Supplementary Materials for "An L^2-norm based ANOVA test for the equality of weakly dependent functional time series"

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A Appendix: Technical proofs

Lemma A.1 is a natural result of the central limit theorem for functional time series developed by Horváth et al. (2013).

Lemma A.1. *Under Assumptions 1, 2 and the null hypothesis, as $n \to \infty$, we have $\mathbf{z}_n(t) \xrightarrow{d} \mathbf{z}(t) \sim GP_k\{\mathbf{0}, diag(c_1, \ldots, c_k)\}$. In particular, when Assumption 5 is satisfied, we have $\mathbf{z}_n(t) \xrightarrow{d} \mathbf{z}(t) \sim GP_k(\mathbf{0}, c\mathbf{I}_k)$.*

Proof of Lemma A.1. From Theorem 1 in Horváth et al. (2013), we know that $z_{ni}(t) = \sqrt{n_i}\{\bar{y}_{i\cdot}(t) - \mu_i(t)\} = \sqrt{n_i}\frac{1}{n_i}\sum_j (y_{ij} - \mu_i) = \frac{1}{\sqrt{n_i}}\sum_j \epsilon_{ij} \xrightarrow{d} \mathrm{GP}\{0, c_i(t,s)\}$, $1 \le i \le k$. Since different groups of samples are independent, we can easily get the conclusion. □

Proof of Lemma 1. Firstly, by Lemma A.1, the continuous mapping theorem for random elements taking values in a Hilbert space (Billingsley 1968, p.34; Cuevas et al. 2004), and the fact that $\mathbf{M}_n \to \mathbf{M} = \mathbf{I}_k - \mathbf{b}\mathbf{b}^\top$ with $\mathbf{b} = (\sqrt{\tau_1}, \ldots, \sqrt{\tau_k})^\top$, we have

$$T_n = \int_{\mathcal{T}} \mathbf{z}_n(t)^\top \mathbf{M}_n \mathbf{z}_n(t) \mathrm{d}t \xrightarrow{d} T_0 = \int_{\mathcal{T}} \mathbf{z}(t)^\top \mathbf{M} \mathbf{z}(t) \mathrm{d}t.$$

Let $\mathbf{z}(t) = [z_1(t), \ldots, z_n(t)]^\top$, then,

$$\begin{aligned}\mathrm{E}(T_0) &= \int_{\mathcal{T}} \mathrm{E}\{\mathbf{z}(t)^\top \mathbf{M}\mathbf{z}(t)\}\mathrm{d}t = \int_{\mathcal{T}} \mathrm{E}\{\sum_\alpha \sum_\beta m_{\alpha\beta} z_\alpha(t) z_\beta(t)\}\mathrm{d}t \\ &= \sum_\alpha m_{\alpha\alpha} \mathrm{tr}\{c_\alpha(s,t)\}.\end{aligned}$$

And,

$$\mathrm{Var}(T_0) = \mathrm{E}(T_0^2) - \mathrm{E}^2(T_0),$$

where

$$\begin{aligned}\mathrm{E}(T_0^2) &= \int_{\mathcal{T}}\int_{\mathcal{T}} \mathrm{E}\{\sum_\alpha \sum_\beta m_{\alpha\beta} z_\alpha(t) z_\beta(t)\}\{\sum_{\alpha_1}\sum_{\beta_1} m_{\alpha_1\beta_1} z_{\alpha_1}(s) z_{\beta_1}(s)\}\mathrm{d}t\mathrm{d}s \\ &= \int_{\mathcal{T}}\int_{\mathcal{T}} \Big[\mathrm{E}\sum_\alpha \{m_{\alpha\alpha}^2 z_\alpha^2(t) z_\alpha^2(s)\} + \mathrm{E}\sum_\alpha \sum_{\alpha_1 \neq \alpha} \{m_{\alpha\alpha} m_{\alpha_1\alpha_1} z_\alpha^2(t) z_{\alpha_1}^2(s)\} \\ &\quad + \mathrm{E}\sum_\alpha \sum_{\beta\neq\alpha}\sum_{\alpha_1}\sum_{\beta_1\neq\alpha_1} \{m_{\alpha\beta} m_{\alpha_1\beta_1} z_\alpha(t) z_\beta(t) z_{\alpha_1}(s) z_{\beta_1}(s)\}\Big]\mathrm{d}t\mathrm{d}s.\end{aligned}$$

The last term of $\mathrm{E}(T_0^2)$ are not equal to 0 only when $\alpha = \alpha_1$, $\beta = \beta_1$ or $\alpha = \beta_1$, $\alpha_1 = \beta$, so we have

$$\begin{aligned}&\int_{\mathcal{T}}\int_{\mathcal{T}} \mathrm{E}\sum_\alpha \sum_{\beta\neq\alpha}\sum_{\alpha_1}\sum_{\beta_1\neq\alpha_1} \{m_{\alpha\beta} m_{\alpha_1\beta_1} z_\alpha(t) z_\beta(t) z_{\alpha_1}(s) z_{\beta_1}(s)\}\mathrm{d}t\mathrm{d}s \\ =&2\sum_\alpha \sum_{\beta\neq\alpha} \int_{\mathcal{T}}\int_{\mathcal{T}} m_{\alpha\beta}^2 \mathrm{E}\{z_\alpha(t) z_\alpha(s)\}\mathrm{E}\{z_\beta(t) z_\beta(s)\}\mathrm{d}t\mathrm{d}s \\ =&2\sum_\alpha \sum_{\beta\neq\alpha} m_{\alpha\beta}^2 \mathrm{tr}(c_\alpha \otimes c_\beta).\end{aligned}$$

From Theorem 4.5(d) of Zhang (2013), we have

$$\int_{\mathcal{T}}\int_{\mathcal{T}} \sum_\alpha m_{\alpha\alpha}^2 \mathrm{E}\{z_\alpha^2(t) z_\alpha^2(s)\}\mathrm{d}t\mathrm{d}s = \sum_\alpha m_{\alpha\alpha}^2 \mathrm{tr}^2(c_\alpha) + 2\sum_\alpha m_{\alpha\alpha}^2 \mathrm{tr}(c_\alpha \otimes c_\alpha).$$

Then we can get

$$\mathrm{Var}(T_0) = 2\sum_\alpha \sum_\beta m_{\alpha\beta}^2 \mathrm{tr}(c_\alpha \otimes c_\beta).$$

$\square$

The following lemma shows that as long as the functional part can be extracted from the covariance function matrix of the pivotal test vector function $\boldsymbol{x}(t) = \mathbf{M}_n^{1/2}\mathbf{z}_n(t)$, the L^2-norm of the test statistic is a χ^2-type mixture plus a constant. This condition is satisfied under both multi-sample homoscedastic case and two-sample heteroscedastic case which allows us to derive the approximate null distribution of T_n.

Lemma A.2. *If $\boldsymbol{x}(t) \sim GP_k\{\boldsymbol{\mu}(t), f(s,t)\boldsymbol{\Sigma}\}$, $t \in \mathcal{T}$ with $\boldsymbol{\mu}(t) \in L^2(\mathcal{T})$, $f(s,t)$ is a function with finite trace, i.e., $tr(f) = \int_{\mathcal{T}} f(t,t)\,dt < \infty$, and $\boldsymbol{\Sigma}$ is a positive semi-definite matrix with $tr(\boldsymbol{\Sigma}) < \infty$, then the squared L^2-norm of $\boldsymbol{x}(t)$ can be expressed as*

$$\int_{\mathcal{T}} ||\boldsymbol{x}(t)||^2\,dt = \sum_{i=1}^{k}\int_{\mathcal{T}} x_i^2(t)\,dt = \sum_{i=1}^{k}\sum_{r=1}^{q}\vartheta_i\lambda_r A_{ir} + \sum_{i=1}^{k}\sum_{r=q+1}^{\infty}\delta_{ir}^2,$$

where $A_{ir} \sim \chi_1^2(\vartheta_i^{-1}\lambda_r^{-1}\delta_{ir}^2)$, $r = 1,\dots,q$, $i = 1,\dots,k$ are independent, ϑ_i, $i = 1,\dots,k$ are the eigenvalues of $\boldsymbol{\Sigma}$, λ_r, $r = 1,\dots,\infty$ are the decreasing-ordered eigenvalues of $f(s,t)$, $\delta_{ir} = \int_{\mathcal{T}} \mu_i^*(t)\phi_r(t)\,dt$, $i = 1,\dots,k$, $r = 1,\dots,\infty$, with $\mu_i^*(t)$ being the i-th entry of $\boldsymbol{\Gamma}^\top\boldsymbol{\mu}(t)$, $\phi_r(t)$, $r = 1,\dots,\infty$ being the associated eigenfunctions of $f(s,t)$ and the columns of $\boldsymbol{\Gamma}$ being the associated eigenvectors of $\boldsymbol{\Sigma}$, and q is the number of all the positive eigenvalues so that $\lambda_q > 0$ and $\lambda_r = 0$, $r > q$.

Proof of Lemma A.2. Since $\boldsymbol{\Sigma}$ is a positive semi-definite matrix, it has the following eigen-decomposition,

$$\boldsymbol{\Sigma} = \boldsymbol{\Gamma}\mathrm{diag}(\vartheta_1,\dots,\vartheta_k)\boldsymbol{\Gamma}^\top$$

where ϑ_i, $i = 1,\dots,k$, are the eigenvalues of $\boldsymbol{\Sigma}$ and the columns of $\boldsymbol{\Gamma}$ are the associated eigenvectors with $\boldsymbol{\Gamma}^\top\boldsymbol{\Gamma} = \mathbf{I}_k$.

Then

$$\boldsymbol{x}^*(t) = [x_1^*(t),\dots,x_k^*(t)]^\top = \boldsymbol{\Gamma}^\top\boldsymbol{x}(t) \sim \mathrm{GP}_k\{\boldsymbol{\Gamma}^\top\boldsymbol{\mu}(t), f(s,t)\mathrm{diag}(\vartheta_1,\dots,\vartheta_k)\}.$$

Note $f(s,t)$ has the following Karhunen-Loève decomposition:

$$f(s,t) = \sum_{r=1}^{q}\lambda_r\phi_r(s)\phi_r(t),$$

where $\lambda_1,\dots,\lambda_q$ are all the decreasingly ordered positive eigenvalues of $f(s,t)$, and $\phi_1(t),\dots,\phi_q(t)$ are the associated orthonormal eigenfunctions of $f(s,t)$ such that

$\int_{\mathcal{T}} \phi_r^2(t)dt = 1$, $\int_{\mathcal{T}} \phi_r(t)\phi_l(t)\mathrm{d}t = 0$, $r \neq l$, and q is the smallest integer such that when $r > q$, $\lambda_r = 0$. Then every entry of $\boldsymbol{x}^*(t)$ has the following Karhunen-Loève expansion:

$$x_i^*(t) = \mu_i^*(t) + \sum_{r=1}^{q} \xi_{ir}\phi_r(t),$$

where μ_i^* is the i-th entry of $\boldsymbol{\Gamma}^\top \boldsymbol{\mu}(t)$ and $\xi_{ir} =< x_i^* - \mu_i^*, \phi_r >$, $r = 1, \ldots, q \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \vartheta_i\lambda_r)$. We can obtain

$$\begin{aligned} ||x_i^*(t)||^2 &= ||\mu_i^*(t)||^2 + 2\sum_{r=1}^{q} \xi_{ir}\delta_{ir} + \sum_{r=1}^{q} \xi_{ir}^2 = \sum_{r=1}^{q}(\xi_{ir} + \delta_{ir})^2 + \sum_{r=q+1}^{\infty} \delta_{ir}^2 \\ &= \sum_{r=1}^{q} \vartheta_i\lambda_r A_{ir} + \sum_{r=q+1}^{\infty} \delta_{ir}^2, \end{aligned}$$

where $A_{ir} = (\xi_{ir} + \delta_{ir})^2/\vartheta_i\lambda_r \sim \chi_1^2(\vartheta_i^{-1}\lambda_r^{-1}\delta_{ir}^2)$, $r = 1, \ldots, q$ are independent as $(\xi_{ir} + \delta_{ir})/\sqrt{\vartheta_i\lambda_r} \sim \mathcal{N}(\delta_{ir}/\sqrt{\vartheta_i\lambda_r}, 1)$.

We conclude that

$$\begin{aligned} \int_{\mathcal{T}} ||\mathbf{x}(t)||^2 \mathrm{d}t = \int_{\mathcal{T}} ||\boldsymbol{\Gamma}^\top \mathbf{x}(t)||^2 \mathrm{d}t &= \sum_{i=1}^{k} \int_{\mathcal{T}} x_i^{*2}(t)\mathrm{d}t \\ &= \sum_{i=1}^{k}\sum_{r=1}^{q} \vartheta_i\lambda_r A_{ir} + \sum_{i=1}^{k}\sum_{r=q+1}^{\infty} \delta_{ir}^2. \end{aligned}$$

□

Proof of Theorem 1. Note that $\mathbf{I}_k - \mathbf{b}\mathbf{b}^\top$ is also an idempotent matrix and it has the following singular value decomposition:

$$\mathbf{I}_k - \mathbf{b}\mathbf{b}^\top = \mathbf{U} \begin{pmatrix} \mathbf{I}_{k-1} & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{pmatrix} \mathbf{U}^\top = \mathbf{U}(\mathbf{I}_{k-1}, \mathbf{0})^\top(\mathbf{I}_{k-1}, \mathbf{0})\mathbf{U}^\top,$$

where the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{I}_k - \mathbf{b}\mathbf{b}^\top$. Then according to the pivotal term of the limit test statistic

$$\boldsymbol{z}^*(t) = (\mathbf{I}_{k-1}, \mathbf{0})\mathbf{U}^\top \boldsymbol{z}(t) \sim \mathrm{GP}_{k-1}(\mathbf{0}, c\mathbf{I}_{k-1}).$$

From Lemma A.2, we can get

$$T_n \xrightarrow{d} \int_{\mathcal{T}} ||\boldsymbol{z}^*(t)||^2 \mathrm{d}t = \sum_{i=1}^{k-1}\sum_{r=1}^{q} \lambda_r A_{ir} = \sum_{r=1}^{q} \lambda_r A_r,$$

where $A_{ir} \sim \chi_1^2$, $r = 1, \ldots, q$, $i = 1, \ldots, k-1$ are independent and λ_r, $r = 1, \ldots, \infty$ are the decreasing-ordered eigenvalues of the common long run covariance $c(s,t)$, with q is the number of all the positive eigenvalues. $\square$

Proof of Theorem 2. Based on Theorem 2 in Horváth et al. (2013), we can get $\int_{\mathcal{T}} \int_{\mathcal{T}} \{\hat{c}(s,t) - c(s,t)\}^2 \mathrm{d}t\mathrm{d}s \xrightarrow{P} 0$. By the continuous mapping theorem for random elements taking values in a Hilbert space, it is easy to obtain $\mathrm{tr}(\hat{c}) \xrightarrow{P} \mathrm{tr}(c)$, $\mathrm{tr}(\hat{c}^{\otimes 2}) \xrightarrow{P} \mathrm{tr}(c^{\otimes 2})$ and $\mathrm{tr}^2(\hat{c}) \xrightarrow{P} \mathrm{tr}^2(c)$. Then this theorem follows immediately. $\square$

Proof of Theorem 3.

$$T_n = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}^\top \left(\mathbf{I}_2 - \mathbf{b}_n \mathbf{b}_n^\top / n\right) \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}^\top \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

where

$$\begin{aligned} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \left(\mathbf{I}_2 - \mathbf{b}_n \mathbf{b}_n^\top / n\right) \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \\ &\xrightarrow{d} \mathrm{GP}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \left(\mathbf{I}_2 - \mathbf{b}_n \mathbf{b}_n^\top / n\right) \begin{bmatrix} c_1(t,s) & 0 \\ 0 & c_2(t,s) \end{bmatrix} \left(\mathbf{I}_2 - \mathbf{b}_n \mathbf{b}_n^\top / n\right) \right\}, \end{aligned}$$

with

$$\begin{aligned} &\left(\mathbf{I}_2 - \mathbf{b}_n \mathbf{b}_n^\top / n\right) \begin{bmatrix} c_1(s,t) & 0 \\ 0 & c_2(s,t) \end{bmatrix} \left(\mathbf{I}_2 - \mathbf{b}_n \mathbf{b}_n^\top / n\right) \\ &= n_1 n_2 / n \begin{pmatrix} \frac{\sqrt{n_2}}{\sqrt{n}} \\ -\frac{\sqrt{n_1}}{\sqrt{n}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{n_2}}{\sqrt{n}} \\ -\frac{\sqrt{n_1}}{\sqrt{n}} \end{pmatrix}^\top \{c_1(s,t)/n_1 + c_2(s,t)/n_2\} \\ &= \boldsymbol{\Gamma} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \boldsymbol{\Gamma}^\top \{n_2 c_1(s,t)/n + n_1 c_2(s,t)/n\}, \end{aligned}$$

where the columns of $\boldsymbol{\Gamma}$ are the associated eigenvectors of $\begin{pmatrix} \frac{\sqrt{n_2}}{\sqrt{n}} \\ -\frac{\sqrt{n_1}}{\sqrt{n}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{n_2}}{\sqrt{n}} \\ -\frac{\sqrt{n_1}}{\sqrt{n}} \end{pmatrix}^\top$.

From Lemma A.2, we can get

$$T_0 = \sum_{i=1}^{q} \lambda_i A_i,$$

where $A_i \sim \chi_1^2$ and λ_i, $i = 1, \ldots, q$ are the eigenvalues of $n_2 c_1(s,t)/n + n_1 c_2(s,t)/n$. $\square$

Proof of Theorem 4. Under the alternative hypothesis (9), we have $\mathbf{z}_n(t) \xrightarrow{d} \mathbf{z}_1(t) \sim \mathrm{GP}_k\{\mathbf{d}(t), c\mathbf{I}_k\}$. Then we can get

$$T_n = \int_{\mathcal{T}} \mathbf{z}_n(t)^\top \mathbf{M}_n \mathbf{z}_n(t) \mathrm{d}t \xrightarrow{d} \int_{\mathcal{T}} \mathbf{z}_1(t)^\top \mathbf{M} \mathbf{z}_1(t) \mathrm{d}t,$$

and the pivotal term of the limit test statistic

$$\boldsymbol{z}_1^*(t) = (\mathbf{I}_{k-1}, \mathbf{0})\mathbf{U}^\top \boldsymbol{z}_1(t) \sim \mathrm{GP}_{k-1}\{(\mathbf{I}_{k-1}, \mathbf{0})\mathbf{U}^\top \mathbf{d}(t), c\mathbf{I}_{k-1}\}.$$

where $\mathbf{M}_n$, $\mathbf{M}$ and $\mathbf{U}$ are defined in Theorem 1. We denote $\mathbf{d}^*(t) = (\mathbf{I}_{k-1}, \mathbf{0})\mathbf{U}^\top \mathbf{d}(t) = [d_1^*(t), \ldots, d_k^*(t)]$.

From Lemma A.2, we can get

$$\begin{aligned} T_n \xrightarrow{d} \int_{\mathcal{T}} ||\boldsymbol{z}_1^*(t)||^2 \mathrm{d}t &= \sum_{i=1}^{k-1} \sum_{r=1}^{q} \lambda_r A_{ir} + \sum_{i=1}^{k-1} \sum_{r=q+1}^{\infty} \delta_{ir}^2 \\ &= \sum_{r=1}^{q} \lambda_r A_r + \sum_{r=q+1}^{\infty} \delta_r^2, \end{aligned}$$

where $A_r = \sum_{i=1}^{k-1} A_{ir} \sim \chi^2_{k-1, \lambda_r^{-1}\delta_r^2}$, $r = 1, \ldots, q$, $i = 1, \ldots, k-1$ are independent, λ_r, $r = 1, \ldots, \infty$ are the decreasing-ordered eigenvalues of the common long run covariance $c(s,t)$ with q being the number of all the positive eigenvalues so that $\lambda_q > 0$ and $\lambda_r = 0$, $r > q$, $\delta_r^2 = \sum_{i=1}^{k-1} \delta_{ir}^2 = || \int_{\mathcal{T}} (\mathbf{I}_{k-1}, \mathbf{0})\mathbf{U}^\top \mathbf{d}(t)\phi_r(t)\mathrm{d}t||^2$, $i = 1, \ldots, k-1$, $r = 1, \ldots, \infty$ with $\phi_r(t)$, $r = 1, \ldots, \infty$ being the associated eigenfunctions of the common long run covariance $c(s,t)$. □

Proof of Theorem 5. The test statistics

$$\begin{aligned} T_n &= \sum_{i=1}^{k} n_i \int_{\mathcal{T}} \{\hat{\mu}_i(t) - \hat{\mu}(t)\}^2 \mathrm{d}t \\ &= \sum_{i=1}^{k} n_i \int_{\mathcal{T}} \{\tilde{\mu}_i(t) - \bar{\tilde{\mu}}(t)\}^2 \mathrm{d}t - 2\sum_{i=1}^{k} n_i \int_{\mathcal{T}} \{\tilde{\mu}_i(t) - \bar{\tilde{\mu}}(t)\}\{\mu_i(t) - \bar{\mu}(t)\} \mathrm{d}t \\ &\quad + \sum_{i=1}^{k} n_i \int_{\mathcal{T}} \{\mu_i(t) - \bar{\mu}(t)\}^2 \mathrm{d}t \\ &:= T_{n0} - 2S_n + \sum_{i=1}^{k} n_i \int_{\mathcal{T}} \{\mu_i(t) - \bar{\mu}(t)\}^2 \mathrm{d}t, \end{aligned}$$

where $\tilde{\mu}_i(t) = \hat{\mu}_i(t) - \mu_i(t)$, $\bar{\tilde{\mu}}(t) = \frac{1}{n}\sum_{1 \le i \le k} n_i \tilde{\mu}_i(t)$, $\bar{\mu}(t) = \frac{1}{n}\sum_{1 \le i \le k} n_i \mu_i(t)$.

Under the alternative hypothesis H_{1n}, note $S_n = \sum_{i=1}^k n_i \int_{\mathcal{T}} \{\tilde{\mu}_i(t) - \bar{\tilde{\mu}}(t)\}\{\mu_i(t) - \bar{\mu}(t)\}\mathrm{d}t = \int_{\mathcal{T}} \begin{bmatrix} \sqrt{n_1}\tilde{\mu}_1(t) \\ \vdots \\ \sqrt{n_k}\tilde{\mu}_k(t) \end{bmatrix}^\top \mathbf{M}_n \begin{bmatrix} \sqrt{n_1}\mu_1(t) \\ \vdots \\ \sqrt{n_k}\mu_k(t) \end{bmatrix} \mathrm{d}t$, and $\begin{bmatrix} \sqrt{n_1}\tilde{\mu}_1(t) \\ \vdots \\ \sqrt{n_k}\tilde{\mu}_k(t) \end{bmatrix} \xrightarrow{d} GP_k\left[\mathbf{0}, \mathrm{diag}\{c_1(s,t), \ldots, c_k(s,t)\}\right]$.

Then $S_n \xrightarrow{d} \mathcal{N}\{0, \mathrm{Var}(S)\}$, where

$$\begin{aligned}
\mathrm{Var}(S) &= \mathrm{E}\left\{\int_{\mathcal{T}} \begin{bmatrix} \sqrt{n_1}\tilde{\mu}_1(t) \\ \vdots \\ \sqrt{n_k}\tilde{\mu}_k(t) \end{bmatrix}^\top \mathbf{M}_n \begin{bmatrix} \sqrt{n_1}\mu_1(t) \\ \vdots \\ \sqrt{n_k}\mu_k(t) \end{bmatrix} \mathrm{d}t\right\}^2 \\
&= \int_{\mathcal{T}}\int_{\mathcal{T}} \begin{bmatrix} \sqrt{n_1}\mu_1(s) \\ \vdots \\ \sqrt{n_k}\mu_k(s) \end{bmatrix}^\top \mathbf{M}_n \mathrm{diag}\{c_1(s,t), \ldots, c_k(s,t)\}\mathbf{M}_n \begin{bmatrix} \sqrt{n_1}\mu_1(t) \\ \vdots \\ \sqrt{n_k}\mu_k(t) \end{bmatrix} \mathrm{d}s\mathrm{d}t \\
&= \int_{\mathcal{T}}\int_{\mathcal{T}} \sum_{i=1}^k \tilde{d}_i(s) c_i(s,t) \tilde{d}_i(t) \mathrm{d}s\mathrm{d}t,
\end{aligned}$$

where $\tilde{d}_i$ is the i-th component of $\tilde{\mathbf{d}}(t) = \mathbf{M}_n \mathbf{d}(t)$, $i = 1, \ldots, k$.

Then the power is

$$\Pr\left(T_n \geq \hat{C}_\alpha\right) = \Pr\left[T_0 - 2S_n + \sum_{i=1}^k n_i \int_{\mathcal{T}} \{\mu_i(t) - \bar{\mu}(t)\}^2 \mathrm{d}t \geq \hat{C}_\alpha\right].$$

Based on Theorem 2, we have $\hat{C}_\alpha \xrightarrow{P} C_\alpha$ where C_α can be $\beta_1\chi^2_{d_1}$ or $\beta_2\chi^2_{d_2} + \beta_0$ and note $\sum_{i=1}^k n_i \int_{\mathcal{T}} \{\mu_i(t) - \bar{\mu}(t)\}^2 \mathrm{d}t = \sum_{i=1}^k \int_{\mathcal{T}} \tilde{d}_i^2(t)\mathrm{d}t = \delta^2$.

The power function is $\Pr(T_0 - 2S_n + \delta^2 \geq C_\alpha) + o(1)$. If $\delta^2 \to \infty$, we now show the above power function tends to 1. When $\mathrm{Var}(S) < \infty$, this is obviously true, and when $\mathrm{Var}(S) \to \infty$,

$$\begin{aligned}
& \Pr\left(S_n \leq \delta^2/2 - C_\alpha/2 + T_0/2\right) + o(1) \\
= \; & \Pr\left[S_n/\sqrt{\mathrm{Var}(S)} \leq \delta^2/\{2\sqrt{\mathrm{Var}(S)}\} - C_\alpha/\{2\sqrt{\mathrm{Var}(S)}\} + T_0/\{2\sqrt{\mathrm{Var}(S)}\}\right] + o(1).
\end{aligned}$$

Note $\sqrt{\mathrm{Var}(S)} = \sqrt{\sum_{i=1}^k \int_{\mathcal{T}}\int_{\mathcal{T}} \tilde{d}_i(s)c_i(s,t)\tilde{d}_i(t)\mathrm{d}s\mathrm{d}t} \leq \sqrt{\lambda_{\max}\sum_{i=1}^k \int_{\mathcal{T}} \tilde{d}_i^2(t)\mathrm{d}t}$, so $\delta^2/\{2\sqrt{\mathrm{Var}(S)}\} \longrightarrow \infty$ where λ_{max} is the largest eigenvalue among all the eigenvalues of the long run covariance function $c_i(s,t)$, $i = 1, \ldots, k$. Thus, we also have power function tends to 1. □

B Appendix: Additional simulations

In this section, we consider the data are observed with missing values and measurement errors by the following model:

$$y_{ij}(t) = \mu_i(t) + \epsilon_{ij}(t) + v_{ij}(t),$$

where $v_{ij}(t), j = 1, \ldots, n_i, i = 1, \ldots, k \overset{\text{i.i.d.}}{\sim} \mathcal{N}[0, \sigma_v(1+t)]$ represent measurement errors, and are independent of $\epsilon_{ij}(t), j = 1, \ldots, n_i, i = 1, \ldots, k$. The functional time series are again sampled discretely at $J = 100$ evenly spaced design time points within $\mathcal{T} = [0, 1]$, but we also randomly remove some design time points so there are about 90 design time points actually observed for each curve. The above settings are similar to those used in Zhang and Chen (2007). For the simulated functional time series above with missing values and measurement errors, we firstly use regression spline method (Ramsay and Silverman 2005, Ch.4; Zhang 2013, Ch.3) to reconstruct the curves, and then apply the tests to the reconstructed data. With all the other settings being the same as in the main paper, we repeat the two simulations represented in the main paper.

The results for the homoscedastic case (repeated Simulation HOM of the main paper) are presented in Tables 1–2, and the results for the heteroscedastic case (repeated Simulation HET of the main paper) are presented in Tables 3–4. It is seen from the results that missing values and measurement errors do have an effect on the performance, especially the powers, of these tests. However, the overall results are very similar to those presented in Tables 1–4 of the main paper, and the main conclusions are also the same.

References

Billingsley, P. (1968). *Convergence of probability measures.* John Wiley & Sons.

Cuevas, A., Febrero, M., and Fraiman, R. (2004). An anova test for functional data. *Computational Statistics & Data Analysis*, 47(1):111–122.

Horváth, L., Kokoszka, P., and Reeder, R. (2013). Estimation of the mean of functional time series and a two-sample problem. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 75(1):103–122.

Table 1: Empirical sizes (in percentages) of our L^2-norm based tests and the existing tests under homoscedasticity.

		Tests for dependent functional data											Tests for independent functional data										
		Homoscedastic						Heteroscedastic					Homoscedastic						Heteroscedastic				
Model	**n**	**L2OM2d**	**L2OM3d**	HRO3d	HRO4d	HRO5d	HRO.9d	**L2TM2d**	HRT3d	HRT4d	HRT5d	HRT.9d	L2OM2i	L2OM3i	HRO3i	HRO4i	HRO5i	HRO.9i	L2TM2i	HRT3i	HRT4i	HRT5i	HRT.9i
	$\mathbf{n}_1$	7.90	7.51	8.43	8.60	9.01	8.69	7.41	11.54	13.39	15.95	14.51	5.43	5.09	5.71	5.16	4.84	4.74	5.37	6.37	6.45	6.21	6.13
	$\mathbf{n}_2$	6.94	6.61	7.46	8.06	8.02	7.97	6.74	9.53	11.39	12.41	12.05	5.29	5.04	5.49	5.27	4.87	4.77	5.16	6.01	5.77	5.48	5.34
IID	$\mathbf{n}_3$	6.67	6.24	6.63	7.04	7.13	6.88	6.51	7.76	8.67	9.43	8.97	5.37	5.09	5.14	5.03	4.48	4.45	5.34	5.27	5.29	4.82	4.77
	$\mathbf{n}_4$	5.90	5.59	6.25	6.37	6.31	6.20	5.78	6.73	7.48	7.98	7.72	5.13	4.88	4.99	4.71	4.19	4.14	5.10	5.11	4.91	4.45	4.35
	ARE	37.05	29.75	43.85	50.35	52.35	48.70	32.20	77.80	104.65	128.85	116.25	6.10	**1.70**	6.75	3.75	8.10	9.50	4.85	13.80	13.00	12.10	11.75
	$\mathbf{n}_1$	9.31	9.03	9.57	9.88	10.08	9.47	8.92	13.72	16.00	18.15	15.17	14.96	14.46	12.07	10.78	10.05	9.91	14.95	13.13	12.17	11.94	11.55
	$\mathbf{n}_2$	7.68	7.40	7.96	8.06	8.17	7.82	7.49	10.19	11.22	12.72	10.66	14.45	14.01	11.64	10.37	9.36	9.17	14.37	12.28	10.98	10.45	10.29
AR	$\mathbf{n}_3$	6.94	6.55	6.88	7.16	7.27	6.99	6.74	8.09	8.71	9.56	8.42	14.99	14.42	11.25	9.95	9.27	9.16	14.93	11.54	10.46	9.78	9.65
	$\mathbf{n}_4$	6.16	5.89	6.28	6.58	6.46	6.41	6.01	7.10	7.88	8.30	7.63	14.02	13.57	11.07	10.13	9.08	8.87	13.99	11.28	10.51	9.44	9.27
	ARE	50.45	**44.35**	53.45	58.40	59.90	53.45	45.80	95.50	119.05	143.65	109.40	192.10	182.30	130.15	106.15	88.80	85.55	191.20	141.15	120.60	108.05	103.80
	$\mathbf{n}_1$	8.64	8.39	9.63	9.75	10.06	9.44	8.40	12.91	15.67	17.95	14.92	12.44	11.94	10.53	9.65	8.83	8.79	12.37	11.44	10.71	10.60	10.37
	$\mathbf{n}_2$	7.26	6.93	7.31	7.43	7.72	7.19	6.98	9.35	10.65	11.69	10.28	11.72	11.24	9.22	7.95	7.09	6.81	11.59	9.67	8.55	7.81	7.62
MA	$\mathbf{n}_3$	6.56	6.31	7.10	7.42	7.20	7.18	6.47	8.15	8.88	9.73	8.62	12.04	11.62	9.89	8.90	7.65	7.63	11.97	10.02	9.15	8.16	8.06
	$\mathbf{n}_4$	5.96	5.63	6.66	7.13	7.28	7.00	5.89	7.70	8.49	8.85	8.46	11.61	11.16	9.51	8.55	7.79	7.70	11.59	9.58	8.66	8.11	7.99
	ARE	42.10	**36.30**	53.50	58.65	61.30	54.05	38.70	90.55	118.45	141.10	111.40	139.05	129.80	95.75	75.25	56.80	54.65	137.60	103.55	85.35	73.40	70.20
ARE		43.20	**36.80**	50.27	55.80	57.85	52.07	38.90	87.95	114.05	137.87	112.35	112.42	104.60	77.55	61.72	51.23	49.90	111.22	86.17	72.98	64.52	61.92

Table 2: Empirical powers (in percentages) of our L^2-norm based tests and the existing tests under homoscedasticity.

			Tests for dependent functional data											Tests for independent functional data										
			Homoscedastic						Heteroscedastic					Homoscedastic						Heteroscedastic				
Model	**n**	ω	**L2OM2d**	**L2OM3d**	HRO3d	HRO4d	HRO5d	HRO.9d	**L2TM2d**	HRT3d	HRT4d	HRT5d	HRT.9d	L2OM2i	L2OM3i	HRO3i	HRO4i	HRO5i	HRO.9i	L2TM2i	HRT3i	HRT4i	HRT5i	HRT.9i
	$\mathbf{n}_1$	0.07	74.20	72.10	9.19	11.74	14.62	12.90	71.46	12.97	17.74	23.65	20.34	62.05	58.93	6.15	7.73	8.60	9.09	60.74	7.09	9.10	10.54	11.20
IID	$\mathbf{n}_2$	0.05	60.55	58.46	7.87	9.94	12.09	11.27	58.70	10.15	13.70	17.08	15.33	50.16	47.28	5.90	6.74	7.59	7.88	49.45	6.46	7.46	8.65	8.98
	$\mathbf{n}_3$	0.04	89.79	88.30	7.23	9.85	11.30	10.58	89.03	8.48	11.71	14.25	13.27	87.54	84.94	5.75	7.05	8.03	8.76	87.30	5.93	7.43	8.46	9.15
	$\mathbf{n}_4$	0.03	63.63	60.93	6.53	8.45	9.38	8.94	62.49	7.45	10.24	11.54	10.96	57.25	54.21	5.35	6.67	6.58	7.27	56.97	5.49	6.88	6.94	7.64
	$\mathbf{n}_1$	0.08	65.46	63.80	10.60	13.70	17.24	12.62	63.30	14.99	20.32	27.37	18.53	93.01	91.51	13.42	15.04	15.83	16.75	92.31	14.65	17.11	18.34	19.30
AR	$\mathbf{n}_2$	0.06	58.92	57.19	8.70	11.92	14.31	10.76	57.32	11.53	15.87	20.02	14.22	93.95	92.60	13.10	14.59	15.43	16.02	93.69	13.83	15.41	16.70	17.36
	$\mathbf{n}_3$	0.04	44.09	42.38	7.40	9.83	11.44	8.98	43.11	8.60	12.09	14.43	11.17	88.33	86.06	11.56	13.20	13.20	13.67	88.03	11.47	13.46	13.86	14.40
	$\mathbf{n}_4$	0.04	80.55	78.74	7.51	10.78	12.42	9.67	79.77	8.15	12.25	14.74	10.92	100.00	99.98	11.90	14.64	15.24	16.38	99.99	12.16	15.00	15.47	16.59
	$\mathbf{n}_1$	0.09	90.18	89.09	10.55	14.33	18.83	13.46	89.01	14.53	21.22	29.01	19.92	99.78	99.69	11.19	13.89	15.93	16.40	99.75	12.42	15.76	18.42	19.13
MA	$\mathbf{n}_2$	0.07	93.51	92.68	8.49	12.63	16.24	11.79	92.48	10.64	16.39	21.93	15.29	100.00	99.99	10.26	13.48	14.95	15.83	99.99	11.03	14.07	16.19	17.10
	$\mathbf{n}_3$	0.04	52.03	50.04	7.31	9.99	11.72	9.52	51.04	8.64	12.46	15.00	11.90	87.21	84.58	10.07	11.53	11.57	12.18	87.06	10.32	12.05	12.28	12.94
	$\mathbf{n}_4$	0.04	90.70	89.38	7.05	10.43	12.73	9.95	90.03	8.19	12.21	15.01	11.71	100.00	100.00	9.67	12.80	13.20	14.30	100.00	9.93	13.22	13.63	14.74

Table 3: Empirical sizes (in percentages) of our L^2-norm based tests and the existing tests under heteroscedasticity.

		Tests for dependent functional data											Tests for independent functional data										
		Homoscedastic						Heteroscedastic					Homoscedastic						Heteroscedastic				
Model	**n**	**L2OM2d**	**L2OM3d**	HRO3d	HRO4d	HRO5d	HRO.9d	**L2TM2d**	HRT3d	HRT4d	HRT5d	HRT.9d	L2OM2i	L2OM3i	HRO3i	HRO4i	HRO5i	HRO.9i	L2TM2i	HRT3i	HRT4i	HRT5i	HRT.9i
IID	$\mathbf{n}_1$	0.87	0.85	0.42	0.44	0.47	0.47	5.23	2.58	2.91	3.47	15.43	0.67	0.64	1.58	1.33	1.07	0.06	4.88	3.77	3.67	3.93	1.14
	$\mathbf{n}_2$	0.83	0.78	0.86	0.73	0.55	0.23	5.44	2.99	3.04	3.57	6.29	0.72	0.69	2.02	1.56	1.23	0.02	5.03	4.43	4.67	4.55	0.56
	ARE	83.00	83.70	87.20	88.30	89.80	93.00	6.70	44.30	40.50	29.60	117.20	86.10	86.70	64.00	71.10	77.00	99.20	**1.50**	18.00	16.60	15.20	83.00
AR	$\mathbf{n}_1$	2.05	1.90	1.12	0.76	0.58	0.50	7.28	4.10	4.08	4.52	16.04	3.13	3.00	5.61	4.52	3.57	0.18	12.20	10.15	9.27	8.77	2.73
	$\mathbf{n}_2$	1.61	1.51	1.41	0.93	0.66	0.27	6.67	3.84	3.68	3.88	6.53	3.13	2.98	6.69	5.01	3.71	0.09	12.26	10.78	9.83	9.36	1.18
	ARE	63.40	65.90	74.70	83.10	87.60	92.30	39.50	20.60	22.40	16.00	125.70	37.40	40.20	23.00	**4.90**	27.20	97.30	144.60	109.30	91.00	81.30	60.90
MA	$\mathbf{n}_1$	1.44	1.33	1.01	0.81	0.57	0.50	6.21	3.71	3.73	4.04	15.47	2.15	2.04	4.35	3.20	2.36	0.20	9.57	8.26	7.34	7.06	1.89
	$\mathbf{n}_2$	1.28	1.22	1.14	0.84	0.70	0.17	5.79	3.52	3.41	3.57	6.85	2.05	1.98	4.74	3.56	2.69	0.08	9.62	8.35	7.75	7.53	1.01
	ARE	72.80	74.50	78.50	83.50	87.30	93.30	20.00	27.70	28.60	23.90	123.20	58.00	59.80	**9.10**	32.40	49.50	97.20	91.90	66.10	50.90	45.90	71.00
ARE		73.07	74.70	80.13	84.97	88.23	92.87	**22.07**	30.87	30.50	23.17	122.03	60.50	62.23	32.03	36.13	51.23	97.90	79.33	64.47	52.83	47.47	71.63

Table 4: Empirical powers (in percentages) of our L^2-norm based tests and the existing tests under heteroscedasticity.

			Tests for dependent functional data											Tests for independent functional data										
			Homoscedastic						Heteroscedastic					Homoscedastic						Heteroscedastic				
Model	**n**	ω	**L2OM2d**	**L2OM3d**	HRO3d	HRO4d	HRO5d	HRO.9d	**L2TM2d**	HRT3d	HRT4d	HRT5d	HRT.9d	L2OM2i	L2OM3i	HRO3i	HRO4i	HRO5i	HRO.9i	L2TM2i	HRT3i	HRT4i	HRT5i	HRT.9i
IID	$\mathbf{n}_1$	0.26	36.69	35.43	0.74	0.65	0.57	2.03	80.43	5.94	8.70	12.58	81.72	33.36	32.27	1.95	1.48	1.48	0.65	79.71	6.96	8.98	12.33	63.52
	$\mathbf{n}_2$	0.20	52.95	51.40	1.14	1.01	1.08	1.99	90.96	6.28	8.66	13.33	83.92	50.57	49.52	2.30	2.18	1.95	0.72	90.93	7.61	9.53	13.79	74.39
AR	$\mathbf{n}_1$	0.28	41.07	39.11	1.24	1.11	1.05	2.77	78.46	7.62	10.77	15.53	85.55	56.17	54.96	6.36	5.01	4.05	1.79	92.19	14.21	16.09	19.80	79.17
	$\mathbf{n}_2$	0.20	41.22	39.11	1.70	1.34	1.37	2.09	78.40	7.55	10.50	14.04	82.16	59.31	58.37	7.46	5.91	5.23	1.38	92.93	14.66	16.15	20.59	77.54
MA	$\mathbf{n}_1$	0.28	42.94	40.99	1.37	0.91	1.05	2.97	81.00	7.37	10.37	15.51	85.83	54.20	53.05	4.44	3.99	3.38	1.59	91.01	11.79	14.11	18.67	78.04
	$\mathbf{n}_2$	0.20	43.89	41.89	1.62	1.25	1.35	2.00	81.85	7.51	9.77	13.85	82.55	56.85	55.78	6.05	4.70	4.36	1.19	92.57	13.35	14.43	18.94	77.87

Ramsay, J. O. and Silverman, B. W. (2005). *Functional Data Analysis*. Spring, New York, 2nd edition.

Zhang, J.-T. (2013). *Analysis of variance for functional data*. CRC Press.

Zhang, J.-T. and Chen, J. (2007). Statistical inferences for functional data. *The Annals of Statistics*, 35(3):1052–1079.