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Two asymptotic approaches for the exponential signal and harmonic noise in Singular Spectrum Analysis

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The general theoretical approach to the asymptotic extraction of the signal series from the perturbed signal with the help of Singular Spectrum Analysis (briefly, SSA) was already outlined in Nekrutkin 2010, SII, v. 3, 297–319.

In this paper we consider the example of such an analysis applied to the increasing exponential signal and the sinusoidal noise. It is proved that if the signal rapidly tends to infinity, then the so-called reconstruction errors of SSA do not uniformly tend to zero as the series length tends to infinity. More precisely, in this case any finite number of last terms of the error series does not tend to any finite or infinite values.

On the contrary, for the "discretization" scheme with the exponential signal bounded from above, all elements of the error series tend to zero. This effect shows that the discretization model can be an effective tool in the theoretical SSA considerations with increasing signals.

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1. INTRODUCTION

Let us start with the general construction described in [2]. Consider the real-valued "signal" series $\mathrm{F}_N = (x_0, \ldots, x_{N-1})$ and choose $1 < L < N-1$. Transfer the series F_N into the Hankel "trajectory" $L \times K$-matrix $\mathbf{H}$ with entries $(\mathbf{H})_{i,j} = x_{i+j-2}$, where $1 \leq j \leq L$, $1 \leq j \leq K$ and $L + K = N + 1$.

It is supposed that $d \stackrel{\text{def}}{=} \operatorname{rank} \mathbf{H} < \min(K, L)$. Denote $\mathbb{U}_0$ the eigenspace corresponding to the zero eigenvalue of the matrix $\mathbf{A} \stackrel{\text{def}}{=} \mathbf{H}\mathbf{H}^{\mathrm{T}}$. Then $d = \dim \mathbb{U}_0^{\perp}$ and $\dim \mathbb{U}_0 = K - d > 0$.

Let $\mathrm{F}_N(\delta) = \mathrm{F}_N + \delta \mathrm{E}_N$ be the perturbed signal, where $\mathrm{E}_N = (e_0, \ldots, e_{N-1})$ is a certain "noise" series and δ stands for a formal perturbation parameter. Then we come to the perturbed matrix $\mathbf{H}(\delta) = \mathbf{H} + \delta\mathbf{E}$ with the Hankel matrix $\mathbf{E}$ produced from the noise series E_N.

If δ is sufficiently small, then the linear space $\mathbb{U}_0^{\perp}(\delta)$ spanned by d main left singular vectors of the matrix $\mathbf{H}(\delta)$ can serve as an approximation to $\mathbb{U}_0^{\perp}$. The quality of this approximation can be measured by the spectral norm $\left\|\mathbf{P}_0^{\perp}(\delta) - \mathbf{P}_0^{\perp}\right\|$, where $\mathbf{P}_0^{\perp}$ and $\mathbf{P}_0^{\perp}(\delta)$ are orthogonal projections on the linear spaces $\mathbb{U}_0^{\perp}$ and $\mathbb{U}_0^{\perp}(\delta)$ correspondingly. Note that $\left\|\mathbf{P}_0^{\perp}(\delta) - \mathbf{P}_0^{\perp}\right\|$ is nothing but the sine of the largest principal angle between unperturbed and perturbed signal subspaces $\mathbb{U}_0^{\perp}$ and $\mathbb{U}_0^{\perp}(\delta)$.

It is well-known that a lot of subspace-based methods of signal processing are relying on the close proximity of $\mathbb{U}_0^{\perp}$ and $\mathbb{U}_0^{\perp}(\delta)$. Still the main goal of Singular Spectrum Analysis (briefly, SSA) is the approximate extraction (or "reconstruction") of the signal F_N from the perturbed signal $\mathrm{F}_N(\delta)$, see [1] for a detailed description.

As it is mentioned in [2, sect. 5], the analysis of the errors of this approximation can be expressed in such a manner. First of all, the "hankelization" (in other terms, "diagonal averaging") operator $\mathcal{S}$ is defined.

If the hankelization operator $\mathcal{S}$ is applied to some $L \times K$ matrix $\mathbf{Y} = \{y_{k,\ell}\}_{k=1,\ell=1}^{L,K}$, then the resulting $L \times K$ matrix $\mathcal{S}\mathbf{Y}$ has equal values denoted by $(\mathcal{S}\mathbf{Y})_j$ on its anti-diagonals $\{(k,\ell) : \text{such that } k + \ell - 2 = j\}$, where $j = 0, \ldots, N-1$, $k = 1, \ldots, L$ and $\ell = 1, \ldots, K$. Besides, $(\mathcal{S}\mathbf{Y})_j$ equals to the average of inputs $y_{k,\ell}$ on this anti-diagonal.

Then, under denotation

$$\Delta_\delta(\mathbf{H}) = \big(\mathbf{P}_0^{\perp}(\delta) - \mathbf{P}_0^{\perp}\big)\mathbf{H}(\delta) + \delta\mathbf{P}_0^{\perp}\mathbf{E}, \tag{1.1}$$

the series $r_0, \ldots, r_{N-1}$ with

$$r_j = r_j(N, L, \delta) = \big(\mathcal{S}\Delta_\delta(\mathbf{H})\big)_j \tag{1.2}$$

is the series of the reconstruction SSA errors.

The peculiarity of the approach introduced in [2] can be explained as follows. The standard way is to fix N and to consider the small perturbation parameter δ. (See for example [3]–[6], where this method is used for several signal-subspace methods including SSA.) Thus the problem becomes linearized.

As it is shown in [2], the general perturbation theory, going back to [7], allows to consider fixed δ and big N. Since

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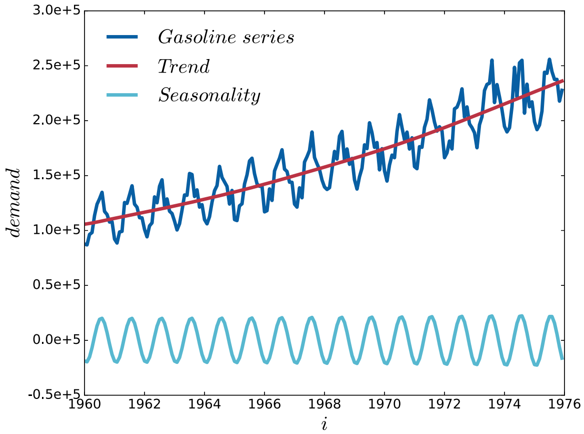

Figure 1. Gasoline demand, monthly Jan 1960 – Dec 1975, gallon millions, Ontario.

SSA mainly deals with long signals, this approach seems to be more close to practice.

This paper is devoted to the example of such an analysis applied to the exponentially growing signal and the harmonic noise. This model is not so far from real-life series. For example, the series "Gasoline demand" (see Fig. 1, data is taken from [8]) can be approximated by the sum of two addends: the increasing trend of the exponential form and the annual periodicity.

Note that both trend and periodicity in Fig. 1 are produced by SSA with $L = N/2 = 96$. Naturally, the trend is reconstructed by the first eigentriple of the decomposition, while the periodicity is produced with the help of eigentriples 2 and 3.

In this paper we deal with the following construction. A certain interval $[0, T]$ is divided into N intervals of length $\Delta = T/N$, and we consider the signal $x_n = e^{\theta\Delta n}$ and the noise $e_n = \cos(\xi n + \varphi)$, so that the perturbed signal has the form

$$f_n = e^{\theta\Delta n} + \delta\cos(\xi n + \varphi), \quad n = 0, \ldots, N-1, \tag{1.3}$$

where $\theta > 0$, $\xi = 2\pi\omega$ with $\omega \in (0, 1/2)$, and $\varphi \in [0, 2\pi)$.

As in [2], we are interested in the behavior of the reconstruction SSA errors for long signals. For this goal we consider two asymptotic schemes as $N \to \infty$.

1. The parameter Δ is fixed, further we put $\Delta = 1$. Then $T = N \to \infty$ and (1.3) has the form

$$f_n = a^n + \delta\cos(\xi n + \varphi), \quad n = 0, \ldots, N-1 \tag{1.4}$$

with $a = e^\theta > 1$.

2. The parameter T is fixed and $\Delta = T/N \to 0$. Then we come to the triangle array of the series

$$f_n = a^{Tn/N} + \delta\cos(\xi n + \varphi), \quad n = 0, \ldots N-1 \tag{1.5}$$

with $N \geq 1$ and the same a. Further we apply the term "discretization" for this scheme.

Note that in both cases $d = \dim \mathbb{U}_0^\perp = \operatorname{rank} \mathbf{H} = 1$ for any $L, K > 1$. Yet there are considerable differences between (1.4) and (1.5). In particular, the signal of the series (1.4) tends to infinity as $N \to \infty$, while $a^{Tn/N} < a^T = \text{const}$ for (1.5). The discussion on the theoretical results for both models as well as on their relation to real-life SSA problems can be found in Section 4 at the end of the paper.

For both models, our interest lies in the asymptotic behavior, as $N \to \infty$, of both $\left\|\mathbf{P}_0^\perp(\delta) - \mathbf{P}_0^\perp\right\|$ and the reconstruction SSA errors. Since the model (1.4) corresponds to the style of all examples in [2], several results about this series can be borrowed from this paper.

In particular, see [2, sect. 3.2.1], it is already shown for the model (1.4), that under the conditions $N \to \infty$ and $\min(L, K) \to \infty$, $\left\|\mathbf{P}_0^\perp(\delta) - \mathbf{P}_0^\perp\right\| = O(Na^{-N})$ and

$$\left\|\mathbf{P}_0^\perp(\delta) - \mathbf{P}_0^\perp - \delta\mathbf{V}_0^{(1)}\right\| = O(N^2 a^{-2N}) \tag{1.6}$$

for any $\delta \in \mathbb{R}$, where

$$\mathbf{V}_0^{(1)} = \mathbf{P}_0\mathbf{E}\mathbf{H}^{\mathrm{T}}\mathbf{S}_0 + \mathbf{S}_0\mathbf{H}\mathbf{E}^{\mathrm{T}}\mathbf{P}_0 \tag{1.7}$$

is the linear term of the expansion of $\mathbf{P}_0^\perp(\delta) - \mathbf{P}_0^\perp$ into power series (see [2, theor. 2.1]), $\mathbf{P}_0$ is the orthogonal projector on the space $\mathbb{U}_0$, and $\mathbf{S}_0$ stands for the pseudoinverse of $\mathbf{H}\mathbf{H}^{\mathrm{T}}$. Besides, $\left\|\mathbf{V}_0^{(1)}\right\| = O\big(Na^{-N}\big)$.

Note that in the case $L \sim \alpha N$ with $\alpha \in (0, 1)$ (see [9] for the discussion on this choice), more careful calculations lead to the precise asymptotic

$$\begin{aligned} \frac{a^N}{\sqrt{N}} \left\|\mathbf{P}_0^\perp(\delta) - \mathbf{P}_0^\perp\right\| \to \\ |\delta|\, \frac{a^2 - 1}{a} \sqrt{\frac{\alpha(a^2 - 1)}{2(a^2 + 1 - 2a\cos\xi)}} \end{aligned} \tag{1.8}$$

as well as to more precise inequality

$$\left\|\mathbf{P}_0^\perp(\delta) - \mathbf{P}_0^\perp - \delta\mathbf{V}_0^{(1)}\right\| = O(N^{3/2} a^{-2N}) \tag{1.9}$$

instead of (1.6). Since we omit here proofs of both (1.8) and (1.9), we use the inequality (1.6) for the series (1.4) in all further considerations.

Section 2 of the paper is devoted to the reconstruction errors $r_j = r_j(N)$ for the model (1.4). Proposition 2.1 shows that $r_j \to 0$ as $N \to \infty$ if, roughly speaking, j is separated from N.

On the contrary, if $j = j(N)$ is close to N, then r_j does not converge to zero. Moreover, the asymptotic behavior of r_j in this case depends on the rationality/irrationality of the frequency $\omega = \xi/2\pi$, see propositions 2.3 and 2.4.

The model (1.5) is studied in Section 3. It is proved in Proposition 3.1 that in this case $\left\|\mathbf{P}_0^\perp(\delta) - \mathbf{P}_0^\perp\right\| = O(N^{-1})$ as $N \to \infty$ for the sufficiently small δ.

Unlike the model (1.4), the reconstruction errors r_j in the discretization scheme tend to zero for all j, see Proposition 3.2. Thus the model (1.5) seems to be more practical than (1.4).

In what follows, we always assume the regular behavior of the parameter $L = L(N)$ as $N \to \infty$. This means that $L \sim \alpha N$ with $\alpha \in (0,1)$. Still several inequalities below are valid under less restrictive condition $\min(L,K) \to \infty$.

2. RECONSTRUCTION ERRORS FOR THE MODEL (1.4)

Consider the series (1.4) and suppose that $L \sim \alpha N$ with $\alpha \in (0,1)$ as $N \to \infty$. Our aim is to study the asymptotic properties of the reconstruction errors (1.1), (1.2) for the perturbed series (1.4). Since the result of the reconstruction does not change if we put $\mathbf{H}^{\mathrm{T}}$ instead of $\mathbf{H}$, we assume that $L \le K$.

The base of the approach is the well-known inequality $\|\mathbf{A}\|_{\max} \le \|\mathbf{A}\|$, where $\|\mathbf{A}\|$ stands for the spectral norm of the matrix $\mathbf{A}$ and $\|\mathbf{A}\|_{\max} = \max|a_{ij}|$ for the matrix $\mathbf{A}$ with entries $\big(\mathbf{A}\big)_{i,j} = a_{ij}$. Therefore, if $\|\mathbf{A}\|$ is small, then $\|\mathcal{S}\mathbf{A}\|_{\max}$ is small as well.

Thus we rewrite (1.1) in the form

$$\begin{aligned}\Delta_\delta(\mathbf{H}) = \big(\mathbf{P}_0^{\perp}(\delta) - \mathbf{P}_0^{\perp} - \delta\mathbf{V}_0^{(1)}\big)\mathbf{H}(\delta) + \\ \delta\mathbf{P}_0^{\perp}\mathbf{E} + \delta\mathbf{V}_0^{(1)}\big(\mathbf{H} + \delta\mathbf{E}\big)\ .\end{aligned} \tag{2.1}$$

It is easy to check that $\|\mathbf{E}\| = O(N)$, $\|\mathbf{H}\| = O(a^N)$ and $\|\mathbf{V}_0^{(1)}\mathbf{E}\| = O(N^2a^{-N})$. Applying (1.6), we see that

$$\big\|\big(\mathbf{P}_0^{\perp}(\delta) - \mathbf{P}_0^{\perp} - \delta\mathbf{V}_0^{(1)}\big)\mathbf{H}(\delta)\big\| = O(N^2a^{-N}).$$

This means that the reconstruction errors have the form

$$r_j = r_j(N) = \delta\big(\mathcal{S}(\mathbf{V}_0^{(1)}\mathbf{H} + \mathbf{P}_0^{\perp}\mathbf{E})\big)_j + O(N^2a^{-N}) \tag{2.2}$$

with $j = 0,\ldots,N-1$.

2.1 Reconstruction errors $r_j(N)$ in the case $N - j \to \infty$

Let us start with the case when $j = j(N)$ is not close to N.

Proposition 2.1. *Let $r_j = r_j(N)$ be defined by (2.2). If $L \sim \alpha N$ as $N \to \infty$ with $\alpha \in (0,1)$ and $N - j \to \infty$, then $r_j(N) \to 0$.*

Proof. All we need is to investigate the asymptotical behavior of the series

$$\begin{aligned}\rho_j = \rho_j(N) \stackrel{\mathrm{def}}{=} \big(\mathcal{S}(\mathbf{V}_0^{(1)}\mathbf{H} + \mathbf{P}_0^{\perp}\mathbf{E})\big)_j = \\ \big(\mathcal{S}(\mathbf{P}_0\mathbf{E}\mathbf{H}^{\mathrm{T}}\mathbf{S}_0\mathbf{H}\big)_j + \big(\mathcal{S}(\mathbf{P}_0^{\perp}\mathbf{E})\big)_j.\end{aligned} \tag{2.3}$$

As it was already mentioned, it is sufficient to consider the case $L \le K$. First of all, for fixed $\xi \in (0,\pi)$, $b > 1$, $\psi \in [0, 2\pi)$ and integer $M \ge 1$ denote

$$\varPhi_M(b,\psi) = \sum_{j=0}^{M-1} b^j\cos(\xi j + \psi) \tag{2.4}$$

and

$$\varUpsilon_{T,M}(b,\psi) = \sum_{j=0}^{T-1} b^j\varPhi_M(b,\xi j + \psi). \tag{2.5}$$

Evidently,

$$\begin{aligned}|\varPhi_M(b,\psi)| \le (b^M-1)/(b-1) \quad \text{and} \\ |\varUpsilon_{T,M}(b,\psi)| \le \frac{(b^M-1)(b^T-1)}{(b-1)^2}\ .\end{aligned} \tag{2.6}$$

Under the denotation $W_M = (1, a, \ldots, a^{M-1})^{\mathrm{T}}$,

$$\mathbf{P}_0^{\perp} = \frac{W_LW_L^{\mathrm{T}}}{\|W_L\|^2} \quad \text{and} \quad \mathbf{S}_0 = \frac{W_LW_L^{\mathrm{T}}}{\|W_L\|^4\|W_K\|^2}, \tag{2.7}$$

and therefore

$$\begin{aligned}\mathbf{P}_0^{\perp}\mathbf{E} + \mathbf{V}_0^{(1)}\mathbf{H} = \frac{W_LW_L^{\mathrm{T}}}{\|W_L\|^2}\,\mathbf{E} + \\ \left(\mathbf{I} - \frac{W_LW_L^{\mathrm{T}}}{\|W_L\|^2}\right)\mathbf{E}\mathbf{H}^{\mathrm{T}}\,\frac{W_LW_L^{\mathrm{T}}}{\|W_L\|^4\|W_K\|^2}\,\mathbf{H} = \\ \frac{W_LW_L^{\mathrm{T}}\mathbf{E}}{\|W_L\|^2} + \frac{\mathbf{E}W_KW_K^{\mathrm{T}}}{\|W_K\|^2} - \varUpsilon_{L,K}(a,\varphi)\,\frac{W_LW_K^{\mathrm{T}}}{\|W_L\|^2\|W_K\|^2} = \\ \mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3.\end{aligned} \tag{2.8}$$

Since $\big(\mathcal{S}(W_LW_K^{\mathrm{T}})\big)_j = a^j$ and

$$\|W_M\|^2 = (a^{2M}-1)/(a^2-1),$$

then

$$\begin{aligned}\Big|\big(\mathcal{S}(\mathbf{J}_3)\big)_j\Big| = |\varUpsilon_{L,K}(a,\varphi)|\ \frac{a^j(a^2-1)^2}{(a^{2L}-1)(a^{2K}-1)} \le \\ \frac{(a^L-1)(a^K-1)}{(a-1)^2}\ \frac{a^j(a^2-1)^2}{(a^{2L}-1)(a^{2K}-1)} = \\ \frac{a^j(a+1)^2}{(a^L+1)(a^K+1)}\ .\end{aligned}$$

Let us now check $\mathbf{J}_1$ and $\mathbf{J}_2$. In view of the equalities

$$\mathbf{E}W_K = \big(\varPhi_K(a,\varphi), \ldots, \varPhi_K(a,(L-1)\xi + \varphi)\big)^{\mathrm{T}} \tag{2.9}$$

and $\mathbf{E}^{\mathrm{T}}W_L = \big(\varPhi_L(a,\varphi), \ldots, \varPhi_L(a,(K-1)\xi + \varphi)\big)^{\mathrm{T}}$, we get that

$$\left(\mathcal{S}(W_L W_L^{\mathrm{T}}\mathbf{E})\right)_j =$$

$$\begin{cases} \frac{1}{j+1}\, Y(0,j) & \text{for } 0 \le j < L, \\ \frac{1}{L}\, Y(0,L) & \text{for } L \le j < K, \\ \frac{1}{N-j}\, Y(j-K+1, N-K) & \text{for } K \le j < N, \end{cases} \tag{2.10}$$

with $Y(i,j) = \sum_{k=i}^{j} a^k \Phi_L(a,(j-k)\xi+\varphi)$. In the same manner,

$$\left(\mathcal{S}(\mathbf{E}W_K W_K^{\mathrm{T}})\right)_j =$$

$$\begin{cases} \frac{1}{j+1}\sum_{k=0}^{j} a^{j-k}\Phi_K(a,k\xi+\varphi) & \text{for } 0 \le j < L, \\ \frac{1}{L}\sum_{k=0}^{L-1} a^{j-k}\Phi_K(a,k\xi+\varphi) & \text{for } L \le j < K, \\ \frac{1}{N-j}\sum_{k=j-K+1}^{N-K} a^{j-k}\Phi_K(a,k\xi+\varphi) & \text{for } K \le j < N. \end{cases}$$

Due to (2.6),

$$\left|\left(\mathcal{S}(W_L W_L^{\mathrm{T}}\mathbf{E})\right)_j\right| \le$$

$$\begin{cases} \frac{1}{j+1}\frac{a^{j+1}-1}{a-1}\frac{a^L-1}{a-1} & \text{for } 0 \le j < L, \\ \frac{1}{L}\left(\frac{a^L-1}{a-1}\right)^2 & \text{for } L \le j < K, \\ \frac{a^{j-K+1}}{N-j}\frac{a^{N-j}-1}{a-1}\frac{a^L-1}{a-1} & \text{for } K \le j < N. \end{cases}$$

and

$$\left|\left(\mathcal{S}(\mathbf{E}W_K W_K^{\mathrm{T}})\right)_j\right| \le$$

$$\begin{cases} \frac{1}{j+1}\frac{a^{j+1}-1}{a-1}\frac{a^K-1}{a-1} & \text{for } 0 \le j < L, \\ \frac{a^j}{L}\frac{a^L-1}{a^L}\frac{1}{a-1}\frac{a^K-1}{a-1} & \text{for } L \le j < K, \\ \frac{a^{j-N+K}}{N-j}\frac{a^{N-j}-1}{a-1}\frac{a^K-1}{a-1} & \text{for } K \le j < N. \end{cases}$$

Therefore,

$$|\rho_j| \le \frac{1}{j+1}\frac{a+1}{a-1}\frac{a^{j+1}-1}{a^L+1} + \frac{1}{j+1}\frac{a+1}{a-1}\frac{a^{j+1}-1}{a^K+1} + \frac{a^j(a+1)^2}{(a^L+1)(a^K+1)}$$

for $0 \le j < L$,

$$|\rho_j| \le \frac{1}{L}\frac{a+1}{a-1}\frac{a^L-1}{a^L+1} + \frac{1}{L}\frac{a+1}{a-1}\frac{a^L-1}{a^L}\frac{a^j}{a^K+1} + \frac{a^j(a+1)^2}{(a^L+1)(a^K+1)}$$

for $L \le j < K$, and

$$|\rho_j| \le \frac{a+1}{a-1}\frac{1}{N-j}\frac{a^L-a^{j-N+L}}{a^L+1} + \frac{a+1}{a-1}\frac{1}{N-j}\frac{a^K-a^{j-N+K}}{a^K+1} + \frac{a^j(a+1)^2}{(a^L+1)(a^K+1)}$$

if $K \le j < N$. Thus there exists a constant C such that for N big enough

$$|\rho_j| \le C \begin{cases} a^{-(L-j)}/(j+1) & \text{for } 0 \le j < L, \\ 1/L & \text{for } L \le j < K, \\ 1/(N-j) + a^{-(N-j)} & \text{for } K \le j < N. \end{cases} \tag{2.11}$$

The proof is complete. □

2.2 Reconstruction errors $r_j(N)$ in the case $N - j = O(1)$

Consider now the case $j = N - 1 - \ell$ with $\ell = O(1)$. Denote $\cos_k(\phi) = \cos(k\xi + \phi)$, $\sin_k(\phi) = \sin(k\xi + \phi)$,

$$G(a,\xi) = \frac{a^2-1}{a(a^2+1-2a\cos\xi)}\ ,$$

and

$$Z_N(a,\xi,\varphi,\ell) = C_1(\ell)\cos_{N-1}(\varphi) + C_2(\ell)\sin_{N-1}(\varphi)$$

with

$$\begin{aligned} C_1(\ell) &= \frac{2}{1+\ell}\left(a\cos(\ell\xi) - a^{-\ell}\cos\xi\right) - \\ &(a^2+1-2a\cos\xi - 2\sin^2\xi)\, G(a,\xi)a^{-\ell} \quad \text{and} \\ C_2(\ell) &= \frac{2}{1+\ell}\left(a\sin(\ell\xi) + a^{-\ell}\sin\xi\right) - \\ &2\sin\xi(a-\cos\xi)\ G(a,\xi)a^{-\ell}\ . \end{aligned} \tag{2.12}$$

Proposition 2.2. *If $\ell = O(1)$ then under the conditions of Proposition 2.1,*

$$r_{N-1-\ell} = \delta\, G(a,\xi)\, Z_N(a,\xi,\varphi,\ell) + O(a^{-L}). \tag{2.13}$$

Proof. The proof is based on formulas (2.3) and (2.8) presented in Proposition 2.1.

For fixed ξ put $P(a,n,\psi) = a\cos_{n-1}(\psi) - \cos_n(\psi)$. Straightforward calculations show that

$$\Phi_M(b,\psi) = \frac{b^M P(b,M,\psi)}{b^2+1-2b\cos\xi} + O(1) \tag{2.14}$$

for fixed b and $M \to \infty$. (Note that Φ_M is defined in (2.4).)

Denote $\psi_n = (K-\ell-n-1)\xi + \varphi$, then $P(a,L,\psi_n) = P(a,N-n-\ell,\varphi)$ and

$$\sum_{n=0}^{\ell} a^{-n} P(a,N-n-\ell,\varphi) = a\cos_{N-1-\ell}(\varphi) - \cos_N(\varphi)a^{-\ell}.$$

Therefore, taking into account that $K \le j = N-1-\ell$ and applying (2.14) with $b = a$, $M = L$ and $\psi = \psi_n$, we get from (2.10)

$$\frac{\big(\mathcal{S}(W_L W_L^{\mathrm{T}}\mathbf{E})\big)_{N-1-\ell}}{\|W_L\|^2} =$$

$$\frac{1}{\|W_L\|^2}\,\frac{1}{\ell+1}\sum_{k=N-K-\ell}^{N-K} a^k \Phi_L(a,(N-1-\ell-k)\xi+\varphi) =$$

$$\sum_{n=0}^{\ell}\Phi_L(a,(K-\ell-n-1)\xi+\varphi)\;\frac{a^{N-K-n}(a^2-1)}{(\ell+1)(a^{2L}+1)} =$$

$$\frac{a^2-1}{\ell+1}\sum_{n=0}^{\ell} a^{L-1-n}\left(\frac{a^L P(a,L,\psi_n)}{a^2+1-2a\cos\xi}+O_L(1)\right)\frac{1}{a^{2L}-1} =$$

$$\frac{G(a,\xi)}{a(\ell+1)}\sum_{n=0}^{\ell} a^{-n}P(a,N-n-\ell,\varphi)+O(a^{-L}) =$$

$$= \frac{G(a,\xi)}{1+\ell}\Big(a\cos_{N-\ell}(\varphi)-\cos_N(\varphi)a^{-\ell}\Big)+O(a^{-L}).$$

In the same manner,

$$\frac{(\mathcal{S}\mathbf{E}W_K W_K^{\mathrm{T}})_{N-1-\ell}}{\|W_K\|^2} =$$

$$\frac{G(a,\xi)}{1+\ell}\Big(a\cos_{N-1-\ell}(\varphi)-\cos_N(\varphi)a^{-\ell}\Big)+O(a^{-K}).$$

Now we pass to $\Upsilon_{L,K}(a,\varphi)$ which is defined in (2.5). It is easy to check that if $b>1$ and $T,M\to\infty$, then

$$\Upsilon_{T,M}(b,\psi) = \frac{(b^{S+1})C(b,S,\psi)}{(b^2+1-2b\cos\xi)^2}+O(b^{\min\{T,M\}}),$$

where $S = T+M-1$ and

$$C(b,S,\psi) = b^2\cos_{S-1}(\psi)-2b\cos_S(\psi)+\cos_{S+1}(\psi).$$

Therefore,

$$\frac{a^{N-1-\ell}\Upsilon_{L,K}(a,\varphi)}{\|W_L\|^2\|W_K\|^2} =$$

$$\frac{a^{2N-\ell}C(a,N,\varphi)}{(a^2+1-2a\cos\xi)^2}\,\frac{(a^2-1)^2}{a^{2N+2}(1-a^{-2L})(1-a^{-2K})}+O(a^{-L}) =$$

$$\frac{G^2(a,\xi)}{a^\ell}\;C(a,N,\varphi)+O(a^{-L}).$$

Since

$$a\cos_{N-1-\ell}(\varphi)-\cos_N(\varphi)a^{-\ell} =$$

$$\big(a\cos(\ell\xi)-a^{-\ell}\cos\xi\big)\cos_{N-1}(\varphi)+$$

$$\big(a\sin(\ell\xi)+a^{-\ell}\sin\xi\big)\sin_N(\varphi)$$

and

$$C(a,N,\varphi) = (a^2-2a\cos\xi+\cos(2\xi))\cos_{N-1}(\varphi)+$$
$$2\sin\xi(a-\cos\xi)\sin_{N-1}(\varphi),$$

we get the result in view of (2.2). □

To analyze the behavior of the right-hand side of (2.13) more precisely, we need some more considerations.

First of all, it is worth mentioning that $C_1(\ell)C_2(\ell)\ne 0$ for any fixed ℓ. Thus we can rewrite the result of Proposition 2.2 in the form

$$r_{N-1-\ell} = \delta F_N(\ell)+O(a^{-L}), \tag{2.15}$$

where

$$F_N(\ell) = D(\ell)\sin\big((N-1)\xi+\varphi_1(\ell)\big) \tag{2.16}$$

with

$$D(\ell) = \frac{a^2-1}{a(a^2+1-2a\cos\xi)}\;\sqrt{C_1^2(\ell)+C_2^2(\ell)} \tag{2.17}$$

and

$$\varphi_1(\ell) = \arccos\left(\frac{C_2(\ell)}{\sqrt{C_1^2(\ell)+C_2^2(\ell)}}\right)+\varphi.$$

Remind that $\xi = 2\pi\omega$ with $\omega\in(0,1/2)$. It is natural that the asymptotic behavior of $r_{N-1-\ell}$ depends on the properties of the frequency ω.

Suppose that $\omega = p/q$, where p and q are coprime natural numbers. For fixed $0\le k<q$ and $\ell\ge 0$ consider the sequence $N_m^{(k)} = mq+k+1,\ m\ge 1$. Since

$$\sin\big(2\pi(N_m^{(k)}-1)p/q+\varphi_1(\ell)\big) = \sin\big(2\pi kp/q+\varphi_1(\ell)\big),$$

then

$$r_{N_m^{(k)}-1-\ell}\to D(\ell)\sin\big(2\pi kp/q+\varphi_1(\ell)\big) \tag{2.18}$$

as $m\to\infty$.

Proposition 2.3. *Let the conditions of Proposition 2.2 be fulfilled. Assume that ℓ is fixed and that ω is a rational number. Denote τ the number of limit points of the series $r_{N-1-\ell}$ as $N\to\infty$. Then $\tau\ge 2$.*

Proof. Since $D(\ell)>0$, it is sufficient to examine the expressions $S(k) = \sin(2\pi kp/q+\varphi_1(\ell))$ with $0\le k<q$. If $S(k) = s = \mathrm{const}$ for all k, then there exist integers m_k such that

$$2\pi kp/q+\varphi_1(\ell) = (-1)^{m_k}\arcsin s+m_k\pi,\ \ k=0,\dots,q-1.$$

Therefore, for $0\le k<q-1$

$$2\pi p/q = \big((-1)^{m_{k+1}}-(-1)^{m_k}\big)\arcsin s+\big(m_{k+1}-m_k\big)\pi =$$
$$\begin{cases} 2\arcsin s+(m_{k+1}-m_k)\pi & \text{for even } m_{k+1} \text{ and odd } m_k,\\ -2\arcsin s+(m_{k+1}-m_k)\pi & \text{for odd } m_{k+1} \text{ and even } m_k.\end{cases}$$

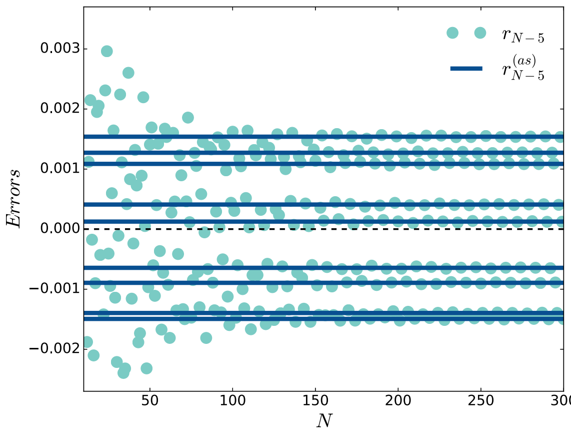

Figure 2. Reconstruction errors r_{N-5} and their limit values $r_{N-5}^{(as)}$ for $\omega = 2/9$, $L = \lfloor 0.35N \rfloor$, $a = 1.05$, $\delta = 0.1$, $\varphi = 0$ and $10 \le N \le 300$.

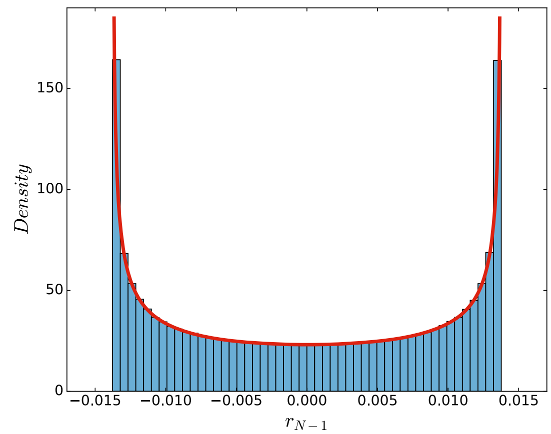

Figure 3. Reconstruction errors r_{N-1} for $\omega = \sqrt{2}/6$, $L = \lfloor 0.35N \rfloor$, $a = 1.05$, $\delta = 0.1$, $\varphi = 0$ and $10^3 \le N \le 10^6$. The histogram and the theoretical density (2.20).

If m_{k+1}, m_k are both odd or even, then

$$2\pi p/q = (m_{k+1} - m_k)\pi.$$

Since $0 < 2\pi p/q < \pi$, then we immediately come to the inequality $s \ne 0$. Suppose now that $\arcsin s > 0$. (The case of $\arcsin s < 0$ can be treated in the same manner.) Then $2\pi p/q \in (0, \pi)$ iff $m_{k+1} - m_k = 1$ and m_{k+1} is odd.

Therefore, for any sequence $\{m_k\}_{k=0}^{q-1}$ with $q > 2$ there exist a pair (m_{k+1}, m_k) with $2\pi p/q \notin (0, \pi)$, and the assertion is proved. □

The convergence (2.18) and the result of Proposition 2.3 are illustrated by Fig. 2. To investigate the case of the irrational ω we use the equidistribution theorem going back to P. Bohl [10] and W. Sierpinski [11].

Theorem 2.1. *If $\alpha \in (0, 1)$ is irrational, then the sequence $z_n = \{n\alpha\}$ is uniformly distributed on $[0, 1]$ in the sense that for any $0 \le a < b \le 1$*

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{[a,b)}(z_i) \to b - a, \quad n \to \infty. \tag{2.19}$$

Proposition 2.4. *Let $\omega \in (0, 1/2)$ be the irrational number and assume $\ell \ge 0$ to be fixed. Then for any $(a, b) \subset [D(-\ell), D(\ell)]$ and any $\delta \ne 0$*

$$\frac{1}{N}\sum_{n=\ell+1}^{N+\ell} \mathbf{1}_{(a,b)}(r_{n-1-\ell}/\delta) \to \int_a^b \frac{1}{\pi\sqrt{D^2(\ell) - u^2}}\, du$$

as $N \to \infty$, where $D(\ell)$ is defined in (2.17).

Proof. In terms of the weak convergence of distributions (see [12] for the entire theory), the convergence (2.19) means that $\mathcal{P}_n \Rightarrow \mathrm{U}(0, 1)$ as $n \to \infty$, where $\mathcal{P}_n$ stands for the uniform distribution on the set $\{z_1, \ldots, z_n\}$, $\mathrm{U}(0, 1)$ is the uniform distribution on $[0, 1]$, and " $\Rightarrow$" is the sign of the weak convergence.

Now let us consider the sequence $\{\beta_n\}_{n\ge1}$ of random variables defined on a certain probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and suppose $\mathcal{L}(\beta_n) = \mathcal{P}_n$ for any n. (Note that here and further $\mathcal{L}(\beta)$ stands for the distribution of the random variable β.) Then (2.19) can be rewritten as $\mathcal{L}(\beta_n) \Rightarrow \mathcal{L}(\upsilon)$ as $n \to \infty$, where $\upsilon \in \mathrm{U}(0, 1)$.

According to the Mapping Theorem [12, theor. 2.7], if $n \to \infty$ then

$$\mathcal{L}\big(h(\beta_n)\big) \Rightarrow \mathcal{L}\big(h(\upsilon)\big)$$

with $h(z) = D(\ell)\sin\big(2\pi z + \varphi_1(\ell)\big)$. Standard calculations show that the random variable $\eta = h(\upsilon)$ has the probability density

$$p_\eta(z) = \frac{\mathbf{1}_{[-D(\ell), D(\ell)]}(z)}{\pi\sqrt{D(\ell)^2 - z^2}}\,, \tag{2.20}$$

where $\mathbf{1}_A(x)$ stands for the indicator function of the set A.

Since $\sin(2\pi j\omega + \phi) = \sin\big(2\pi\{j\omega\} + \phi\big)$ for any integer $j \ge 1$, this means that for any $a < b$

$$\frac{1}{N}\sum_{n=\ell+1}^{N+\ell} \mathbf{1}_{(a,b)}\big(F_n(\ell)\big) \to \int_a^b p_\eta(u)du,$$

where $F_N(\ell)$ is defined in (2.16) and $N \to \infty$. In view of (2.15) the assertion is proved. □

The result of Proposition 2.4 is illustrated by Fig. 3.

Remark 2.1. Propositions 2.3 and 2.4 show that for fixed ℓ and any $\omega \in (0, 1/2)$ the reconstruction error $r_{N-1-\ell}$ does not converge to any limit value as $L/N \to \alpha \in (0, 1)$. The case $\omega = 1/2$ can be studied in the same manner and gives the analogous result.

3. RECONSTRUCTION ERRORS FOR THE MODEL (1.5)

Now we deal with the discretization of the exponential signal, described in the Introduction. More precisely, we consider the constant $T > 0$ and the triangular array of the series

$$f_n = f_n^{(N)} = a^{nT/N} + \delta \cos(\xi n + \varphi) \tag{3.1}$$

with $n = 0, \ldots, N-1$ and $N = 1, 2, \ldots$ under the assumption that $N \to \infty$ and $L \sim \alpha N$ with $\alpha \in (0,1)$. As in the previous section, we suppose that $a > 1$, $\xi \in (0, \pi)$ and $\varphi \in [0, 2\pi)$.

Of course, we can apply all general formulas of Section 2 if we use $a^{jT/N}$ instead of a^j. For example, now we put $W_M = \big(1, a^{T/N}, \ldots, a^{(M-1)T/N}\big)^{\mathrm{T}}$ instead of the denotation $W_M = \big(1, a, \ldots, a^{M-1}\big)^{\mathrm{T}}$ that was used in Section 2.

In particular, since $\operatorname{rank} \mathbf{H} = 1$, then the unique positive eigenvalue μ of the matrix $\mathbf{H}\mathbf{H}^{\mathrm{T}}$ has the form

$$\mu = \|W_L\|^2 \, \|W_K\|^2 = \frac{(a^{2LT/N} - 1)(a^{2KT/N} - 1)}{(a^{2T/N} - 1)^2} \,. \tag{3.2}$$

To study the discretization model (3.1), we apply two general inequalities demonstrated in [2]. Here we put these statements in the form adapted to our problem. Denote

$$\mathbf{B}(\delta) = \delta\big(\mathbf{H}\mathbf{E}^{\mathrm{T}} + \mathbf{E}\mathbf{H}^{\mathrm{T}}\big) + \delta^2 \mathbf{E}\mathbf{E}^{\mathrm{T}} = \delta \mathbf{A}^{(1)} + \delta^2 \mathbf{A}^{(2)} \tag{3.3}$$

and let μ be defined by (3.2).

Theorem 3.1. ([2, theor. 2.3]). *If $\delta_0 > 0$ and $\|\mathbf{B}(\delta)\|/\mu < 1/4$ for any $\delta \in (-\delta_0, \delta_0)$, then*

$$\big\|\mathbf{P}_0^{\perp}(\delta) - \mathbf{P}_0^{\perp}\big\| \le 4C \, \frac{\|\mathbf{S}_0 \mathbf{B}(\delta) \mathbf{P}_0\|}{1 - 4\|\mathbf{B}(\delta)\|/\mu} \quad \text{with} \quad C = e^{1/6}/\sqrt{\pi}.$$

Theorem 3.2. ([2, theor. 2.5]) *Put*

$$B(\delta) = |\delta| \, \|\mathbf{A}^{(1)}\| + \delta^2 \|\mathbf{A}^{(2)}\|$$

and assume that $\delta_0 > 0$, $B(\delta_0) = \mu/4$ and $|\delta| < \delta_0$. Denote $\mathbf{A}_0^{(2)} = \mathbf{P}_0 \mathbf{A}^{(2)} \mathbf{P}_0$.

Then $\|\delta \mathbf{A}_0^{(2)}\| < 1$ and the matrix $\mathbf{I} - \delta \mathbf{A}_0^{(2)}$ is invertible. Besides, under denotation $\mathbf{L}(\delta) = \mathbf{L}_1(\delta) + \mathbf{L}_1^{\mathrm{T}}(\delta)$ with

$$\mathbf{L}_1(\delta) = \frac{\mathbf{P}_0^{\perp} \mathbf{B}(\delta) \mathbf{P}_0}{\mu} \Big(\mathbf{I} - \delta \mathbf{A}_0^{(2)}/\mu\Big)^{-1}, \tag{3.4}$$

the inequality

$$\big\|\mathbf{P}_0^{\perp}(\delta) - \mathbf{P}_0^{\perp} - \mathbf{L}(\delta)\big\| \le 16 \, C \frac{\|\mathbf{S}_0 \mathbf{B}(\delta)\| \|\mathbf{S}_0 \mathbf{B}(\delta) \mathbf{P}_0\|}{1 - 4\|\mathbf{B}(\delta)\|/\mu} \tag{3.5}$$

is valid with the same C as in Theorem 3.1.

3.1 The convergence of $\big\|\mathbf{P}_0^{\perp}(\delta) - \mathbf{P}_0^{\perp}\big\|$

We start with the norm of the operator (3.3) for the series (3.1).

Lemma 3.1. *Assume that $N \to \infty$ and $L \sim \alpha N$ with $\alpha \in (0,1)$. Then there exist $\delta_0 > 0$, N_0 and C such that*

$$\|\mathbf{B}(\delta)\|/\mu \le B(\delta)/\mu \le C\delta^2 < 1/4$$

for any δ with $|\delta| \le \delta_0$ and $N > N_0$.

Proof. First of all, $\mathbf{E}\mathbf{E}^{\mathrm{T}} = O(N^2)$ as $N \to \infty$. Applying (2.9) we get that

$$\begin{aligned} &\|\mathbf{H}\mathbf{E}^{\mathrm{T}} + \mathbf{E}\mathbf{H}^{\mathrm{T}}\| \le 2\|\mathbf{E}W_K\| \, \|W_L\| \le \\ &2 \sqrt{\sum_{\ell=0}^{L-1} \varPhi_K^2(a^{T/N}, \xi\ell + \varphi) \, \frac{a^{2LT/N} - 1}{a^{2T/N} - 1}} \,. \end{aligned} \tag{3.6}$$

It can be checked that $|\varPhi_K(a^{T/N}, \psi)| \le C$ with a certain constant $C = C(a, T, \alpha, \xi)$ that does not depend on ψ. For the further use we denote

$$C_1 = \max\big(C(a, T, \alpha, \xi), C(a, T, 1-\alpha, \xi)\big). \tag{3.7}$$

Since

$$\frac{a^{2LT/N} - 1}{a^{2T/N} - 1} \sim \frac{a^{2\alpha T} - 1}{2T \ln a} \, N$$

as $N \to \infty$, then it follows from (3.6) that

$$\|\mathbf{B}(\delta)\| \le B(\delta) \le O(N^2).$$

In view of the asymptotic

$$\mu = \frac{(a^{2\alpha T} - 1)(a^{2(1-\alpha)T} - 1)}{4T^2 \ln^2 a} \, N^2 + o(N^2), \tag{3.8}$$

the proof is complete. □

Proposition 3.1. *Under the conditions of Lemma 3.1, $\big\|\mathbf{P}_0^{\perp}(\delta) - \mathbf{P}_0^{\perp}\big\| = O(N^{-1})$ as $N \to \infty$.*

Proof. Due to Theorem 3.1 and Lemma 3.1, it is sufficient to proof that

$$\|\mathbf{S}_0 \mathbf{B}(\delta)\| = O(1/N). \tag{3.9}$$

By (2.7),

$$\begin{aligned} \mathbf{S}_0 \mathbf{B}(\delta) = \delta \, &\frac{\mathbf{D}_1}{\|W_L\|^2 \|W_K\|^2} + \delta \, \frac{\mathbf{D}_2}{\|W_L\|^4 \|W_K\|^2} + \\ &\delta^2 \, \frac{\mathbf{D}_3}{\|W_L\|^4 \|W_K\|^2} \end{aligned} \tag{3.10}$$

with

$$\begin{aligned} \mathbf{D}_1 = W_L W_K^{\mathrm{T}} \mathbf{E}^{\mathrm{T}}, \quad \mathbf{D}_2 = W_L W_L^{\mathrm{T}} \mathbf{E} W_K W_L^{\mathrm{T}}, \quad \text{and} \\ \mathbf{D}_3 = W_L W_L^{\mathrm{T}} \mathbf{E}\mathbf{E}^{\mathrm{T}}. \end{aligned} \tag{3.11}$$

Consider summands in the righthand side of (3.10) separately. First of all,

$$\mathbf{D}_3\mathbf{D}_3^{\mathrm{T}} = W_L W_L^{\mathrm{T}}\mathbf{E}\mathbf{E}^{\mathrm{T}}\mathbf{E}\mathbf{E}^{\mathrm{T}}W_L W_L^{\mathrm{T}} \le W_L W_L^{\mathrm{T}} \sum_{i=0}^{L-1}\left(\sum_{j=0}^{L-1} a^{jT/N}\Psi_K(\varphi,i,j)\right)^2,$$

where

$$\Psi_M(\psi,k,\ell) = \sum_{j=0}^{M-1}\cos_{j+k}(\psi)\cos_{j+\ell}(\psi). \tag{3.12}$$

Since

$$\sum_{j=0}^{L-1} a^{jT/N}\Psi_K(\varphi,i,j) = \frac{K}{2}\,\Phi_L(a^{T/N},-i\xi)+ \frac{\sin(K\xi)}{2\sin\xi}\,\Phi_L(a^{T/N}, i\xi+(K-1)\xi+2\varphi), \tag{3.13}$$

we get

$$\|\mathbf{D}_3\|^2 \le \frac{a^{2TL/N}-1}{a^{2T/N}-1}\left(C_1^2\,\frac{K^2L}{4}+O(LK)\right) = \frac{a^{2\alpha T}-1}{2T\ln a}\,C_1^2\,\frac{(1-\alpha)^2\alpha}{4}\,N^4+o(N^4)$$

with C_1 defined in (3.7).

Thus $\|\mathbf{D}_3\|\left(\|W_L\|^4\|W_K\|^2\right)^{-1} = O(1/N)$. In the same manner,

$$\mathbf{D}_2 = W_L W_L^{\mathrm{T}}\sum_{i=0}^{L-1} a^{iT/N}\Phi_K(a^{T/N}, i\xi+\varphi)$$

and

$$\|\mathbf{D}_2\| \le C_1\sqrt{\frac{a^{2\alpha T}-1}{2}}\;\frac{a^{\alpha T}-1}{(T\ln a)^{3/2}}\,N^{3/2}+o(N^{3/2}).$$

Therefore, $\|\mathbf{D}_2\|\left(\|W_L\|^4\|W_K\|^2\right)^{-1} = O(1/N)$. Lastly,

$$\mathbf{D}_1\mathbf{D}_1^{\mathrm{T}} = W_L W_K^{\mathrm{T}}\mathbf{E}^{\mathrm{T}}\mathbf{E}W_K W_L^{\mathrm{T}} = W_L W_L^{\mathrm{T}}\sum_{i=0}^{L-1}\Phi_K^2(a^{T/N}, i\xi+\varphi).$$

Since $\sum_{i=0}^{L-1}\Phi_K^2(a^{T/N}, i\xi+\varphi) \le LC_1^2$, then

$$\|\mathbf{D}_1\|^2 \le \frac{a^{2T_0L/N}-1}{a^{2T_0/N}-1}\,C_1^2 L = \alpha\,\frac{a^{2\alpha T}-1}{2T\ln a}C_1^2\,N^2+o(N^2),$$

$\|\mathbf{D}_1\|\left(\|W_L\|^2\|W_K\|^2\right)^{-1} = O(1/N)$ and the proof is complete. □

3.2 Reconstruction errors

To investigate the reconstruction errors we use the same idea as in Section 2 but deal with the inequality (3.5) instead of (1.6) and use the expression

$$\Delta_\delta(\mathbf{H}) = \left(\mathbf{P}_0^{\perp}(\delta)-\mathbf{P}_0^{\perp}-\mathbf{L}(\delta)\right)\mathbf{H}(\delta)+ \delta\mathbf{P}_0^{\perp}\mathbf{E}+\mathbf{L}(\delta)\mathbf{H}+\delta\mathbf{L}(\delta)\mathbf{E}\,. \tag{3.14}$$

instead of (2.1). For this goal, we need the following supplementary assertions.

Lemma 3.2. *Denote* $\mathbf{Z} = \delta\mathbf{A}_0^{(2)}/\mu = \delta\mathbf{P}_0\mathbf{E}\mathbf{E}^{\mathrm{T}}\mathbf{P}_0/\mu$. *Then there exists a constant* C_2 *such that*

$$\|\mathbf{Z}\|_{\max} \le |\delta|C_2/N. \tag{3.15}$$

Proof. First of all,

$$\left(\mathbf{A}_0^{(2)}\right)_{m,\ell} = \left(\mathbf{E}\mathbf{E}^{\mathrm{T}}\right)_{m,\ell} - \frac{\left(\mathbf{E}\mathbf{E}^{\mathrm{T}}W_L W_L^{\mathrm{T}}\right)_{m,\ell}}{\|W_L\|^2} - \frac{\left(W_L W_L^{\mathrm{T}}\mathbf{E}\mathbf{E}^{\mathrm{T}}\right)_{m,\ell}}{\|W_L\|^2} + \frac{\left(W_L W_L^{\mathrm{T}}\mathbf{E}\mathbf{E}^{\mathrm{T}}W_L W_L^{\mathrm{T}}\right)_{m,\ell}}{\|W_L\|^4}.$$

Note that $\left(\mathbf{E}\mathbf{E}^{\mathrm{T}}\right)_{m,\ell} = \Psi_K(\varphi,m,\ell)$, where $\Psi_M(\psi,k,\ell)$ is defined in (3.12). Analogously,

$$\left(\mathbf{E}\mathbf{E}^{\mathrm{T}}W_L W_L^{\mathrm{T}}\right)_{m,\ell} = a^{\ell T/N}\sum_{j=0}^{L-1} a^{jT/N}\Psi_K(\varphi,m,j),$$

$$\left(W_L W_L^{\mathrm{T}}\mathbf{E}\mathbf{E}^{\mathrm{T}}\right)_{m,\ell} = a^{mT/N}\sum_{j=0}^{L-1} a^{jT/N}\Psi_K(\varphi,j,\ell), \quad \text{and}$$

$$\left(W_L W_L^{\mathrm{T}}\mathbf{E}\mathbf{E}^{\mathrm{T}}W_L W_L^{\mathrm{T}}\right)_{m,\ell} = a^{(\ell+m)T/N}\sum_{k,j=0}^{L-1} a^{(j+k)T/N}\Psi_K(\varphi,k,j).$$

In view of (3.13),

$$\frac{\left(\mathbf{E}\mathbf{E}^{\mathrm{T}}W_L W_L^{\mathrm{T}}\right)_{m,\ell}}{\|W_L\|^2} + \frac{\left(W_L W_L^{\mathrm{T}}\mathbf{E}\mathbf{E}^{\mathrm{T}}\right)_{m,\ell}}{\|W_L\|^2} + \frac{\left(W_L W_L^{\mathrm{T}}\mathbf{E}\mathbf{E}^{\mathrm{T}}W_L W_L^{\mathrm{T}}\right)_{m,\ell}}{\|W_L\|^4} = O(1).$$

Since $\Psi_K(\varphi,m,\ell) = K\cos(\xi(m-\ell))/2+O(1)$ as $K\to\infty$, then

$$\left(\mathbf{A}_0^{(2)}\right)_{m,\ell} = \frac{\cos((m-\ell)\xi)}{2}\,K+o(1)$$

uniformly in $0\le m,\ell\le L-1$. Therefore, see (3.8),

$$\left(\mathbf{Z}\right)_{m,\ell} \sim \delta\cos((m-\ell)\xi)\;\frac{2(1-\alpha)T^2\ln^2 a}{(a^{2\alpha T}-1)(a^{2(1-\alpha)T}-1)}\;\frac{1}{N}$$

as $N\to\infty$ and the assertion proved. □

Remark 3.1. As a consequence of the inequality (3.15) we get that for any $n \geq 1$

$$\|\mathbf{Z}^n\|_{\max} \leq |\delta|^n C_2^n / N.$$

Since $\|\mathbf{Z}^n\|_{\max} \leq L\|\mathbf{Z}^{n-1}\|_{\max}\|\mathbf{Z}\|_{\max}$ and $L\|\mathbf{Z}\|_{\max} \leq |\delta| C_2$, this fact can be proved with the help of a simple induction. Therefore, if $|\delta| < 1/C_2$, then

$$\Big\|\sum_{n\geq 1}\mathbf{Z}^n\Big\|_{\max} \leq \sum_{n\geq 1}\|\mathbf{Z}^n\|_{\max} \leq \frac{|\delta|C_2}{1-|\delta|C_2}\,\frac{1}{N}\,. \tag{3.16}$$

Lemma 3.3. *If the series $f_n = f_n^{(N)}$ is defined by (3.1), $N \to \infty$ and $L \sim \alpha N$ with $\alpha \in (0,1)$, then*

1) $\|\mathbf{B}(\delta)\mathbf{H}\|_{\max} = O(N)$;
2) $\|\mathbf{S}_0\mathbf{B}(\delta)\|_{\max} = O(1/N^2)$;
3) $\|\mathbf{B}(\delta)\mathbf{S}_0\mathbf{E}\|_{\max} = O(1/N^2)$ and
4) $\|\mathbf{P}_0^{\perp}\mathbf{E}\|_{\max} = O(1/N)$.

Proof. 1) The matrix $\mathbf{B}(\delta)\mathbf{H}$ can be rewritten as follows:

$$\mathbf{B}(\delta)\mathbf{H} = \delta\|W_L\|^2\mathbf{J}_1 + \delta\mathbf{J}_2 + \delta^2\mathbf{J}_3$$

with $\mathbf{J}_1 = \mathbf{E}W_K W_K^{\mathrm{T}}$, $\mathbf{J}_2 = W_L W_K^{\mathrm{T}}\mathbf{E}^{\mathrm{T}}W_L W_K^{\mathrm{T}}$ and $\mathbf{J}_3 = \mathbf{E}\mathbf{E}^{\mathrm{T}}W_L W_K^{\mathrm{T}}$.

Applying the equalities (2.5) and (3.12), (3.13) we get that

$$\|\mathbf{J}_1\|_{\max} = \max_{k<L,\,\ell<K}\big|\varPhi_K(a^{T/N}, k\xi+\varphi)a^{\ell T/N}\big| \leq C_1\, a^{(1-\alpha)T} + o(1),$$

$$\|\mathbf{J}_2\|_{\max} = \big|\varUpsilon_{L,\,K}(a^{T/N},\varphi)\big| \max_{k<L;\ell<K} a^{(k+\ell)T/N} \leq C_1\,\frac{a^T(a^{\alpha T}-1)}{T\ln a}\,N + o(N),$$

$$\|\mathbf{J}_3\|_{\max} = \max_{k<L,\,\ell<K}\left|\sum_{j=0}^{L-1} a^j\varPsi_K(\varphi,k,j)a^{\ell T/N}\right| \leq C_1\,\frac{(1-\alpha)a^{(1-\alpha)T}}{2}\,N + o(N)$$

with C_1 defined in (3.7). Thus the first assertion is proved.

2) The expression for $\mathbf{S}_0\mathbf{B}(\delta)$ is presented in (3.10), (3.11). It can be checked that

$$\|\mathbf{D}_1\|_{\max} = \max_{k<L;\ell<L}\big|\varPhi_K(a^{T/N}, k\xi+\varphi)a^{\ell T/N}\big| \leq C_1\, a^{\alpha T} + o(1),$$

$$\|\mathbf{D}_2\|_{\max} = \big|\varUpsilon_{L,K}(a^{T/N},\varphi)\big| \max_{k,\ell<L} a^{(k+\ell)T/N} \leq C_1\frac{a^{2\alpha T}(a^{\alpha T}-1)}{T\ln a}\,N + o(N), \quad \text{and}$$

$$\|\mathbf{D}_3\|_{\max} = \max_{k,\ell<L}\left|\sum_{j=0}^{L-1} a^j\varPsi_K(\varphi,k,j)a^{\ell T/N}\right| \leq C_1\,\frac{(1-\alpha)a^{\alpha T}}{2}\,N + o(N).$$

Applying (3.10) we see that $\|\mathbf{S}_0\mathbf{B}(\delta)\|_{\max} = O(1/N^2)$.

3) In the same manner,

$$\mathbf{B}(\delta)\mathbf{S}_0\mathbf{E} = \delta\mathbf{E}\mathbf{H}^{\mathrm{T}}\mathbf{S}_0\mathbf{E} + \delta\mathbf{H}\mathbf{E}\mathbf{S}_0\mathbf{E} + \delta^2\mathbf{E}\mathbf{E}^{\mathrm{T}}\mathbf{S}_0\mathbf{E} = \frac{1}{\|W_L\|^4\|W_K\|^2}\big(\delta\|W_L\|^2\mathbf{E}W_K W_L^{\mathrm{T}}\mathbf{E} + \delta W_L W_K^{\mathrm{T}}\mathbf{E}^{\mathrm{T}}W_L W_L^{\mathrm{T}}\mathbf{E} + \delta^2\mathbf{E}\mathbf{E}^{\mathrm{T}}W_L W_L^{\mathrm{T}}\mathbf{E}\big)$$

with

$$\big\|\mathbf{E}W_K W_L^{\mathrm{T}}\mathbf{E}\big\|_{\max} = \max_{k<L;\ell<K}\big|\varPhi_K(a^{T/N}, k\xi+\varphi)\varPhi_L(a^{T/N}, \ell\xi+\varphi)\big| = O(1),$$

$$\big\|W_L W_K^{\mathrm{T}}\mathbf{E}^{\mathrm{T}}W_L W_L^{\mathrm{T}}\mathbf{E}\big\|_{\max} = \big|\varUpsilon_{L,K}(a^{T/N},\varphi)\big| \max_{k<L;\ell<K}\big|a^{kT/N}\varPhi_L(a^{T/N}, \ell\xi+\varphi)\big| \leq C_1^2\,\frac{a^{\alpha T}(a^{\alpha T}-1)}{T\ln a}\,N + o(N), \quad \text{and}$$

$$\big\|\mathbf{E}\mathbf{E}^{\mathrm{T}}W_L W_L^{\mathrm{T}}\mathbf{E}\big\|_{\max} = \max_{k<L;\ell<K}\left|\sum_{j=0}^{L-1} a^j\varPsi_K(\varphi,k,j)\varPhi_L(a^{T/N}, \ell\xi+\varphi)\right| \leq NC_1^2\,(1-\alpha)/2 + o(N).$$

Therefore, $\|\mathbf{B}(\delta)\mathbf{S}_0\mathbf{E}\|_{\max} = O(1/N^2)$.

4) Since $\mathbf{P}_0^{\perp}\mathbf{E} = W_L W_L^{\mathrm{T}}\mathbf{E}/\|W_L\|^2$, then

$$\|\mathbf{P}_0^{\perp}\mathbf{E}\|_{\max} = \frac{1}{\|W_L\|^2}\max_{k<K;\ell<L}\big|\varPhi_L(a^{T/N}, k\xi+\varphi)a^{\ell T/N}\big| \leq C_1\,\frac{2T\ln a}{a^{2\alpha T}-1}\,a^{\alpha T}\,\frac{1}{N} + o(1/N),$$

and the proof is complete. □

Proposition 3.2. *Denote $r_j = r_j(N,\delta)$ the reconstruction error for the term $x_j = a^{jT/N}$ of the perturbed series $f_j = x_j + \delta\cos(\xi j + \varphi)$ with $a > 1$, $\xi \in (0,\pi)$ and $\varphi \in [0, 2\pi)$.*

If $N \to \infty$ and $L = \alpha N + o(N)$ with $0 < \alpha < 1$, then there exists $\delta^* > 0$ such that $r_j = O(1/N)$ uniformly in $0 \leq j < N$ for any δ with $|\delta| < \delta^*$.

Proof. First of all, due to (3.14),

$$\mathbf{P}_0^{\perp}(\delta)\mathbf{H}(\delta) - \mathbf{P}_0^{\perp}\mathbf{H} = \big(\mathbf{P}_0^{\perp}(\delta) - \mathbf{P}_0^{\perp} - \mathbf{L}(\delta)\big)\mathbf{H}(\delta) + \mathbf{L}(\delta)\mathbf{H} + \delta\mathbf{L}(\delta)\mathbf{E} + \delta\mathbf{P}_0^{\perp}\mathbf{E}. \tag{3.17}$$

Then, see Lemma 3.1, the inequality (3.5) holds for any δ such that $|\delta| < \delta_0$. It follows from (3.5) and (3.9) that $\big\|\big(\mathbf{P}_0^{\perp}(\delta) - \mathbf{P}_0^{\perp} - \mathbf{L}(\delta)\big)\big\| = O(1/N^2)$ for $|\delta| < \delta_0$, and therefore

$$\big\|\big(\mathbf{P}_0^{\perp}(\delta) - \mathbf{P}_0^{\perp} - \mathbf{L}(\delta)\big)\mathbf{H}(\delta)\big\| \leq \big\|\big(\mathbf{P}_0^{\perp}(\delta) - \mathbf{P}_0^{\perp} - \mathbf{L}(\delta)\big)\big\|\,\|\mathbf{H}(\delta)\| = O(1/N).$$

Thus we must check the three last terms in the righthand side of (3.17).

Note that $\|\mathbf{P}_0^{\perp}\mathbf{E}\|_{\max} = O(1/N)$, see Lemma 3.3. Let us consider operators $\mathbf{L}(\delta)\mathbf{H}$ and $\mathbf{L}(\delta)\mathbf{E}$. As in Lemma 3.2, put $\mathbf{Z} = \delta\mathbf{A}_0^{(2)}/\mu$. Since $\mathbf{P}_0\mathbf{H} = \mathbf{0}$, then $\mathbf{Z}\mathbf{H} = \mathbf{0}$ and

$$\mathbf{L}_1(\delta)\mathbf{H} = \mathbf{S}_0\mathbf{B}(\delta)\mathbf{P}_0\big(\mathbf{I}-\mathbf{Z}\big)^{-1}\mathbf{H} = \mathbf{S}_0\mathbf{B}(\delta)\mathbf{P}_0\sum_{m\geq 0}\mathbf{Z}^m\mathbf{H} = \mathbf{0}.$$

Therefore,

$$\mathbf{L}(\delta)\mathbf{H} = \mathbf{L}_1^{\mathrm{T}}(\delta)\mathbf{H} = \Big(\mathbf{I}-\mathbf{Z}\Big)^{-1}\mathbf{P}_0\mathbf{B}(\delta)\mathbf{H}/\mu = \frac{1}{\mu}\left(\mathbf{I}+\sum_{m\geq 1}\mathbf{Z}^m\right)\mathbf{P}_0\mathbf{B}(\delta)\mathbf{H} = \frac{1}{\mu}\mathbf{P}_0\mathbf{B}(\delta)\mathbf{H} + \frac{1}{\mu}\sum_{m\geq 1}\mathbf{Z}^m\mathbf{P}_0\mathbf{B}(\delta)\mathbf{H} = \frac{1}{\mu}\mathbf{P}_0\mathbf{B}(\delta)\mathbf{H} + \frac{1}{\mu}\left(\sum_{m\geq 1}\mathbf{Z}^m\right)\mathbf{B}(\delta)\mathbf{H}.$$

Remind that $\mathbf{P}_0^{\perp} = W_L W_L^{\mathrm{T}}/\|W_L\|^2$ and $\|W_L W_L^{\mathrm{T}}\|_{\max} \leq a^{2T}$. Thus $\|\mathbf{P}_0^{\perp}\|_{\max} = O(1/N)$ and, in view of Lemma 3.3,

$$\big\|\mathbf{P}_0\mathbf{B}(\delta)\mathbf{H}/\mu\big\|_{\max} = \big\|\big(\mathbf{I}-\mathbf{P}_0^{\perp}\big)\mathbf{B}(\delta)\mathbf{H}/\mu\big\|_{\max} \leq \|\mathbf{B}(\delta)\mathbf{H}\|_{\max}/\mu + \big\|\mathbf{P}_0^{\perp}\mathbf{B}(\delta)\mathbf{H}\big\|_{\max}/\mu \leq \|\mathbf{B}(\delta)\mathbf{H}\|_{\max}/\mu + L\,\big\|\mathbf{P}_0^{\perp}\big\|_{\max}\;\|\mathbf{B}(\delta)\mathbf{H}\|_{\max}/\mu = O(1/N).$$

Besides, if additionally $|\delta|C_2 < 1$, then, due to (3.8) and (3.16),

$$\Big\|\frac{1}{\mu}\Big(\sum_{m\geq 1}\mathbf{Z}^m\Big)\mathbf{B}(\delta)\mathbf{H}\Big\|_{\max} \leq \frac{L}{\mu}\Big\|\sum_{m\geq 1}\mathbf{Z}^m\Big\|_{\max}\;\Big\|\mathbf{B}(\delta)\mathbf{H}\Big\|_{\max} \leq \frac{L}{\mu}\,\frac{|\delta|C_2}{1-|\delta|C_2}\,\frac{1}{N}\;\Big\|\mathbf{B}(\delta)\mathbf{H}\Big\|_{\max} = O(1/N).$$

As the result, $\|\mathbf{L}(\delta)\mathbf{H}\|_{\max} = O(1/N)$.

By definition, $\mathbf{L}(\delta)\mathbf{E} = \mathbf{L}_1(\delta)\mathbf{E} + \mathbf{L}_1^{\mathrm{T}}(\delta)\mathbf{E}$ with

$$\mathbf{L}_1(\delta)\mathbf{E} = \mathbf{S}_0\mathbf{B}(\delta)\mathbf{P}_0\big(\mathbf{I}-\mathbf{Z}\big)^{-1}\mathbf{E} \quad\text{and}\quad \mathbf{L}_1^{\mathrm{T}}(\delta)\mathbf{E} = \big(\mathbf{I}-\mathbf{Z}\big)^{-1}\mathbf{P}_0\mathbf{B}(\delta)\mathbf{S}_0\mathbf{E}.$$

The equality $\|\mathbf{L}_1^{\mathrm{T}}(\delta)\mathbf{E}\|_{\max} = O(1/N^2)$ can be proved in the same manner as $\|\mathbf{L}_1^{\mathrm{T}}(\delta)\mathbf{H}\|_{\max} = O(1/N)$, but with the help of the equality $\|\mathbf{B}(\delta)\mathbf{S}_0\mathbf{E}\|_{\max} = O(1/N^2)$, see Lemma 3.3. Now note that

$$\mathbf{L}_1(\delta)\mathbf{E} = \mathbf{S}_0\mathbf{B}(\delta)\mathbf{P}_0\mathbf{E} + \mathbf{S}_0\mathbf{B}(\delta)\mathbf{P}_0\bigg(\sum_{m\geq 1}\mathbf{Z}^m\bigg)\mathbf{E} = \mathbf{S}_0\mathbf{B}(\delta)\mathbf{P}_0\mathbf{E} + \mathbf{S}_0\mathbf{B}(\delta)\bigg(\sum_{m\geq 1}\mathbf{Z}^m\bigg)\mathbf{E}.$$

Taking into account (3.16) and the equalities $\|\mathbf{E}\|_{\max} = O(1)$, $\|\mathbf{S}_0\mathbf{B}(\delta)\|_{\max} = O(1/N^2)$, we get

$$\Big\|\mathbf{S}_0\mathbf{B}(\delta)\Big(\sum_{m\geq 1}\mathbf{Z}^m\Big)\mathbf{E}\Big\|_{\max} \leq L^2\,\|\mathbf{S}_0\mathbf{B}(\delta)\|_{\max}\,\Big\|\sum_{m\geq 1}\mathbf{Z}^m\Big\|_{\max}\,\|\mathbf{E}\|_{\max} = O(1/N).$$

Lastly,

$$\|\mathbf{S}_0\mathbf{B}(\delta)\mathbf{P}_0\mathbf{E}\|_{\max} \leq \|\mathbf{S}_0\mathbf{B}(\delta)\mathbf{E}\|_{\max} + \|\mathbf{S}_0\mathbf{B}(\delta)\mathbf{P}_0^{\perp}\mathbf{E}\|_{\max} \leq L\|\mathbf{S}_0\mathbf{B}(\delta)\|_{\max}\|\mathbf{E}\|_{\max} + L^2\|\mathbf{S}_0\mathbf{B}(\delta)\|_{\max}\,\|\mathbf{P}_0^{\perp}\|_{\max}\,\|\mathbf{E}\|_{\max} = O(1/N).$$

Therefore, $\|\mathbf{L}(\delta)\mathbf{E}\|_{\max} = O(1/N)$. Finally, the uniform norm $\|\cdot\|_{\max}$ of each addend in the sum (3.14) has the order $O(1/N)$. Since $\|\mathcal{S}\mathbf{C}\|_{\max} \leq \|\mathbf{C}\|_{\max}$ for any matrix $\mathbf{C}$, the proof is complete. □

4. COMMENTS AND REMARKS

Let us discuss the results of the paper and their relation to SSA problems.

Of course, the general model "exponential signal plus harmonic noise" corresponds to simply structured cases of time series. Still this model is not far from some real-life time series such as the "gasoline demand" series.

Our goal is to extract the signal from this sum with the help of Singular Spectrum Analysis and to study the precision of this extraction. As a rule (see [1] or [9] for details), SSA needs a big number N of observations, therefore the corresponding theoretical approach must be asymptotical as $N \to \infty$. Thus we come to the problem of the asymptotic analysis of reconstruction errors $r_j(N)$, $0 \leq j < N$.

Here we consider two analytical models for "exponential signal plus harmonic noise" series. The first, see (1.4), is rather straightforward and corresponds to the general scheme of [1] and [2]: the signal tends to infinity as $N \to \infty$ while the frequency of the harmonic noise does not depend on N.

Then (see propositions 2.1 and 2.2–2.4) for any amplitude of the noise series, $r_j(N) \to 0$ if $N - j \to \infty$ and $r_j(N) \nrightarrow 0$ if $N - j - O(1)$.

This means that $\|R_N\|_{\max} \stackrel{\mathrm{def}}{=} \max_j |r_j(N)|$ does not tend to zero as $N \to \infty$. Of course, the max-norm $\|\cdot\|_{\max}$ applied to the series R_N of reconstruction errors can be considered as a too strong measure of the SSA accuracy. (Note that more usual $\|R_N\|_2 = \sqrt{\sum_j r_N^2(j)/N}$ tends to zero here.) Still the last terms of the reconstructed signal are of special interest, since just these terms are used in many forecasting algorithms.

The second model is more complicated. The goal of this model is to describe the situation when the range of the increasing signal is relatively small while the number of periods of the additive harmonic is big. As it was already mentioned, the second condition is usual in SSA. The first seems to be more or less adequate for real-life series if we consider big N. (Naturally, it is difficult to expect that the trend of a real-life series can rapidly increase for a very long period of time.)

Note that both conditions seem to be fulfilled for the "gasoline demand" series: the trend of this series grows rather slowly over the period of observations, while the number of periods of the annual harmonic is relatively big.

The model (1.5) is the attempt to formalize these conditions. Here we deal with the triangle array of the perturbed signals, where all exponential signals are uniformly bounded from above while the harmonic noises are the same as in (1.4). In other words, the "discretization" is applied only to the exponential trend.

It is remarkable, that the accuracy of the standard model (1.4) fails for last terms of the series, while the second model (1.5) shows asymptotically good results for all terms, provided that the amplitude of the noise component is sufficiently small.

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