from the eigenvalues which may be interpreted as frequencies of vibration.

In constructing a one-semester or two-quarter course, we suggest selecting sections from the list below, keeping the indicated priorities in mind. In addition, Section 1.1 should be covered if your students are weak in ODEs. Sections which are marked with stars can or should be covered in only 2 hours, whereas most instructors will want to spend about 3 hours on the other sections. Leave time for tests and going over some of the homework. Chapters 8 and 9 are probably best left for a second semester or possibly as sources of projects for advanced, gifted and/or highly motivated students. In some schools where students have strong backgrounds or interests in computers one may wish to cover Chapter 8 in lieu of Chapter 7.

Crucial sections: 1.2, 1.3*, 3.1, 3.2, 4.1, 4.2, 4.3*, 5.1, 5.2, 6.1*, 6.2*, 6.3

Highly desirable sections: 2.1, 3.3*, 5.3*, 6.4, 6.5, 7.1*, 7.2*, 7.3*, 7.4

Luxury sections: 2.2, 2.3, 2.4, 3.4, 4.4, 7.5

Differences between the International Press and Van Nostrand editions

The main difference between the current edition and the Van Nostrand edition is that the book has now been reformatted in TeX to improve readability and appearance. In doing this, we have very nearly preserved the pagination. Most of the figures have now been "vectorized" so that they could be smoothly rescaled; the book is currently printed at 600 dpi instead of 300 dpi. In many places we have improved the clarity of the exposition. Numerous small errors and a few substantial errors have been corrected. Occasionally some informative remarks and additional problems have been added. While we are not aware of any errors that remain, we would very much appreciate hearing of any errors or suggestions for improvements (email to bleecker@math.hawaii.edu and/or george@math.hawaii.edu).

We owe a special thanks to PDE and geometry expert, Field's Medalist, colleague and editor S.T. Yau and coeditor Julie Lynch for having great confidence in the book. We have made every effort to insure that this confidence has not been misplaced.

Chapter 1

Review and Introduction

In this chapter, we review those aspects of ordinary differential equations (ODEs) which will be needed in the sequel. We also provide an overview of the applications of partial differential equations (PDEs), and introduce the reader to some elementary techniques, such as separation of variables. The review of ODEs in Section 1.1 is self-contained, since experience dictates that a remedial study of this material is often sorely needed. Even those whose mathematical knowledge of ODEs is sufficient may find the applied examples and problems (dealing with biology, fluid flow, electronics, mechanical vibrations, resonance, etc.) interesting and challenging. Section 1.2 gives the reader a perspective on the uses of PDEs in various scientific applications, such as gravitation, electrostatics, thermodynamics, acoustics, and minimal soap film surfaces. Some of the material (e.g., the use of Green's functions and integral operators), will not be universally appreciated upon a first reading. Indeed, students will find certain aspects (such as the superposition principle) of Section 1.2 more illuminating at later stages in their course of study. In Section 1.3, the studies of ODEs and PDEs are contrasted, with regard to the differences in the typical forms for general solutions. We illustrate how side conditions are used to extract particular solutions from general ones. Moreover, the method of separation of variables is also covered in this section.