

Asymptotic theory for differentially private generalized β -models with parameters increasing*

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Modelling edge weights play a crucial role in the analysis of network data, which reveals the extent of relationships among individuals. Due to the diversity of weight information, sharing these data has become a complicated challenge in a privacy-preserving way. In this paper, we consider the case of the non-denoising process to achieve the trade-off between privacy and weight information in the generalized β -model. Under the edge differential privacy with a discrete Laplace mechanism, the Z-estimators from estimating equations for the model parameters are shown to be consistent and asymptotically normally distributed. The simulations and a real data example are given to further support the theoretical results.

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1. INTRODUCTION

With the rapid development of computer and network technology, the analysis of network data has aroused widespread concerns in various fields. Unfortunately, collecting, storing, analyzing and sharing these data is challenging, due to the privacy of individuals (e.g., financial transactions). Besides, more privacy protection may reduce the validity of data [Duncan et al. (2004)]. Many approaches have been proposed to guarantee the trade-off between individual privacy and the utility of published data, which focus on data encryption, identity authentication, data perturbation [Samarati and Sweeney (1998); Fung et al. (2007); Machanavajjhala et al. (2006); Ghinita et al. (2008); Li et al. (2007); Aggarwal and Yu (2007)]. Dwork (2006a) proposed a rigorous notion of privacy named ϵ -differential privacy to control strong worst-case privacy risks. More formally, adding or removing

a single record in the dataset does not have a serious effect on the outcome of any analysis. Starting from Dwork (2006a), various types of data and queries were widely applied by researchers under differential privacy constraints [Holohan et al. (2017); McSherry and Talwar (2007); Wasserman and Zhou (2010)].

Random graphs are powerful statistical tools in the study of network data. These graph models are based on degree sequences d , which are used in modelling the realistic networks. In the undirected case, the β -model is well-known for the binary network, renamed by Chatterjee et al. (2011). Many scholars have focused on the studies of the β -model [Jackson (2008); Lauritzen (2008); Blitzstein and Diaconis (2011)]. Chatterjee et al. (2011) proved the existence and consistency of the maximum likelihood estimator (MLE) of the β -model as the number of parameters goes to infinity. Yan and Xu (2013) further derived its asymptotic normality. On the other hand, edge weights reveal the strength of relationships among individuals, which are critical for understanding many phenomena. For example, in friendship networks, we can assign close friends with a higher weight and acquaintances or normal friends with a lower weight, which are also referred to as the strong tie and weak tie reported by Granovetter (1993). To this point, Hillar et al. (2013) studied the maximum entropy distributions on weighted graphs with the β -model as the special case and proved the consistency of the MLE under the assumption that all parameters are bounded by a constant; Yan et al. (2015) proved the asymptotic normality of the MLE.

In the privacy analysis of network data, the raw data is published via pre-processing so that the confidential and sensitive information is captured as less as possible. One of the popular approaches is to add some noises e into the degree sequence d . For example, Hay et al. (2009) applied the Laplace noise-addition mechanism to release the degree partition of a graph, and designed to reduce the error with the ℓ_2 -norm between the true and released degree partitions. However, the process of adding noises is often ignored when summary statistics are published in a privacy-preserving way. As a result, the estimated parameters may not be consistent, even not exist [e.g., Hay et al. (2009)]. Duchi et al. (2018) illustrated that the estimator operated on private data has a larger error than the non-private estimator. Based on privatized data, estimating summary statistics and estimating parameters of models are totally different problems.

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To this point, [Karwa et al. \(2016\)](#) paid attention to the noise addition process through the denoised method to achieve valid inference and obtained the consistency and asymptotic normality of a differential privacy estimator in the β -model.

In this paper, we adopt the non-denoised method to establish the asymptotic properties of the Z-estimator of the parameter in the generalized β -model with finite discrete weighted edges under the discrete Laplace mechanism, which is different from the work of [Karwa et al. \(2016\)](#). Moreover, [Karwa et al. \(2016\)](#) only considered binary edges. In some scenarios, edge weights play important roles in the analysis of network data. For example, weighted social networks often provide a more realistic representation of the complex social interactions among individuals than binary networks [e.g., [Farine \(2014\)](#)]. Furthermore, edge weights may further increase the risk of privacy disclosure, due to the diversity of weight-related information. For instance, edge weights represent the numbers of co-written papers in a coauthorship network. A hacker can easily identify an author via the total number of published papers [[Li et al. \(2016\)](#)]. In the generalized β -model, each node is assigned one parameter, so the number of parameters increases with n . The asymptotic properties for the increasing dimensional Z-estimator cannot directly be followed from the classical Z-estimation theory; see chapter 5 of [van der Vaart \(1998\)](#). Therefore, based on [Yan and Xu \(2013\)](#), we alternatively show that the Z-estimator of the parameter involving the noisy degree sequence is asymptotically consistent and normally distributed in the generalized β -model under edge differential privacy constraints.

The organization of this paper is as follows. In Section 2, we first introduce some notations and definitions of our results. Subsequently, we obtain the asymptotic normality of the Z-estimator in the generalized β -model involving noisy degree sequence $\bar{\mathbf{d}} = \mathbf{d} + \mathbf{e}$, where \mathbf{d} is the sufficient statistic and \mathbf{e} are some noises from the discrete Laplace distribution. In Section 3, we give some simulation results to support our theories. We further present a data example application, which is from a community of 27 Grevy's zebras. A summary follows in Section 4. All proofs are contained in Appendix.

2. MAIN RESULTS

2.1 Notations

For a vector $\mathbf{x} = (x_1, \dots, x_n)^T \in R^n$, the ℓ_∞ -norm of \mathbf{x} is denoted by $\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$. For an $n \times n$ matrix $J = (J_{ij})$, $\|J\|_\infty$ denotes the matrix norm induced by the $\|\cdot\|_\infty$ -norm on vectors in R^n :

$$\|J\|_\infty = \max_{\mathbf{x} \neq 0} \frac{\|J\mathbf{x}\|_\infty}{\|\mathbf{x}\|_\infty} = \max_{1 \leq i \leq n} \sum_{j=1}^n |J_{ij}|,$$

i.e., the maximum absolute row sum norm.

We define another matrix norm $\|\cdot\|$ for a matrix $A = (a_{i,j})$ by $\|A\| := \max_{i,j} |a_{i,j}|$, and let $\|\mathbf{x}\|_1 = \sum_i |x_i|$ be the

ℓ_1 -norm for a general vector \mathbf{x} . We say that $a_n = \Omega(r_n)$ if there exists a real constant $c > 0$ and there exists an integer constant $n_0 \geq 1$ such that $a_n \geq cr_n$ for every $n \geq n_0$.

Let D be an open convex subset of R^n . An $n \times n$ function matrix $G(\mathbf{x})$ whose elements $G_{ij}(\mathbf{x})$ are functions on a vector \mathbf{x} , is Lipschitz continuous w.r.t the max norm on D if there exists a real number κ such that for any $\mathbf{v} \in R^n$ and any $\mathbf{x}, \mathbf{y} \in D$,

$$\|G(\mathbf{x})(\mathbf{v}) - G(\mathbf{y})(\mathbf{v})\|_\infty \leq \kappa \|\mathbf{x} - \mathbf{y}\|_\infty \|\mathbf{v}\|_\infty,$$

where κ may depend on n but it is independent of \mathbf{x} and \mathbf{y} . For every fixed n , κ is a constant.

2.2 Edge differential privacy

In the contexts of network data, there are two main variants of differential privacy: edge differential privacy (EDP) [[Nissim et al. \(2007\)](#)] and node differential privacy (NDP) [[Hay et al. \(2009\)](#), [Kasiviswanathan et al. \(2013\)](#)], which are based on the different definitions of graph neighbors. Specifically, EDP guarantees that released databases do not reveal the addition or removal of a special edge, while NDP hides the addition or removal of a node (along with its edges) in a graph G . In this paper, we refer to EDP, where two graphs G and G' are said to be neighbors if they differ in exactly one edge.

Definition 2.1 (Edge differential privacy). Let $\varepsilon \geq 0$ be a privacy parameter. A randomized mechanism (or a family of conditional probability distributions) $\mathcal{Q}(\cdot|G)$ is ε -edge differentially private if

$$\sup_{G, G' \in \mathcal{G}, \delta(G, G')=1} \sup_{S \in \mathcal{S}} \frac{\mathcal{Q}(S|G)}{\mathcal{Q}(S|G')} \leq e^\varepsilon,$$

where \mathcal{G} is the set of all undirected graphs of interest on n nodes, $\delta(G, G')$ is the number of edges on which G and G' differ, \mathcal{S} is the set of all possible outputs (or the support of \mathcal{Q}).

The above definition of EDP is based on ratios of probabilities. Generally, the data curator chooses an appropriate privacy parameter ε to achieve the trade-off between privacy and validity. As the value of ε is extremely small, more privacy is protected. Under EDP constraints, changing one record in the dataset cannot affect seriously on the distribution of the output. For example, a hospital can release some medical information about their patients to the public, while simultaneously ensuring very high levels of privacy in the case of EDP. This is because EDP offers a guarantee no matter whether or not the patient participates in the study, the probability of a possible output is almost the same. As a result, an attacker can not find whether a single individual is in the original database or not. As we know, the effective implementation of ε -differential privacy is associated with the magnitude of additional random noise.

To this end, [Dwork et al. \(2006b\)](#) introduced the notion of global sensitivity, which is referred to as the maximum ℓ_1 -norm among various dataset pairs (G, G') .

Definition 2.2 (Global sensitivity). Let $f : \mathcal{G} \rightarrow \mathcal{R}^k$. The global sensitivity of f is defined as

$$\Delta_G(f) = \max_{\delta(G, G')=1} \|f(G) - f(G')\|_1,$$

where $\|\cdot\|_1$ is the ℓ_1 -norm for vector.

Although there are many mechanisms for releasing the output of any function f under differential privacy, the Laplace mechanism is the most common one. [Karwa et al. \(2016\)](#) presented a discrete Laplace mechanism to achieve edge differential privacy, which is given below.

Let $f : \mathcal{G} \rightarrow \mathcal{Z}^k$, and let Z_1, \dots, Z_k be independent and identically distributed discrete Laplace random variables with probability mass function defined by

$$P(Z = z) = \frac{1 - \lambda}{1 + \lambda} \lambda^{|z|}, z \in \mathcal{Z}, \lambda \in (0, 1).$$

Then the algorithm which outputs $f(G) + (Z_1, \dots, Z_k)$ with inputs G is ε -edge differentially private, where $\varepsilon = -\Delta_G(f) \log \lambda$.

Based on the definition of differential privacy, [Dwork et al. \(2006b\)](#) found that any function of a differentially private mechanism is also differentially private, as follow: Let f be an output of an ε -differentially private mechanism and g be any function. Then $g(f(G))$ is also ε -differentially private. This result indicates that any post-processing done on the noisy degree sequence obtained as an output of a differentially private mechanism is also differentially private.

More generally, we may consider the *skew discrete Laplace mechanism*. When the positive noises and negative noises arising with different probability law, the skew discrete Laplace distribution [[Kozubowski and Inusah \(2006\)](#)] as a discretization of non-symmetric Laplace distribution could be used. The skew Laplace distribution is useful in applications to communications, engineering, and finance and economics, see [Kotz et al. \(2012\)](#) and references therein. For more information on the *skew discrete Laplace mechanism*, see the supplementary material for details.

2.3 Estimation

Let \mathcal{G}_n be a simple undirected graph including n nodes. Let a_{ij} be the weight of edge (i, j) , $1 \leq i \neq j \leq n$, taking values from the set $\{0, 1, \dots, q-1\}$. Let $A = (a_{ij})$ be the adjacency matrix of \mathcal{G}_n . Note that \mathcal{G}_n has no self-loops, $a_{ii} = 0$. Define $d_i = \sum_{j \neq i} a_{ij}$ and $\mathbf{d} = (d_1, \dots, d_n)^T$ as the degree sequence of \mathcal{G}_n . The density or probability mass function on \mathcal{G}_n with respect to some canonical measure ν has the exponential-family random graph models with the degree sequence as sufficient statistic, i.e.,

$$p(\mathcal{G}_n; \boldsymbol{\alpha}) = \exp(\boldsymbol{\alpha}^T \mathbf{d} - Z(\boldsymbol{\alpha})),$$

where $Z(\boldsymbol{\alpha})$ is the normalizing constant, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)^T$ is a vector parameter.

We assume that the edge weights $\{a_{ij}\}$ are independently multinomial random variables with the probability mass function:

$$(1) \quad P(a_{ij} = a) = \frac{e^{a(\alpha_i + \alpha_j)}}{\sum_{k=0}^{q-1} e^{k(\alpha_i + \alpha_j)}}, \quad a = 0, 1, \dots, q-1.$$

where $q \geq 2$ is a fixed number of the class. Thus the likelihood of $\boldsymbol{\alpha}$ is

$$L(\boldsymbol{\alpha}) \propto \prod_{j \neq i} \prod_{a=0}^{q-1} [P(a_{ij} = a)]^{1(a_{ij}=a)},$$

which gives the log-likelihood of $\boldsymbol{\alpha}$,

$$\log L(\boldsymbol{\alpha}) \propto \sum_{j \neq i} \sum_{a=0}^{q-1} \left[1(a_{ij} = a) \cdot \left\{ a(\alpha_i + \alpha_j) - \log \left(\sum_{k=0}^{q-1} e^{k(\alpha_i + \alpha_j)} \right) \right\} \right].$$

So the log-partition function in (1) is given by $Z(\boldsymbol{\alpha}) = \sum_{j \neq i} \log \sum_{k=0}^{q-1} e^{k(\alpha_i + \alpha_j)}$. This model is a direct generalization of the β -model, which only considers the dichotomous edges.

Moreover, the first order condition for the log-likelihood function w.r.t. α_i are

$$\frac{\partial \log L(\boldsymbol{\alpha})}{\partial \alpha_i} = \sum_{j \neq i} a_{ij} - \sum_{j=1; j \neq i} \sum_{a=0}^{q-1} \frac{ae^{a(\alpha_i + \alpha_j)}}{\sum_{k=0}^{q-1} e^{k(\alpha_i + \alpha_j)}}, i = 1, \dots, n.$$

Let e_i be a noise independently drawn from discrete Laplace distribution with parameter λ_n . We output $\bar{d}_i := d_i + e_i$ using the discrete Laplace mechanism. However, the degree $d_i = \sum_{j \neq i} a_{ij}$ of vertex i is not attainable, since the observed degree contains unknown noise in private data set. We resort to the moment equations which are given by the following system of functions:

$$F_i(\boldsymbol{\alpha}) = \bar{d}_i - E(\bar{d}_i) = \bar{d}_i - E(d_i), \quad i = 1, \dots, n,$$

$$F(\boldsymbol{\alpha}) = (F_1(\boldsymbol{\alpha}), \dots, F_n(\boldsymbol{\alpha}))^T.$$

Under this case, since adding or removing an edge can change the degree of at most two nodes, by 1 each, the global sensitivity for the degree sequence \mathbf{d} is 2. Therefore, we have the privacy parameter

$$\varepsilon_n := -\Delta_G(f) \log \lambda_n = -2 \log \lambda_n.$$

So, $\lambda_n = \exp(-\frac{\varepsilon_n}{2})$.

We use $\hat{\boldsymbol{\alpha}}$ to denote the Z-estimator of $\boldsymbol{\alpha}$ satisfying $F(\hat{\boldsymbol{\alpha}}) = 0$. Since the noises e_i 's ($i = 1, 2, \dots, n$) are independently drawn from symmetric discrete Laplace distribution

with parameter λ_n , $E(e_i) = 0$. Note that d_i is a sum of edge weights a_{ij} 's ($j = 1, \dots, n, j \neq i$). So we have

$$\begin{aligned} E(\bar{d}_i) &= E(d_i + e_i) = E(d_i) = \sum_{j=1; j \neq i}^n E(a_{ij}) \\ &= \sum_{j=1; j \neq i}^n \sum_{k=0}^{q-1} kP(a_{ij} = k) = \sum_{j=1; j \neq i}^n \sum_{k=0}^{q-1} \frac{ke^{k(\alpha_i + \alpha_j)}}{\sum_{k=0}^{q-1} e^{k(\alpha_i + \alpha_j)}}. \end{aligned}$$

Therefore the moment-based estimating equations with noisy degree sequences are

$$(2) \quad \bar{d}_i = \sum_{j=1; j \neq i}^n \sum_{a=0}^{q-1} \frac{ae^{a(\hat{\alpha}_i + \hat{\alpha}_j)}}{\sum_{k=0}^{q-1} e^{k(\hat{\alpha}_i + \hat{\alpha}_j)}}, \quad i = 1, \dots, n.$$

2.4 Consistency and asymptotical normality

In this section, we obtain that the Z-estimator of the parameter involving noisy degree sequence is asymptotically consistent and normally distributed.

Given $m, M > 0$, we say an $n \times n$ matrix $V_n = (v_{ij})$ belongs to the matrix class $\mathcal{L}_n(m, M)$ if V_n is a symmetric nonnegative matrix satisfying

$$v_{ii} = \sum_{j=1, j \neq i}^n v_{ij}; \quad M \geq v_{ij} = v_{ji} \geq m > 0, i \neq j.$$

Generally, the inverse of V_n , V_n^{-1} , does not have a closed form. Yan and Xu (2013) proposed a simple matrix $\bar{S} = (\bar{s}_{ij})$ to approximate V_n^{-1} , where $\bar{s}_{ij} = \frac{\delta_{ij}}{v_{ii}} - \frac{1}{v_{..}}$, δ_{ij} is the Kronecker delta function, and $v_{..} = \sum_{i,j}^n (1 - \delta_{ij})v_{ij} = \sum_{i=1}^n v_{ii}$.

Similar to Yan et al. (2016b), let the parameter vector $\alpha = (\alpha_1, \dots, \alpha_n)^T$ belong to the symmetric parameter space

$$D = \{\alpha \in R^n : -Q_n \leq \alpha_i + \alpha_j \leq Q_n, 1 \leq i < j \leq n\},$$

where $\{Q_n\}$ is the sequence of upper bound of the parameters. Let $F'(\alpha)$ be the Jacobian matrix of $F(\alpha)$ at α , then for $i, j = 1, \dots, n$,

$$\begin{aligned} \frac{\partial F_i}{\partial \alpha_i} &= \sum_{j=1; j \neq i}^n \frac{\sum_{0 \leq k < l \leq q-1} (k-l)^2 e^{(k+l)(\alpha_i + \alpha_j)}}{(\sum_{a=0}^{q-1} e^{a(\alpha_i + \alpha_j)})^2}, \\ \frac{\partial F_i}{\partial \alpha_j} &= \frac{\sum_{0 \leq k < l \leq q-1} (k-l)^2 e^{(k+l)(\alpha_i + \alpha_j)}}{(\sum_{a=0}^{q-1} e^{a(\alpha_i + \alpha_j)})^2}, \quad j = 1, \dots, n; j \neq i. \end{aligned}$$

Here we need assume $V_n := F'(\alpha) \in \mathcal{L}_n(m, M)$, i.e., $v_{ii} = \frac{\partial F_i}{\partial \alpha_i}$ and $v_{ij} = \frac{\partial F_i}{\partial \alpha_j}$, where

$$(3) \quad m = (2(1 + e^{Q_n}))^{-1} \text{ and } M = \frac{q^2}{2}.$$

First, we give the convergence rate of the ℓ_∞ error which directly shows the consistency of the parameters under some mild conditions.

Theorem 2.1. Consider the discrete Laplace mechanism with $\lambda_n = \exp(-\epsilon_n/2)$ and assume that $\alpha \in D$ and $e^{Q_n} = o((n/\log n)^{\frac{1}{12}})$, where $D = \{\alpha \in R^n : -Q_n \leq \alpha_i + \alpha_j \leq Q_n, \text{ for } 1 \leq i < j \leq n\}$. If $\epsilon_n \geq c\sqrt{\log n}$ (denoted as $\epsilon_n = \Omega(\sqrt{\log n})$), where $c \geq 4$ is a constant, then as n goes to infinity, $\hat{\alpha}$ exists and satisfies

$$(4) \quad \|\hat{\alpha} - \alpha\|_\infty = O_p(e^{3Q_n} \sqrt{\frac{\log n}{n}}) = o_p(1).$$

Remark 2.1. In Theorem 2.1, we use Newton's method to obtain the existence and consistency of $\hat{\alpha}$. This indicates that the Z-estimator of the parameter α involving a noisy sequence is accurate under the non-denoised process. If Q_n is bounded and thus α is a sparse vector, this convergence matches the minimax optimal upper bound $\|\hat{\beta} - \beta^*\|_\infty = O_p(\sqrt{\frac{\log p}{n}})$ for the Lasso estimator in the linear model with $p = n - 1$ -dimensional true parameter vector β^* and the sample size n , see Lounici (2008).

Second, we get the asymptotic normality of the estimator in the restricted parameter space under the slower rate condition for e^{Q_n} compared with the rate for e^{Q_n} in Theorem 2.1, as follow.

Theorem 2.2. Consider the discrete Laplace mechanism with $\lambda_n = \exp(-\epsilon_n/2)$ and assume that the conditions in Theorem 2.1 hold. If we assign a smaller $e^{Q_n} = o(n^{1/18}/(\log n)^{1/9})$, then as n goes to infinity, for any fixed $r \geq 1$, the vector

$$(v_{11}^{1/2}(\hat{\alpha}_1 - \alpha_1), \dots, v_{rr}^{1/2}(\hat{\alpha}_r - \alpha_r)) \xrightarrow{d} N_r(\mathbf{0}, \mathbf{I}_r)$$

where \mathbf{I}_r is a $r \times r$ identity matrix.

Remark 2.2. By Theorem 2.2, for any fixed i , the convergence rate of $\hat{\alpha}_i$ is $1/v_{ii}^{1/2}$, when $\epsilon_n = \Omega(\sqrt{\log n})$. Since $(n-1)(2(1 + e^{Q_n}))^{-1} \leq v_{ii} \leq (n-1)q^2/2$, the convergence rate is between $O(n^{-1/2}e^{Q_n/2})$ and $O(n^{-1/2})$, which is the same as the non-privacy estimator [Yan et al. (2016b)].

The proofs of Theorems 2.1 and 2.2 are postponed in the Appendix section. After deriving the theoretical results, numerical studies are carried out in the next section to verify the asymptotic properties of the Z-estimate. Theorem 2.2 can be also used to construct a confidence interval for the parameters. For instance, an approximate $1 - \alpha$ confidence interval for $\alpha_i - \alpha_j$ is $\hat{\alpha}_i - \hat{\alpha}_j \pm Z_{1-\alpha/2}(1/\hat{v}_{ii} + 1/\hat{v}_{jj})^{1/2}$, where $Z_{1-\alpha/2}$ is the $1 - \alpha$ -quantile of the standard normal distribution, \hat{v}_{ii} and \hat{v}_{jj} are the Z-estimates of v_{ii} and v_{jj} by replacing all α_i with their Z-estimates.

3. NUMERICAL STUDIES

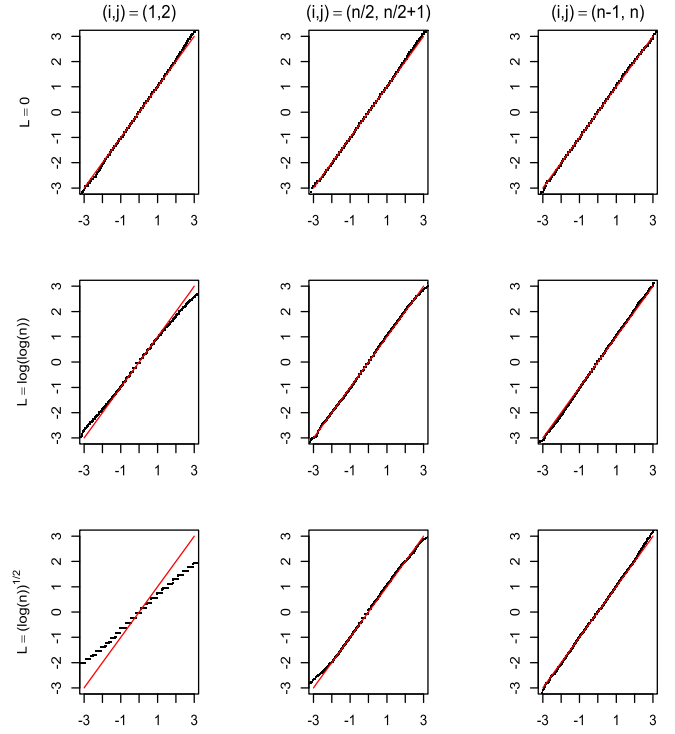
3.1 Simulations

We first consider simulations under a discrete weight $q = 3$. In this case, we evaluate asymptotic properties by simulating finite sample data in finite networks. We consider the changes of n, ε and L . Based on Yan and Leng (2015) and Yan et al. (2016a), the setting of the true parameter vector α^* takes a linear form. Specifically, we set $\alpha_i^* = (n - i + 1)L/n$, for $i = 1, \dots, n$. We discuss three distinct values for L , $L = 0, \log(\log n), (\log n)^{1/2}$, respectively. We simulate three distinct values for ε : one is fixed ($\varepsilon = 2$) and the other two values tend to zero with n , i.e., $\varepsilon = \log(n)/n^{1/4}, \log(n)/n^{1/2}$. Here we discuss three values for n , $n = 100, 200$ and 500 . Each simulation is repeated 10,000 times.

By Theorem 2.2, $\hat{\xi}_{ij} = [\hat{\alpha}_i - \hat{\alpha}_j - (\alpha_i^* - \alpha_j^*)]/(1/\hat{v}_{ii} + 1/\hat{v}_{jj})^{1/2}$ converges to the standard normal distribution, where \hat{v}_{ii} is the estimator of v_{ii} by replacing α_i^* with $\hat{\alpha}_i$. Hence, we apply the quantile-quantile (QQ) plot to demonstrate the asymptotic normality of $\hat{\xi}_{ij}$. Three special pairs $(1, 2), (n/2, n/2 + 1)$ and $(n - 1, n)$ for (i, j) are presented in Figure 1. Further, we list the coverage probability of the 95% confidence interval, the length of the confidence interval, and the frequency that the estimate does not exist.

For $\varepsilon = 2, \log(n)/n^{1/4}$, the QQ-plots under $n = 100, 200$ and 500 are similar. Thus, we here only show the QQ-plots for $\hat{\xi}_{ij}$ under the case of $\varepsilon = 2$ and $n = 100$ in Figure 1 to save space. In Figure 1, the horizontal and vertical axes are the theoretical and empirical quantiles, respectively, and the red lines correspond to the reference lines $y = x$. From Figure 1, we see that for fixed pair $(i, j) = (1, 2)$, the empirical quantiles coincide well with the ones of the standard normality for noisy estimates (i.e., $\hat{\xi}_{ij}$) expect for $L = (\log n)^{1/2}$. When $L = (\log n)^{1/2}$, notable deviations exist for pair $(1, 2)$ in Figure 1. For other pairs $(n/2, n/2 + 1)$ and $(n - 1, n)$, the approximation of asymptotic normality is good when $L = 0, \log(\log n), (\log n)^{1/2}$. While $\varepsilon = \log(n)/n^{1/2}$, the approximation of asymptotic normality respecting to $\hat{\xi}_{ij}$ is bad, see Figure 1 of the supplementary material.

The coverage probability of the 95% confidence interval for $\alpha_i^* - \alpha_j^*$, the length of the confidence interval, and the frequency that the estimate does not exist, are reported in Table 1. The length of the confidence interval is related to L and n . That is, the length increases as L increases, or the length decreases as n increases. Under the case of $\varepsilon = 2, \log(n)/n^{1/4}$, the coverage frequencies of pair $(1, 2)$ are higher than the nominal level 95% expect for $L = 0$; for other pairs $(n/2, n/2 + 1)$ and $(n - 1, n)$, the coverage frequencies are all close to the nominal level 95% for all L , where the ones are the closest at $n = 500$. For $\varepsilon = \log(n)/n^{1/2}$, the coverage frequencies are lower than the nominal level 95% for all L . This indicates that as ε reduces to a specific value (e.g., $\log(n)/n^{1/2}$), notable deviations exist between



(a) finite discrete weights ($q = 3$)

Figure 1. The QQ plots ($n = 100, \varepsilon = 2$).

the coverage frequencies and the nominal level 95%, especially the probabilities of the non-existent estimates are very high when $L = (\log n)^{1/2}$.

Second, we compare the simulation results between with the denoising process [Karwa et al. (2016)] and without the denoising process in the case of $q = 2$. Here the settings of $\alpha^*, \bar{\alpha}^*, L$ and ε are the same as those in the first simulation. Here, we only consider $n = 100, 200$ without 500. Each simulation is repeated 10,000 times.

According to the results in Karwa et al. (2016), $\bar{\xi}_{ij} = [\bar{\alpha}_i - \bar{\alpha}_j - (\alpha_i^* - \alpha_j^*)]/(1/\bar{v}_{ii} + 1/\bar{v}_{jj})^{1/2}$ converges to the standard normal distributions, where $\bar{\alpha}_i$ is the estimate of α_i with the denoising process and \bar{v}_{ii} is the estimate of v_{ii} by replacing α_i with $\bar{\alpha}_i$. We apply the quantile-quantile (QQ) plot and record the coverage probability of the 95% confidence interval, the length of confidence interval, and the frequency that the estimate does not exist, to compare the performance of $\hat{\xi}_{ij}$ and $\bar{\xi}_{ij}$. The QQ-plots are shown in Figure 2 and numeric comparison results are given in Table 2. In Figure 2, the QQ-plots for both $\hat{\xi}_{ij}$ denoted by the red color and $\bar{\xi}_{ij}$ denoted by the blue color are very close and coincide well with the ones of the standard normality when $\varepsilon = 2, \log(n)/n^{1/4}$ and $L \leq \log(\log n)$. (We only show the QQ-plots of $\varepsilon = 2$ and $n = 100$ in Figure 2 to save space and the other cases are similar.) This indicates that the parameter estimates are nearly the same with and without the

Table 1. Estimated coverage probabilities ($\times 100\%$) of $\alpha_i^* - \alpha_j^*$ for pair (i, j) as well as the length of confidence intervals (in square brackets), and the probabilities ($\times 100\%$) that the estimate does not exist (in parentheses)

n	(i, j)	$L = 0$	$L = \log(\log n)$	$L = (\log n)^{1/2}$
$\epsilon = 2$				
100	(1, 2)	94.63[0.35](0)	96.75[0.81](0.41)	99.75[1.13](31.79)
	(50, 51)	94.80[0.35](0)	94.79[0.55](0.41)	95.07[0.73](31.79)
	(99, 100)	94.90[0.35](0)	94.040.41	94.55[0.46](31.79)
200	(1, 2)	94.40[0.25](0)	97.68[0.62](0)	99.86[0.85](6.42)
	(50, 51)	94.51[0.25](0)	94.54[0.41](0)	95.74[0.54](6.42)
	(99, 100)	94.44[0.25](0)	94.45[0.30](0)	94.92[0.33](6.42)
500	(1, 2)	95.17[0.16](0)	98.62[0.42](0)	99.98[0.57](0.03)
	(50, 51)	94.79[0.16](0)	94.67[0.27](0)	96.81[0.36](0.03)
	(99, 100)	95.05[0.16](0)	94.91[0.19](0)	95.20[0.21](0.03)
$\epsilon = \log(n)/n^{1/4}$				
100	(1, 2)	94.41[0.35](0)	95.80[0.81](1.42)	99.48[1.12](51.83)
	(50, 51)	94.40[0.35](0)	94.07[0.56](1.42)	93.96[0.73](51.83)
	(99, 100)	94.39[0.35](0)	93.66[0.41](1.42)	94.12[0.46](51.83)
200	(1, 2)	94.40[0.25](0)	96.90[0.62](0.03)	99.70[0.84](17.92)
	(50, 51)	94.34[0.25](0)	94.10[0.41](0.03)	95.13[0.54](17.92)
	(99, 100)	94.24[0.25](0)	94.09[0.30](0.03)	94.42[0.33](17.92)
500	(1, 2)	95.03[0.16](0)	98.30[0.42](0)	99.94[0.57](0.73)
	(50, 51)	94.60[0.16](0)	94.53[0.27](0)	96.40[0.36](0.73)
	(99, 100)	94.92[0.16](0)	94.82[0.19](0)	95.00[0.21](0.73)
$\epsilon = \log(n)/n^{1/2}$				
100	(1, 2)	88.65[0.35](0)	83.92[0.82](61.94)	87.50[1.03](99.60)
	(50, 51)	88.84[0.35](0)	81.56[0.57](61.94)	77.50[0.76](99.60)
	(99, 100)	88.00[0.35](0)	84.73[0.42](61.94)	67.50[0.47](99.60)
200	(1, 2)	90.03[0.25](0)	81.25[0.63](40.95)	86.21[0.78](99.71)
	(50, 51)	89.57[0.25](0)	81.69[0.42](40.95)	72.41[0.53](99.71)
	(99, 100)	89.27[0.25](0)	87.33[0.30](40.95)	93.10[0.33](99.71)
500	(1, 2)	91.52[0.16](0)	83.64[0.43](12.78)	93.55[0.56](99.38)
	(50, 51)	91.38[0.16](0)	84.41[0.28](12.78)	75.81[0.37](99.38)
	(99, 100)	91.59[0.16](0)	89.46[0.19](12.78)	87.10[0.21](99.38)

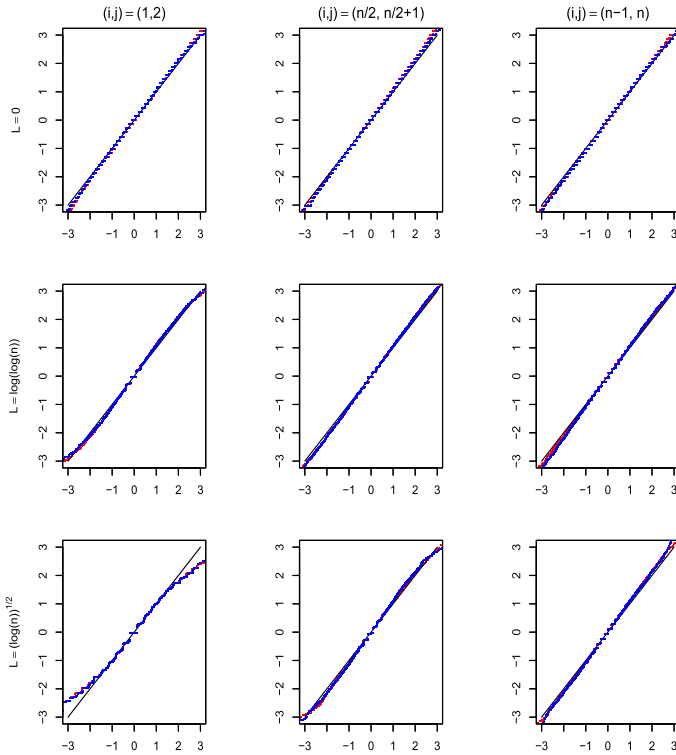
denoising process. However, when $\epsilon = \log(n)/n^{1/2}$, the approximation of asymptotic normality of both $\hat{\xi}_{ij}$ and $\bar{\xi}_{ij}$ is not good, see Figure 2 of the supplementary material.

In Table 2, Type ‘‘A’’ and ‘‘B’’ represent the estimates without and with the denoised process, respectively. From this table, we can see that the difference between both estimates is very small. Similar to the analysis of Table 1, the length of confidence interval increases as L increases and decreases as n increases. Under the case of $\epsilon = 2, \log(n)/n^{1/4}$, the coverage frequencies of all pairs are all close to the nominal level 95% when $L = 0, \log(\log n)$; for $L = (\log n)^{1/2}$, both the non-denoised and denoised estimates often failed to exist for $n = 100$, while $n = 200$ the non-existent frequencies of estimates are lower. For $\epsilon = \log(n)/n^{1/2}$, the coverage frequencies for both non-denoised and denoised estimates exist a great gap compared with the nominal level

95% for all L , and the probabilities of the non-existent estimates also increase as L increases.

3.2 Real data example

We use the affiliation network dataset in Sundaresan et al. (2007) as a data example. As discussed in Haratym (2017), it remains an interesting issue that the animals should have some sort of privacy rights. In some ways, society has already begun moving in that direction. This network dataset is based on a study of a community of 28 Grevy’s zebras. Sundaresan et al. (2007) showed that Grevy’s zebra individuals are more selective in their choices of associates, tending to form bonds with others in the same reproductive state. In the dataset, Grevy’s zebras are labelled from 1 to 28, and 111 edges with finite weight $q = 3$. The edge weight of 0 denotes



(a) finite discrete weights ($q = 2$)

Figure 2. The QQ plots of the non-denoised and denoised estimates ($n = 100, \epsilon = 2$).

that a pair of zebras never appeared during the study, the edge weight of 1 denotes that a pair of zebras appeared together at least once, while the edge weight of 2 indicates a statistically significant tendency of pairs to appear together. On the other hand, the estimate $\hat{\alpha}$ does not exist when the degree of a vertex is zero. Hence we removed the vertex 8 whose degree is zero before analysis. The network with the left 27 vertices is shown in Figure 3. We chose the privacy parameter ϵ as 1. Figure 4 reports the scatter plots of noisy degree sequence $\bar{\mathbf{d}}$ vs the estimates $\hat{\alpha}$ for the 27 Grevey’s zebra dataset. From Figure 4, the value of $\hat{\alpha}$ increases as the number of $\bar{\mathbf{d}}$ increases. Furthermore, the estimates can reveal a trend in these zebras’ choices of associates. The larger the estimates $\hat{\alpha}$, the more zebras have associates or the higher the frequency of pairs to appear together. As shown in Figure 4, the number of zebras’ associates and the frequency of pairs to appear together are more and higher under the case that $\hat{\alpha}$ is around zero. Table 3 reports the estimates, the 95% confidence interval, the corresponding standard errors and the noisy degree sequence. In Table 3, the larger estimates correspond to the larger noisy degrees. The largest degree is 23 for vertex 4, which also has the largest estimate 0.447. On the other hand, the vertex for 22 with the smallest estimate -2.260 , has degree 2 in Table 3.

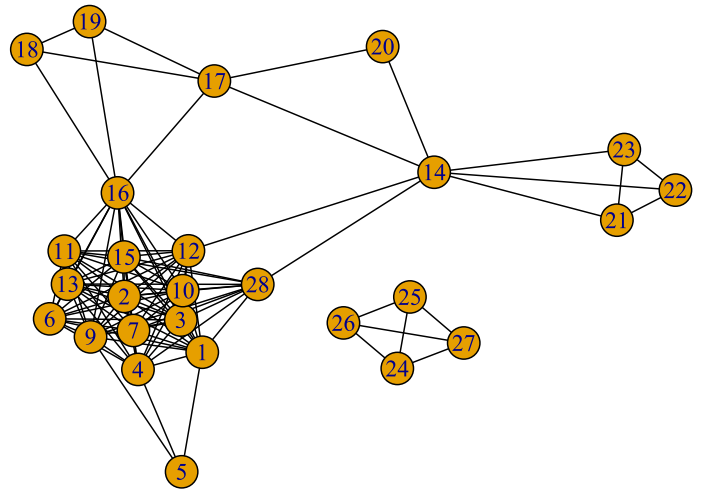
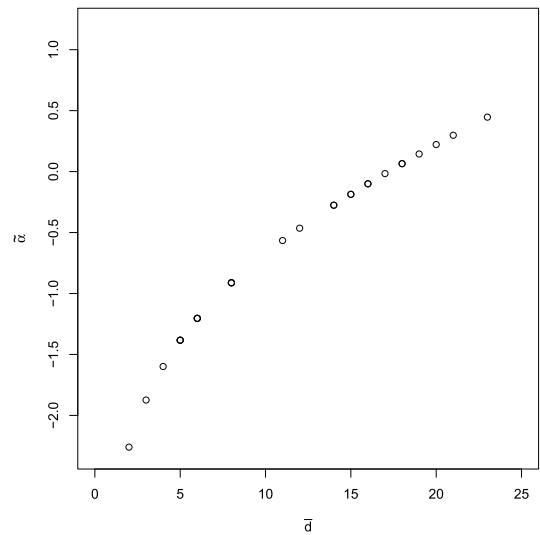


Figure 3. the Sundaresan, Fischhoff, Dushoff, Rubenstein Zebra Affiliation network becomes a community network of 27 Greavey’s zebras with 111 edges, after removing the isolated vertex 8. There are 3 edge weights representing the trend of a pair of zebras appeared together during the study: 0 means none; 1 means at least once; 2 means very significantly.



(a) $n = 27, \epsilon = 1$.

Figure 4. The scatter plots ($\bar{\mathbf{d}}$ vs $\hat{\alpha}$). The $\bar{\mathbf{d}}$ denotes the noisy degree sequence and $\hat{\alpha}$ denotes the corresponding estimate.

4. SUMMARY AND FUTURE STUDY

In this paper, we have established the uniform consistency and asymptotic normality of the Z-estimator in the generalized β -model involving noisy degree sequence $\bar{\mathbf{d}} = \mathbf{d} + \mathbf{e}$, where \mathbf{d} is the sufficient statistic and \mathbf{e} are some noises from discrete Laplace distribution. By using the Newton-Kantorovich theorem, we try to ignore adding noisy process, and obtain the existence and consistency of the Z-estimator

Table 2. Estimated coverage probabilities ($\times 100\%$) of $\alpha_i^* - \alpha_j^*$ for pair (i, j) as well as the length of confidence intervals (in square brackets), and the probabilities ($\times 100\%$) that the estimate does not exist (in parentheses). Type "A" denotes the estimate with the non-denoised process and "B" denotes the estimate with the denoised process

n	(i, j)	Type	$L = 0$	$L = \log(\log(n))$	$L = (\log n)^{1/2}$
$\epsilon = 2$					
100	(1,2)	A	93.62[0.57](0)	93.38[1.01](1.25)	97.38[1.46](43.88)
		B	93.79[0.57](0)	93.76[1.01](1.30)	97.35[1.47](44.17)
	(50,51)	A	93.44[0.57](0)	93.57[0.76](1.25)	93.16[0.94](43.88)
		B	93.62[0.57](0)	93.48[0.76](1.30)	93.19[0.94](44.17)
	(99,100)	A	93.42[0.57](0)	93.63[0.63](1.25)	93.16[0.68](43.88)
		B	93.82[0.57](0)	92.90[0.63](1.30)	93.43[0.68](44.17)
200	(1,2)	A	94.55[0.40](0)	93.80[0.75](0.03)	96.60[1.11](9.94)
		B	94.85[0.40](0)	94.04[0.75](0.03)	96.75[1.11](10.55)
	(100,101)	A	95.09[0.40](0)	94.28[0.55](0.03)	93.93[0.68](9.94)
		B	94.77[0.40](0)	94.58[0.55](0.03)	93.87[0.68](10.55)
	(199,200)	A	94.89[0.40](0)	94.20[0.45](0.03)	93.75[0.48](9.94)
		B	94.28[0.40](0)	94.23[0.45](0.03)	93.56[0.48](10.55)
$\epsilon = \log(n)/n^{1/4}$					
100	(1,2)	A	92.62[0.58](0)	91.31[1.02](4.46)	96.04[1.46](65.58)
		B	92.74[0.58](0)	91.74[1.02](5.14)	95.99[1.45](66.34)
	(50,51)	A	92.56[0.58](0)	91.88[0.76](4.46)	91.11[0.95](65.58)
		B	92.67[0.58](0)	92.00[0.76](5.14)	91.21[0.95](66.34)
	(99,100)	A	92.70[0.58](0)	92.58[0.63](4.46)	91.81[0.68](65.58)
		B	92.78[0.58](0)	91.79[0.64](5.14)	92.13[0.68](66.34)
200	(1,2)	A	94.14[0.40](0)	92.03[0.76](0.19)	95.34[1.12](26.08)
		B	94.31[0.40](0)	92.69[0.76](0.21)	95.00[1.12](26.44)
	(100,101)	A	94.72[0.40](0)	93.40[0.55](0.19)	92.48[0.68](26.08)
		B	94.21[0.40](0)	93.46[0.55](0.21)	92.62[0.68](26.44)
	(199,200)	A	94.46[0.40](0)	93.51[0.45](0.19)	93.06[0.48](26.08)
		B	93.92[0.40](0)	93.46[0.45](0.21)	92.71[0.48](26.44)
$\epsilon = \log(n)/n^{1/2}$					
100	(1,2)	A	79.34[0.58](0.24)	72.51[1.05](88.03)	100.00[1.44](99.97)
		B	78.94[0.58](0.24)	75.06[1.05](87.29)	66.67[1.27](99.97)
	(50,51)	A	78.51[0.58](0.24)	73.68[0.81](88.03)	100.00[0.87](99.97)
		B	79.20[0.58](0.24)	72.46[0.80](87.29)	100.00[0.82](99.97)
	(99,100)	A	78.81[0.58](0.24)	75.86[0.65](88.03)	100.00[0.70](99.97)
		B	78.73[0.58](0.24)	75.30[0.66](87.29)	100.00[0.67](99.97)
200	(1,2)	A	82.64[0.41](0)	69.26[0.80](70.75)	100.00[1.23](99.96)
		B	82.51[0.41](0)	70.91[0.79](71.16)	100.00[0.83](99.96)
	(50,51)	A	83.03[0.41](0)	74.50[0.57](70.75)	50.00[0.94](99.96)
		B	82.74[0.41](0)	75.24[0.57](71.16)	50.00[0.64](99.96)
	(99,100)	A	82.66[0.41](0)	77.88[0.45](70.75)	100.00[0.50](99.96)
		B	82.54[0.41](0)	79.96[0.45](71.16)	50.00[0.47](99.96)

Table 3. Sundaesan, Fischhoff, Dushoff, Rubenstein Zebra Affiliation network dataset: the estimate $\hat{\alpha}$, 95% confidence intervals (in square brackets), and their standard errors (in parentheses)

Vertex	$\hat{\alpha}_i$	Degree	Vertex	$\hat{\alpha}_i$	Degree
$\epsilon = 1$					
1	0.065[-0.477, 0.606](0.276)	18	16	-0.186[-0.751, 0.379](0.288)	15
2	0.298[-0.229, 0.825](0.269)	21	17	-1.204[-1.994, -0.414](0.403)	6
3	-0.276[-0.851, 0.300](0.294)	14	18	-1.383[-2.242, -0.524](0.438)	5
4	0.447[-0.075, 0.968](0.266)	23	19	-1.599[-2.554, -0.643](0.488)	4
5	-0.912[-1.611, -0.213](0.356)	8	20	-1.204[-1.994, -0.414](0.403)	6
6	-0.186[-0.751, 0.379](0.288)	15	21	-1.383[-2.242, -0.524](0.438)	5
7	-0.276[-0.851, 0.300](0.294)	14	22	-2.260[-3.611, -0.910](0.689)	2
9	0.065[-0.477, 0.606](0.276)	18	23	-0.912[-1.611, -0.213](0.356)	8
10	0.144[-0.391, 0.680](0.273)	19	24	-1.874[-2.975, -0.773](0.562)	3
11	-0.100[-0.656, 0.456](0.284)	16	25	-0.464[-1.067, 0.138](0.307)	12
12	-0.016[-0.565, 0.532](0.280)	17	26	-1.204[-1.994, -0.414](0.403)	6
13	-0.100[-0.656, 0.456](0.284)	16	27	-0.912[-1.611, -0.213](0.356)	8
14	-1.383[-2.242, -0.524](0.438)	5	28	-0.566[-1.186, 0.054](0.316)	11
15	0.222[-0.309, 0.753](0.271)	20			

satisfying the equation (2). Furthermore, we give some simulation results to illustrate the authenticity of our obtained results under a non-denoised process. Although the simulation results in Section 3 show that the approximation of asymptotic normality behaves well under certain conditions, our theoretical results may be improved but not only in the generalized β -model.

It should be noted that the discrete Laplace random variable is the difference of two i.i.d. geometric distributed random variables, see Proposition 3.1 in Inusah and Kozubowski (2006). The geometric distribution as the class of infinitely divisible distribution is a special case of discrete compound Poisson distribution. The difference of geometric noise-addition mechanism can be flexibly extended to the difference between two i.i.d. (or independent) discrete compound Poisson random variables, see Definition 4.2 of Zhang et al. (2014). In fact, the difference of two independent discrete compound Poisson random variables follows the infinitely divisible distributions with integer support, see Chapter IV of Steutel and van Harn (2003). The frequently employed discrete Laplace noise $\{e_i\}$ in differential privacy, may be further optimally selected from other flexible discrete distributions to achieve effective privacy protection in the future.

APPENDIX: PROOFS

A.1 Proof of Theorem 2.1

The proof of Theorem 2.1 is based on two steps.

Step 1: we need two lemmas below.

Lemma A.1. Consider the discrete Laplace mechanism with $\lambda_n = \exp(-\epsilon_n/2)$, if $\epsilon_n = \Omega(\sqrt{\log n})$ and $e^{Q_n} =$

$o((n/\log n)^{\frac{1}{2}})$, then as n goes to infinity, for any fixed $r \geq 1$,

$$\left(\frac{\bar{d}_1 - E(d_1)}{\sqrt{v_{11}}}, \dots, \frac{\bar{d}_r - E(d_r)}{\sqrt{v_{rr}}}\right) \xrightarrow{d} N_r(\mathbf{0}, \mathbf{I}_r).$$

Lemma A.1 indicates that the components of $(\bar{d}_1 - E(d_1), \dots, \bar{d}_r - E(d_r))$ are asymptotically independent and normally distributed with variances v_{11}, \dots, v_{rr} , respectively.

Lemma A.2 (Karwa et al. (2016), Proposition E). Let e_1, \dots, e_n be i.i.d random variables drawn from discrete Laplace distribution with probability mass function defined by

$$P(e_i = e) = \frac{1 - \lambda}{1 + \lambda} \lambda^{|e|}, e \in \mathcal{Z}, \lambda \in (0, 1).$$

Then we have $E(e_i) = 0$ and $\text{Var}(e_i) = \frac{2\lambda}{(1 - \lambda)^2}$. Moreover,

$$P(|e_i| > c) = \frac{2\lambda^{c+1}}{1 + \lambda}, \quad E|e_i| = \frac{2\lambda}{1 - \lambda^2};$$

$$P(\max_i |e_i| > c) = 1 - \left(1 - \frac{2\lambda^{c+1}}{1 + \lambda}\right)^n.$$

Proof of Lemma A.1. By $\bar{d}_i = d_i + e_i$, we can analysis the asymptotic normality of Lemma A.1 in two parts, i.e., $(d_i - E(d_i))/v_{ii}^{1/2}$ and $(e_i - E(e_i))/v_{ii}^{1/2}$. On the one hand, Yan et al. (2015) have verified the result of the first part by Liapounov's central limit theorem [Chung (2001)]. On the other hand, we can easily obtain the stochastic order of the second part by Chebyshev inequality.

Let $\bar{d}_i = d_i + e_i$, $i = 1, \dots, r$, then

$$(5) \quad \frac{\bar{d}_i - E(d_i)}{\sqrt{v_{ii}}} = \frac{d_i - E(d_i)}{\sqrt{v_{ii}}} + \frac{e_i}{\sqrt{v_{ii}}}, \quad i = 1, \dots, r.$$

Now, we only discuss the property of $\frac{e_i}{\sqrt{v_{ii}}}$. In fact, by Chebyshev inequality and Lemma A.2, for any constant $a > 0$, as n goes to infinity, we have

$$\begin{aligned} P\left(\left|\frac{e_i}{\sqrt{v_{ii}}}\right| \geq a\right) &= P(|e_i| \geq a\sqrt{v_{ii}}) \leq \frac{\text{Var}(e_i)}{a^2 v_{ii}} \\ \text{[by (3)]} &\leq \frac{2(1 + e^{Q_n})}{a^2(n-1)} \times \frac{2e^{-\frac{\varepsilon_n}{2}}}{(1 - e^{-\frac{\varepsilon_n}{2}})^2} \\ &\leq O\left(\frac{e^{Q_n}}{n}\right) \rightarrow 0. \end{aligned}$$

by noticing that $e^{-\frac{\varepsilon_n}{2}} < 1$ for all n . \square

Step 2: we apply the Newton-Kantorovich theorem [Gragg and Tapia (1974)] to obtain the existence and consistency of the estimator satisfying the equation (2).

For a subset $C \subset R^n$, let C^0 and \bar{C} denote the interior and closure of C in R^n , respectively. Let $\Omega(\mathbf{x}, r)$ denote the open ball $\{\mathbf{y} : \|\mathbf{y} - \mathbf{x}\| < r\}$, and $\bar{\Omega}(\mathbf{x}, r)$ be its closure.

Proposition A.1 (Gragg and Tapia (1974)). Let $F(\mathbf{x}) = (F_1(\mathbf{x}), \dots, F_n(\mathbf{x}))^T$ be a function vector on $\mathbf{x} \in R^n$. Assume that the Jacobian matrix $F'(\mathbf{x})$ is Lipschitz continuous on an open convex set D with the Lipschitz constant κ . Given $\mathbf{x}_0 \in D$, assume that $[F'(\mathbf{x}_0)]^{-1}$ exists,

$$\begin{aligned} \|[F'(\mathbf{x}_0)]^{-1}\|_\infty &\leq \aleph, \\ \|[F'(\mathbf{x}_0)]^{-1}F'(\mathbf{x}_0)\|_\infty &\leq \delta, h = 2\aleph\kappa\delta \leq 1, \\ \Omega(\mathbf{x}_0, t^*) \subset D^0, t^* &:= \frac{2}{h}(1 - \sqrt{1-h})\delta = \frac{2}{1 + \sqrt{1-h}}\delta \leq 2\delta, \end{aligned}$$

where \aleph and δ are positive constants that may depend on \mathbf{x}_0 and the dimension n of \mathbf{x}_0 . Then the Newton iteration $\mathbf{x}_{k+1} = \mathbf{x}_k - [F'(\mathbf{x}_k)]^{-1}F(\mathbf{x}_k)$ exists and $\mathbf{x}_k \in \Omega(\mathbf{x}_0, t^*) \subset D^0$ for all $k \geq 0$; $\hat{\mathbf{x}} = \lim \mathbf{x}_k$ exists, $\hat{\mathbf{x}} \in \bar{\Omega}(\mathbf{x}_0, t^*) \subset D$ and $F(\hat{\mathbf{x}}) = 0$.

Besides, we also need the following four lemmas to prove Theorem 2.1. Specifically, Lemmas A.3–A.5 are served for Newton-Kantorovich theorem, and Lemma A.6 is based on Hoeffding's inequality and Lemma A.2.

Lemma A.3 (Yan and Xu (2013)). If $V_n \in \mathcal{L}_n(m, M)$, and n is large enough, then

$$\|V_n^{-1} - \bar{S}_n\| \leq \frac{cM^2}{m^3(n-1)^2},$$

where $(\bar{S}_n)_{ij} := \frac{\delta_{ij}}{v_{ii}} - \frac{1}{v_{..}}$, $v_{..} := \sum_{i=1}^n v_{ii}$, c is a constant that not depends on M, m and n .

To establish the form $\|[F'(\boldsymbol{\alpha})]^{-1}F(\boldsymbol{\alpha})\|_\infty \leq \delta$ in Theorem 2.1, we first use a simple matrix $S_n = (s_{ij})$ to approximate V_n^{-1} . The upper bound of the approximation error is given below.

Lemma A.4. Assume that $\boldsymbol{\alpha} \in D$, where $D = \{\boldsymbol{\alpha} \in R^n : -Q_n \leq \alpha_i + \alpha_j \leq Q_n, \text{ for } 1 \leq i < j \leq n\}$. If n is large enough, then

$$\|V_n^{-1} - S_n\| \leq \frac{c_1(1 + e^{Q_n})^3}{(n-1)^2},$$

where $V_n := F'(\boldsymbol{\alpha})$, $(S_n)_{ij} := \frac{\delta_{ij}}{v_{ii}}$, c_1 is a constant that not depends on e^{Q_n} and n .

Proof. By (3), $v_{..} = \sum_{i=1}^n v_{ii} \geq n(n-1)m$ and Lemma A.3, we can easily obtain

$$\begin{aligned} \|V_n^{-1} - S_n\| &\leq \|V_n^{-1} - \bar{S}_n\| + \|\bar{S}_n - S_n\| \\ &\leq \frac{cM^2}{m^3(n-1)^2} + \frac{1}{n(n-1)m} \\ &\leq \left(c + \frac{m^2}{M^2}\right) \frac{M^2}{m^3(n-1)^2} \\ &\leq (c+1) \frac{M^2}{m^3(n-1)^2} = \frac{c_1(1 + e^{Q_n})^3}{(n-1)^2}, \end{aligned}$$

where c_1 is a constant that not depends on e^{Q_n} and n . \square

To confirm the value of \aleph in Newton-Kantorovich theorem, we use triangle inequality and Lemma A.4 to obtain the upper bound of V_n^{-1} , as follow.

Lemma A.5. Assume that $\boldsymbol{\alpha} \in D$, where $D = \{\boldsymbol{\alpha} \in R^n : -Q_n \leq \alpha_i + \alpha_j \leq Q_n, \text{ for } 1 \leq i < j \leq n\}$. If n is large enough, then

$$\|V_n^{-1}\|_\infty \leq \frac{c_2(1 + e^{Q_n})^3}{n-1},$$

where $V_n := F'(\boldsymbol{\alpha})$, $(S_n)_{ij} := \frac{\delta_{ij}}{v_{ii}}$, c_2 is a constant that not depends on e^{Q_n} and n .

Proof. By (3) and Lemma A.4, we obtain

$$\begin{aligned} \|V_n^{-1}\|_\infty &\leq \|V_n^{-1} - S_n\|_\infty + \|S_n\|_\infty \\ &\leq \frac{c_1 n(1 + e^{Q_n})^3}{(n-1)^2} + \frac{2(1 + e^{Q_n})}{n-1} \\ &\leq \frac{c_2(1 + e^{Q_n})^3}{n-1}, \end{aligned}$$

where c_2 is a constant that not depends on e^{Q_n} and n . \square

The following lemma guarantees that the upper bound of $\|\bar{\mathbf{d}} - E(\mathbf{d})\|_\infty$ is the magnitude of $(n \log n)^{1/2}$.

Lemma A.6. Let $\kappa_n = 2(q-1)\sqrt{(n-1)\log(n-1)}$. If $\varepsilon_n = \Omega(\sqrt{\log n})$, then with probability approaching one as $n \rightarrow \infty$,

$$(6) \quad \max_{1 \leq i \leq n} |\bar{d}_i - E(d_i)| \leq 2(q-1)\sqrt{(n-1)\log(n-1)}.$$

Proof. Let $\kappa_n = 2(q-1)\sqrt{(n-1)\log(n-1)}$, then

$$(7) \quad \begin{aligned} & P(\max_i |\bar{d}_i - E(d_i)| \geq \kappa_n) \\ & \leq P(\max_i |d_i - E(d_i)| \geq \frac{\kappa_n}{2}) + P(\max_i |e_i| \geq \frac{\kappa_n}{2}). \end{aligned}$$

Here, the inequality (7) is divided into two parts and is given respective discussions. For the first part, we have

$$P(\max_i |d_i - E(d_i)| \geq \frac{\kappa_n}{2}) \leq \sum_i P(|d_i - E(d_i)| \geq \frac{\kappa_n}{2}).$$

By Hoeffding's inequality, it implies

$$(8) \quad \begin{aligned} & P(|d_i - E(d_i)| \geq \frac{\kappa_n}{2}) \\ & \leq 2 \exp\left(-\frac{2\left(\frac{\kappa_n}{2}\right)^2}{(n-1)(q-1)^2}\right) \\ & = 2 \exp\left(-\frac{2(q-1)^2(n-1)\log(n-1)}{(n-1)(q-1)^2}\right) \\ & = \frac{2}{(n-1)^2}. \end{aligned}$$

For the second part, by definition of maximum and $\lambda_n = \exp(-\epsilon_n/2)$, we obtain

$$\begin{aligned} P(\max_i |e_i| \geq \frac{\kappa_n}{2}) &= 1 - \prod_{i=1}^n P(|e_i| \leq \frac{\kappa_n}{2}) \\ (\text{By Lemma A.2}) &= 1 - \left(1 - \frac{2\lambda_n^{\lfloor \frac{\kappa_n}{2} \rfloor + 1}}{1 + \lambda_n}\right)^n \\ &= 1 - \left(1 - \frac{2e^{-\epsilon_n(\lfloor \frac{\kappa_n}{2} \rfloor + 1)/2}}{1 + e^{-\frac{\epsilon_n}{2}}}\right)^n. \end{aligned}$$

For $x \in (0, 1)$, $f(x) = 1 - (1-x)^n$ is an increasing function on x . Then we get

$$1 - \left(1 - \frac{2e^{-\epsilon_n(\lfloor \frac{\kappa_n}{2} \rfloor + 1)/2}}{1 + e^{-\frac{\epsilon_n}{2}}}\right)^n \leq 1 - (1 - 2e^{-\epsilon_n(\lfloor \frac{\kappa_n}{2} \rfloor + 1)/2})^n.$$

For $x \in (0, 1)$, $(1-x)^n \geq 1 - nx$. Then we also get

$$\begin{aligned} 1 - \left(1 - \frac{2e^{-\epsilon_n(\lfloor \frac{\kappa_n}{2} \rfloor + 1)/2}}{1 + e^{-\frac{\epsilon_n}{2}}}\right)^n &\leq 1 - (1 - 2ne^{-\epsilon_n(\lfloor \frac{\kappa_n}{2} \rfloor + 1)/2}) \\ &= 2ne^{-\epsilon_n(\lfloor \frac{\kappa_n}{2} \rfloor + 1)/2} \\ &\leq 2ne^{-\epsilon_n \lfloor \frac{\kappa_n}{2} \rfloor / 2}. \end{aligned}$$

By $\epsilon_n = \Omega(\sqrt{\log n})$, we obtain $\epsilon_n \geq c\sqrt{\log n} \geq c\sqrt{\frac{\log n}{n}}$, and thus $\epsilon_n \lfloor \frac{\kappa_n}{2} \rfloor \geq 4 \log n$. This implies

$$(9) \quad P(\max_i |e_i| \geq \frac{\kappa_n}{2}) \leq \frac{2n}{n^2}.$$

Therefore, by (8) and (9), with probability approaching one as $n \rightarrow \infty$, we have

$$\max_i |\bar{d}_i - E(d_i)| \leq 2(q-1)\sqrt{(n-1)\log(n-1)}. \quad \square$$

Proof of Theorem 2.1. In the Newton's iterative step, putting the initial value $\alpha^0 := \alpha$. Let $V_n = F'(\alpha) \in \mathcal{L}_n(m, M)$ and $W_n = V_n^{-1} - S_n$. Let $F(\alpha) = \bar{\mathbf{d}} - E(\mathbf{d})$, by Lemma A.4 and (6), we get

$$\begin{aligned} \|[F'(\alpha)]^{-1}F(\alpha)\|_\infty &\leq n\|W_n\|\|F(\alpha)\|_\infty + \max_i \frac{|F_i(\alpha)|}{v_{ii}} \\ &\leq (n\|W_n\| + \frac{1}{v_{ii}})\|F(\alpha)\|_\infty \\ &\leq \left(\frac{c_1n(1+e^{Q_n})^3}{(n-1)^2} + \frac{2(1+e^{Q_n})}{n-1}\right)\|F(\alpha)\|_\infty \\ &\leq \left(\frac{c_1n(1+e^{Q_n})^3}{(n-1)^2} + \frac{2(1+e^{Q_n})}{n-1}\right) \\ &\quad \times 2(q-1)\sqrt{(n-1)\log(n-1)} \\ &\leq c_3(1+e^{Q_n})^3\sqrt{\frac{\log(n-1)}{n-1}}, \end{aligned}$$

where c_3 is a constant.

Combining Lemma A.5, we can set $\delta = c_3(1+e^{Q_n})^3\sqrt{\frac{\log(n-1)}{n-1}}$ and $\aleph = \frac{c_2(1+e^{Q_n})^3}{n-1}$ in Newton-Kantorovich theorem.

Next, we indicate that the Jacobian matrix $F'(\alpha)$ is Lipschitz continuous with $\kappa = 4(q-1)^3(n-1)$. Here, our method is similar to Yan et al. (2016b).

Let

$$g_{ij}(\alpha) = \left(\frac{\partial^2 F_i}{\partial \alpha_1 \partial \alpha_j}, \dots, \frac{\partial^2 F_i}{\partial \alpha_n \partial \alpha_j}\right)^T.$$

By some computations, we have

$$\begin{aligned} \frac{\partial^2 F_i}{\partial \alpha_i^2} &= \sum_{j=1; j \neq i}^n \frac{\sum_{k \neq l, a} (k-l)^2 (k+l-2a) e^{(k+l+a)(\alpha_i + \alpha_j)}}{2(\sum_{a=0}^{q-1} e^{a(\alpha_i + \alpha_j)})^3}, \\ \frac{\partial^2 F_i}{\partial \alpha_j \alpha_i} &= \frac{\sum_{k \neq l, a} (k-l)^2 (k+l-2a) e^{(k+l+a)(\alpha_i + \alpha_j)}}{2(\sum_{a=0}^{q-1} e^{a(\alpha_i + \alpha_j)})^3}. \end{aligned}$$

As $\sum_{k \neq l, a} e^{(k+l+a)(\alpha_i + \alpha_j)} \leq (\sum_{a=0}^{q-1} e^{a(\alpha_i + \alpha_j)})^3$, we have

$$(10) \quad \left|\frac{\partial^2 F_i}{\partial \alpha_i^2}\right| \leq (q-1)^3(n-1), \quad \left|\frac{\partial^2 F_i}{\partial \alpha_j \alpha_i}\right| \leq (q-1)^3.$$

Thus $\|g_{ii}(\alpha)\|_1 \leq 2(n-1)(q-1)^3$.

If $i \neq j$ and $k \neq i, j$,

$$\frac{\partial^2 F_i}{\partial \alpha_k \partial \alpha_j} = 0.$$

Then we get $\|g_{ij}(\alpha)\|_1 \leq 2(q-1)^3, i \neq j$. Therefore, by the mean-value for vector-valued functions (Lang (1993)),

p. 341), for a vector \mathbf{v} ,

$$\begin{aligned} & \max_i \left\{ \sum_{j=1}^n \left[\frac{\partial F_i}{\partial \alpha_j}(\mathbf{x}) - \frac{\partial F_i}{\partial \alpha_j}(\mathbf{y}) \right] v_j \right\} \\ & \leq \|\mathbf{v}\|_\infty \max_i \sum_{j=1}^n \left| \frac{\partial F_i}{\partial \alpha_j}(\mathbf{x}) - \frac{\partial F_i}{\partial \alpha_j}(\mathbf{y}) \right| \\ & = \|\mathbf{v}\|_\infty \max_i \sum_{j=1}^n \left| \int_0^1 g_{ij}(t\mathbf{x} + (1-t)\mathbf{y})(\mathbf{x} - \mathbf{y}) dt \right| \\ & \leq 4(q-1)^3(n-1)\|\mathbf{v}\|_\infty\|\mathbf{x} - \mathbf{y}\|_\infty. \end{aligned}$$

Thereby, we have

$$\begin{aligned} h & = 2\aleph\kappa\delta \\ & = \frac{8c_2(1+e^{Q_n})^3}{n-1} \\ & \cdot (q-1)^3(n-1)c_3(1+e^{Q_n})^3\sqrt{\frac{\log(n-1)}{n-1}}, \\ & = 8c_2c_3(q-1)^3(1+e^{Q_n})^6\sqrt{\frac{\log(n-1)}{n-1}} \\ & = O(e^{6Q_n}\sqrt{\frac{\log n}{n}}) = o(1). \end{aligned}$$

Thus, all conditions in the Newton-Kantorovich theorem are satisfied. Since the inequality (6) holds with probability approaching one, then (4) is fulfilled. \square

A.2 Proof of Theorem 2.2

The aim of proving Theorem 2.2 is to establish the following equation

$$(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha})_i = [S_n(\bar{\mathbf{d}} - E(\mathbf{d}))]_i + o_p(n^{-1/2}).$$

This will follow directly from $[W_n\{\bar{\mathbf{d}} - E(\mathbf{d})\}]_i = o_p(n^{-1/2})$, $n \rightarrow \infty$, and by Theorem 2.1. To this end, we introduce the following lemma.

Lemma A.7. Assume that $\boldsymbol{\alpha} \in D$, where $D = \{\boldsymbol{\alpha} \in R^n : -Q_n \leq \alpha_i + \alpha_j \leq Q_n, \text{ for } 1 \leq i < j \leq n\}$. Let $\varepsilon_n = \Omega(\sqrt{\log n})$, $e^{Q_n} = o((n/\log n)^{\frac{1}{12}})$, and $U_n = \text{cov}[W_n\{\bar{\mathbf{d}} - E(\mathbf{d})\}]$. Then

$$[W_n\{\bar{\mathbf{d}} - E(\mathbf{d})\}]_i = o_p(n^{-1/2}).$$

Proof. Let $\bar{V}_n = \text{cov}\{\bar{\mathbf{d}} - E(\mathbf{d})\}$, $V_n = \text{cov}\{\mathbf{d} - E(\mathbf{d})\}$ and $E_n = \text{cov}\{\mathbf{e}\}$. For $1 \leq i \leq n$, the random variables d_i and e_i

are mutually independent, then

$$\begin{aligned} & \text{cov}(\bar{d}_i - E(d_i), \bar{d}_j - E(d_j)) \\ & = \text{cov}(d_i + e_i - E(d_i), d_j + e_j - E(d_j)) \\ & = \text{cov}(d_i - E(d_i), d_j + e_j - E(d_j)) \\ & \quad + \text{cov}(e_i, d_j + e_j - E(d_j)) \\ & = \text{cov}(d_i - E(d_i), d_j - E(d_j)) + \text{cov}(d_i - E(d_i), e_j) \\ & \quad + \text{cov}(e_i, d_j - E(d_j)) + \text{cov}(e_i, e_j) \\ & = \text{cov}(d_i - E(d_i), d_j - E(d_j)) + \text{cov}(e_i, e_j). \end{aligned}$$

Two cases are discussed:

Case 1. If $i \neq j$, then $\text{cov}(\bar{d}_i - E(d_i), \bar{d}_j - E(d_j)) = \text{cov}(d_i - E(d_i), d_j - E(d_j))$;

Case 2. If $i = j$, then $\text{cov}(\bar{d}_i - E(d_i), \bar{d}_j - E(d_j)) = \text{Var}(d_i - E(d_i)) + \text{Var}(e_i)$.

Thus the elements of the matrix \bar{V}_n are denoted by $\bar{v}_{ij} = v_{ij}$, $\bar{v}_{ii} = v_{ii} + \text{Var}(e_i)$, $1 \leq i \neq j \leq n$. Let $U_n = \text{cov}[W_n\{\bar{\mathbf{d}} - E(\mathbf{d})\}]$ with $W_n = V_n^{-1} - S_n$, then

$$\begin{aligned} U_n & = W_n \bar{V}_n W_n^T = W_n(V_n + E_n)W_n^T \\ & = W_n V_n W_n^T + W_n E_n W_n^T. \end{aligned}$$

On the one hand, $W_n V_n W_n^T = (V_n^{-1} - S_n) - S_n(I_n - V_n S_n)$, where $n \times n$ matrix I_n is an identity matrix.

By (3), we obtain

$$(11) \quad | \{S_n(I_n - V_n S_n)\}_{ij} | = \left| \frac{(\delta_{ij} - 1)v_{ij}}{v_{ii}v_{jj}} \right| \leq \frac{2q^2(1+e^{Q_n})^2}{(n-1)^2}.$$

By Lemma A.4 and (11), we have

$$\begin{aligned} \|W_n V_n W_n^T\| & \leq \frac{c_1(1+e^{Q_n})^3}{(n-1)^2} + \frac{2q^2(1+e^{Q_n})^2}{(n-1)^2} \\ & \leq O\left(\frac{e^{3Q_n}}{n^2}\right). \end{aligned}$$

On the other hand,

$$\begin{aligned} \|(W_n E_n W_n^T)_{ij}\| & = \left\| \sum_{k=1}^{n-1} w_{ik} e_k w_{kj} \right\| \\ & \leq \max_k |e_k| \sum_{k=1}^n |w_{ik}| |w_{kj}| \\ & \leq n \max_k |e_k| \|W_n\|^2 \\ & \leq \frac{2ne^{-\frac{\varepsilon}{2}}}{(1-e^{-\frac{\varepsilon}{2}})^2} \times \frac{c_1^2(1+e^{Q_n})^6}{(n-1)^4} \\ & \leq O\left(\frac{e^{6Q_n}}{n^3}\right). \end{aligned}$$

Hence, $\|U_n\| \leq O\left(\frac{e^{3Q_n}}{n^2}\right)$. Furthermore, by Chebyshev inequality, for any constant $a > 0$, we get

$$\begin{aligned} & P\left(\frac{[W_n\{\bar{\mathbf{d}} - E(\mathbf{d})\}]_i}{n^{-1/2}} \geq a\right) \\ &= P\left([W_n\{\bar{\mathbf{d}} - E(\mathbf{d})\}]_i \geq an^{-1/2}\right) \\ &\leq \frac{n[\text{cov}\{W_n(\bar{\mathbf{d}} - E(\mathbf{d}))\}]_i}{a^2} \leq O\left(\frac{e^{3Q_n}}{n}\right). \end{aligned}$$

Then while $e^{Q_n} = o((n/\log n)^{1/2})$,

$$P\left(\frac{[W_n\{\bar{\mathbf{d}} - E(\mathbf{d})\}]_i}{n^{-1/2}} \geq a\right) \rightarrow 0,$$

as $n \rightarrow \infty$. Therefore,

$$[W_n\{\bar{\mathbf{d}} - E(\mathbf{d})\}]_i = o_p(n^{-1/2}). \quad \square$$

Proof of Theorem 2.2. Let $\hat{r}_{ij} = \hat{\alpha}_i + \hat{\alpha}_j - \alpha_i - \alpha_j$. Under the conditions in Theorem 2.1, we have from the consistency property

$$\max_{i \neq j} |\hat{r}_{ij}| = O_p(e^{3Q_n} \sqrt{\frac{\log n}{n}}).$$

Let $u(t) = \sum_{a=0}^{q-1} \frac{ae^{at}}{\sum_{k=0}^{q-1} e^{kt}}$. For $i = 1, \dots, n$, by the Taylor's expansion, we get

$$\begin{aligned} \bar{d}_i - E(d_i) &= \sum_{j \neq i} (u(\hat{\alpha}_i + \hat{\alpha}_j) - u(\alpha_i + \alpha_j)) \\ &= \sum_{j \neq i} [u'(\alpha_i + \alpha_j)((\hat{\alpha}_i + \hat{\alpha}_j) - (\alpha_i + \alpha_j))] + h_i, \end{aligned}$$

where $h_i = (1/2) \sum_{j \neq i} u''(\hat{r}_{ij})[(\hat{\alpha}_i + \hat{\alpha}_j) - (\alpha_i + \alpha_j)]^2$ and $\hat{r}_{ij} = t_{ij}(\alpha_i + \alpha_j) + (1 - t_{ij})(\hat{\alpha}_i + \hat{\alpha}_j)$, $t_{ij} \in (0, 1)$. Writing the above expressions into a matrix, we have

$$\bar{\mathbf{d}} - E(\mathbf{d}) = V_n(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}) + \mathbf{h},$$

thus

$$\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha} = V_n^{-1}(\bar{\mathbf{d}} - E(\mathbf{d})) + V_n^{-1}\mathbf{h},$$

where $\mathbf{h} = (h_1, \dots, h_n)^T$.

By (10), we know $|h_i| \leq \frac{1}{2}(n-1)(q-1)^3 \hat{r}_{ij}^2$. Therefore,

$$\begin{aligned} |(V_n^{-1}\mathbf{h})_i| &= |(S_n\mathbf{h})_i| + |(W_n\mathbf{h})_i| \\ &\leq \max_i \frac{|h_i|}{v_{ii}} + \|W_n\| \sum_i |h_i| \leq O\left(e^{9Q_n} \frac{\log n}{n}\right), \end{aligned}$$

If $e^{Q_n} = o\left(\frac{n^{1/18}}{(\log n)^{1/9}}\right)$, then $(V_n^{-1}\mathbf{h})_i = o(n^{-1/2})$.

By Theorem 2.1 and Lemma A.7, for $i = 1, \dots, r$, we have

$$\begin{aligned} (\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha})_i &= [S_n(\bar{\mathbf{d}} - E(\mathbf{d}))]_i + o_p(n^{-1/2}) \\ &= \frac{\bar{d}_i - E(d_i)}{v_{ii}} + o_p(n^{-1/2}). \end{aligned}$$

Hence, Theorem 2.2 follows directly from Lemma A.1. Finally, we conclude the proof by multiplying $\sqrt{v_{ii}}$ to left and right of the last display. \square

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SUPPLEMENTARY MATERIAL

Supplement to ‘‘Asymptotic Theory for Differentially Private Generalized β -models with Parameters Increasing.’’ The supplementary material contains a brief introduction of *skew discrete Laplace mechanism* in Subsection 2.2, and the QQ plots of parameter estimates with $\epsilon = \log(n)/n^{1/2}$ in Subsection 3.1 (http://intlpress.com/site/pub/files/_supp/sii/2020/0013/0003/SII-2020-0013-0003-s003.pdf).

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