Supplementary Material for "An Adaptive Spatial-Sign-Based Test for Mean Vectors of Elliptically Distributed High-Dimensional Data"

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In this supplementary material, we present two additional simulation studies. Here we consider the moving average model where the jth component of $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{ip})^\top$ is generated in the following way:

$$x_{ij} = \mu_j + \rho_1 v_{ij} + \cdots + \rho_p v_{i(j+p-1)}, j = 1, 2, \ldots, p; i = 1, 2, \ldots, n,$$

where μ_j is the jth component of $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_p)^\top$ and v_{ij}'s are mean 0 and variance 1 random variables. The mean vector $\boldsymbol{\mu}$ is specified the same way as in the simulation studies of the main paper when the empirical sizes and powers of the tests are considered. For the dependence between the components of $\boldsymbol{x}_i$, we consider the following two cases:

1. Partial dependence: $\rho_1 = 2.883$, $\rho_2 = 2.794$, $\rho_3 = 2.849$, $\rho_j = 0$ for $j = 4, \ldots, p$ and are kept fixed throughout the simulation study.
2. Full dependence: the ρ_j's are all generated from $U[2, 3]$ and kept fixed.

Note that the simulated high-dimensional samples are less (highly) correlated in the partial (full) dependence case.

1 Additional Simulation 1

In this additional simulation study, the following two models are used for the innovations $\{v_{ij}\}$:

- Model 1. $v_{ij},\ j = 1, \ldots, p \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$.
- Model 2. $v_{ij} = w_{ij}/\sqrt{2},\ j = 1, \ldots, p$ with $w_{ij},\ j = 1, \ldots, p \overset{\text{i.i.d.}}{\sim} \mathrm{t}_4$.

The resulting empirical sizes and powers of the NEW, WPL, BS and CQ tests are displayed in Tables 1 and 2 respectively. The estimated approximate degrees of freedom of the NEW test are displayed in Table 3. In terms of size control, the NEW test again outperforms the WPL, BS and CQ tests and in terms of power, the four tests are generally comparable.

2 Additional Simulation 2

In this additional simulation study, the following two models are used for the innovations $\{v_{ij}\}$:

- Model 3. $v_{ij},\ j = 1, \ldots, p$, i.i.d. follow the normal mixture $0.9\mathcal{N}(0, 1) + 0.1\mathcal{N}(0, 9)$.
- Model 4. $v_{ij} = w_{ij}/\sqrt{3}, j = 1, 2, \ldots, p$, with $\boldsymbol{w}_i = (w_{i1}, w_{i2}, \ldots, w_{ip})^\top$ following a multivariate t-distribution with mean $\boldsymbol{0}$, correlation matrix $\boldsymbol{I}_p$, and 3 degrees of freedom.

Table 1: Empirical sizes (%) of the four tests in Additional Simulation 1.

Dependence			Partial				Full			
Model	p	n	NEW	WPL	BS	CQ	NEW	WPL	BS	CQ
1	50	30	5.32	6.86	6.92	6.92	5.67	7.15	7.70	7.69
		60	5.08	6.29	6.32	6.32	5.57	7.04	7.40	7.39
		120	5.13	6.46	6.43	6.43	5.30	6.89	6.75	6.76
	500	30	5.28	5.75	5.82	5.73	6.06	7.85	7.94	7.94
		60	5.19	5.78	5.74	5.74	5.24	6.95	7.30	7.31
		120	5.21	5.67	5.58	5.58	5.23	6.97	7.17	7.18
	1000	30	5.08	5.35	5.33	5.33	5.96	7.65	7.95	7.95
		60	5.24	5.53	5.60	5.62	5.51	7.11	7.37	7.37
		120	4.88	5.14	5.15	5.15	5.05	6.68	6.77	6.77
2	50	30	5.15	6.39	5.71	6.53	5.85	7.65	7.91	7.93
		60	5.11	6.42	5.89	6.43	5.39	7.40	7.30	7.32
		120	5.14	6.33	6.13	6.51	5.36	7.22	7.43	7.48
	500	30	5.35	5.82	4.56	5.70	5.72	7.45	7.74	7.77
		60	5.03	5.60	4.88	5.60	5.37	6.99	7.16	7.16
		120	5.08	5.47	5.26	5.62	5.43	7.31	7.22	7.20
	1000	30	4.56	4.83	3.65	4.93	5.71	7.45	7.78	7.73
		60	4.76	5.10	4.40	5.20	5.05	6.66	7.09	7.13
		120	4.96	5.36	4.76	5.24	5.45	7.16	7.48	7.49
ARE			3.59	16.10	15.14	16.36	9.91	43.98	48.29	48.41

Table 2: Empirical powers (%) of the four tests in Additional Simulation 1.

Dependence			Partial					Full				
Model	p	n	δ	NEW	WPL	BS	CQ	δ	NEW	WPL	BS	CQ
1	50	30	8	95.09	95.91	96.33	96.36	64	93.82	95.43	95.83	95.85
		60	6	97.43	98.08	98.39	98.38	44	93.17	94.80	95.88	95.88
		120	4	95.52	96.44	96.75	96.75	30	90.62	92.73	94.10	94.08
	500	30	12	93.76	94.17	94.29	94.26	600	89.80	92.34	93.45	93.41
		60	9	97.09	97.36	97.39	97.39	450	93.85	95.51	96.20	96.18
		120	6	94.42	94.92	95.09	95.09	300	91.48	93.74	94.49	94.49
	1000	30	14	94.55	94.83	94.99	94.95	1250	93.18	94.85	95.45	95.44
		60	10	94.84	95.18	95.23	95.24	900	94.52	95.94	96.48	96.51
		120	7	94.76	95.08	95.13	95.14	600	91.01	93.12	94.08	94.07
2	50	30	8	96.83	97.37	95.71	96.63	64	94.25	95.81	95.44	95.51
		60	6	98.63	98.86	97.94	98.17	44	93.33	95.00	94.74	94.75
		120	4	96.84	97.35	96.20	96.54	30	93.22	94.89	94.58	94.62
	500	30	12	94.13	94.56	92.35	94.11	600	90.73	93.03	93.27	93.27
		60	9	97.57	97.78	96.90	97.56	450	94.25	95.75	96.20	96.19
		120	6	94.89	95.45	94.23	94.84	300	91.62	93.75	94.33	94.33
	1000	30	14	94.45	94.72	92.68	94.70	1250	92.75	94.46	95.24	95.26
		60	10	95.31	95.61	94.27	95.36	900	94.40	95.73	96.20	96.21
		120	7	94.99	95.25	94.46	95.17	600	91.33	93.49	94.50	94.48

Table 3: Estimated approximate degrees of freedom of the NEW test in Additional Simulation 1.

		Model 1		Model 2	
p	n	Partial	Full	Partial	Full
50	30	25.80	3.34	25.76	3.38
	60	25.25	3.30	25.22	3.35
	120	25.01	3.13	24.98	3.18
500	30	248.19	3.37	248.38	3.38
	60	243.03	3.33	242.95	3.29
	120	240.76	3.25	240.85	3.26
1000	30	495.05	3.40	494.91	3.40
	60	485.20	3.28	485.38	3.31
	120	480.59	3.25	480.60	3.24

Table 4: Empirical sizes (%) of the four tests in Additional Simulation 2.

Dependence			Partial				Full			
Model	p	n	NEW	WPL	BS	CQ	NEW	WPL	BS	CQ
3	50	30	5.53	6.64	1.97	6.83	5.65	7.20	6.94	7.61
		60	5.18	6.42	3.15	6.76	5.60	7.25	7.29	7.67
		120	4.99	6.14	4.25	6.62	5.36	7.20	6.72	7.00
	500	30	4.94	5.40	0.24	5.72	5.91	7.89	7.46	8.35
		60	5.18	5.69	0.03	5.76	5.58	7.43	7.06	7.55
		120	5.80	6.27	0.14	6.12	5.44	7.06	6.73	6.96
	1000	30	5.24	5.62	0.24	5.58	5.87	7.67	6.86	7.51
		60	5.02	5.37	0.03	5.69	5.65	7.43	6.79	7.22
		120	5.12	5.53	0.00	5.68	4.81	6.65	7.07	7.39
4	50	30	5.68	6.95	1.77	6.75	6.16	7.78	6.69	7.88
		60	5.52	6.68	2.42	6.64	5.80	7.57	6.65	7.55
		120	5.37	6.64	3.02	6.63	5.33	7.05	6.09	6.79
	500	30	5.46	5.95	0.01	6.23	6.16	7.58	7.08	8.21
		60	5.04	5.61	0.03	5.44	5.61	7.45	6.36	7.42
		120	5.21	5.64	0.07	5.72	5.26	7.07	6.66	7.32
	1000	30	5.02	5.34	0.00	5.29	6.07	7.98	6.52	7.78
		60	5.19	5.47	0.00	5.36	5.49	7.17	6.82	7.81
		120	5.21	5.58	0.00	5.38	5.57	7.40	6.89	7.52
ARE			5.38	18.82	80.70	20.22	13.00	47.59	36.31	50.60

The simulation results are presented in Tables 4–6. In terms of size control, the NEW test again performs best, and in terms of power, the NEW and WPL tests are comparable and outperform the BS and CQ tests.

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Table 5: Empirical powers (%) of the four tests in Additional Simulation 2.

Dependence			Partial					Full			
Model	p	n	δ	NEW	WPL	BS	CQ	δ	NEW	WPL	BS	CQ
3	50	30	8.0	91.34	92.63	59.89	77.47	70	94.26	95.79	84.93	86.28
		60	6.0	94.80	95.64	73.06	81.66	50	95.41	96.51	86.63	87.43
		120	4.0	91.96	93.31	70.90	76.53	40	98.75	99.10	93.27	93.63
	500	30	14.0	97.48	97.70	8.37	84.16	700	95.29	96.67	85.41	86.85
		60	9.0	93.47	93.84	7.86	73.48	500	95.62	96.65	85.84	86.83
		120	7.0	97.81	97.98	35.26	83.46	350	95.73	96.91	86.72	87.09
	1000	30	16.0	97.09	97.26	4.51	81.00	1250	88.14	90.67	76.56	78.09
		60	11.0	96.81	96.97	1.42	77.31	950	93.12	94.86	81.96	82.94
		120	7.0	89.49	90.05	2.63	64.97	600	86.66	89.51	73.73	74.42
4	50	30	5.0	93.37	94.21	47.90	69.48	45	96.73	97.70	79.47	81.94
		60	4.0	98.43	98.74	60.52	77.76	30	95.14	96.31	73.12	75.21
		120	2.6	96.21	96.93	53.55	68.10	20	93.66	95.24	68.22	69.76
	500	30	8.0	96.06	96.37	5.34	64.42	500	98.65	99.09	85.15	87.36
		60	5.0	88.89	89.70	2.95	47.93	300	95.22	96.42	73.86	75.85
		120	3.8	94.44	94.96	6.37	52.86	200	93.81	95.43	68.24	69.83
	1000	30	9.0	94.32	94.57	0.64	59.20	850	95.01	96.23	75.82	78.71
		60	6.0	91.17	91.56	0.66	49.59	700	99.05	99.39	84.70	86.49
		120	4.6	96.45	96.64	1.57	55.50	450	97.42	98.18	77.15	78.84

Table 6: Estimated approximate degrees of freedom of the NEW test in Additional Simulation 2.

		Model 3		Model 4	
p	n	Partial	Full	Partial	Full
50	30	25.78	3.29	25.78	3.35
	60	25.24	3.22	25.24	3.19
	120	25.01	3.23	25.02	3.19
500	30	248.38	3.45	247.99	3.42
	60	243.08	3.29	243.28	3.30
	120	240.76	3.26	240.77	3.21
1000	30	495.58	3.41	495.27	3.41
	60	484.73	3.27	485.08	3.29
	120	480.66	3.23	480.63	3.23