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The generalized half-t distribution

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In this paper, we introduce a new distribution as a scale mixture of the generalized half normal (GHN) distribution proposed by [3] and the generalized gamma (GG) distribution. Since the half-t (HT) distribution given in [10] is a special case of the new distribution, we call the new distribution as "generalized half-t (GHT)" distribution. We derive the probability density function (pdf) of the GHT distribution and study some of its properties. We give maximum likelihood (ML) estimators for its parameters based on the Expectation-Maximization (EM) algorithm. We provide a small simulation study to show the performances of the ML estimators for GHT distribution. Also, we give a real data example to illustrate the modeling performance of the proposed distribution over the GHN and HT distributions.

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1. INTRODUCTION

A new family of lifetime distribution with two parameters was introduced by [3]. Since they noticed that the cumulative distribution function (cdf) of the new distribution resembles to the cdf of the half normal distribution, they called it generalized half normal (GHN) distribution. In their study, they compared the GHN distribution with other lifetime distributions such as Weibull, gamma, lognormal and Birnbaum-Saunders.

In this paper, we introduce the scale mixture of the GHN distribution and the generalized gamma (GG) distribution [5, 8]. The new distribution has four parameters and contains the half-t (HT) distribution given in [10]. We call the new distribution as generalized half-t (GHT) distribution. As a result of the scale mixture approach, the new distribution will have heavier tail than the GHN distribution. Therefore, it will be a robust alternative to the GHN distribution to model lifetime data sets that may have outliers.

The rest of the paper is organized as follows. In Section 2, we introduce the GHT distribution and study some of its distributional properties. Mainly in this section, we give the probability density function (pdf), cdf, hazard function and provide the moments of this distribution. In Section 3, we give the maximum likelihood (ML) estimators based on the Expectation-Maximization (EM) algorithm. In Sections 4 and 5, we give a simulation study and a real data example to illustrate the performance of proposed distribution, respectively. Finally, Section 6 is devoted to conclusions.

2. GHT DISTRIBUTION: DEFINITION AND ITS DISTRIBUTIONAL PROPERTIES

Theorem 2.1. *Let the random variable X have a GHN distribution with the parameters $\alpha > 0$ and $\theta > 0$ ($X \sim GHN(\theta, \alpha)$) with the pdf*

$$f_X(x) = \frac{\sqrt{2}}{\sqrt{\pi}}\left(\frac{\alpha}{x}\right)\left(\frac{x}{\theta}\right)^{\alpha} e^{-\frac{1}{2}\left(\frac{x}{\theta}\right)^{2\alpha}}, \quad x \geq 0. \tag{1}$$

Let the random V have a GG distribution with the parameters $\alpha > 0$, $\beta > 0$ and $\frac{\eta}{2} > 0$ $\left(V \sim GG\left(\alpha, \beta, \frac{\eta}{2}\right)\right)$ with the pdf

$$f_V(v) = \frac{\alpha}{\beta^{\frac{\eta}{2}}\Gamma\left(\frac{\eta}{2\alpha}\right)} v^{\frac{\eta}{2}-1} e^{-\frac{1}{2}\left(\frac{v}{\beta}\right)^{\alpha}}, \quad x \geq 0 \tag{2}$$

and assume that X and $V$are independent. Then, the distribution of the random variable T

$$T = \frac{X}{\sqrt{V/\eta}} \tag{3}$$

has the following pdf

$$\begin{aligned} f_T(t;\ \alpha,\ \beta,\ \theta,\ \eta) &= \frac{\sqrt{2}\alpha t^{\alpha-1}}{\sqrt{\pi}\eta^{\alpha/2}\theta^{\alpha}\beta^{\eta/2}} \frac{\Gamma\left(\frac{\eta}{2\alpha} - \frac{1}{\alpha} + \frac{3}{2}\right)}{\Gamma\left(\frac{\eta}{2\alpha}\right)} \\ &\times \left[\frac{1}{\beta^{\alpha}}\left(1 + \frac{t^{2\alpha}\beta^{\alpha}}{2\eta^{\alpha}\theta^{2\alpha}}\right)\right]^{-\frac{\eta}{2\alpha}-\frac{1}{2}}. \end{aligned} \tag{4}$$

Proof. Using the pdfs of GHN and GG distributions given in (1) and (2), the joint pdf of X and V can be written

$$f_{X,V}(x,v) = \frac{\sqrt{2}}{\sqrt{\pi}}\left(\frac{\alpha}{x}\right)\left(\frac{x}{\theta}\right)^{\alpha} e^{-\frac{1}{2}\left(\frac{x}{\theta}\right)^{2\alpha}} \frac{\alpha}{\beta^{\frac{\eta}{2}}\Gamma\left(\frac{\eta}{2\alpha}\right)} v^{\frac{\eta}{2}-1} e^{-\frac{1}{2}\left(\frac{v}{\beta}\right)^{\alpha}}.$$

Using the transformation $x = t\sqrt{\frac{v}{\eta}}$, the joint pdf of T and V can be obtained

(5) $f_{T,V}(t,v)$

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$$= \frac{\sqrt{2}\alpha^2 t^{\alpha-1} v^{\frac{\alpha}{2}+\frac{\eta}{2}-1}}{\sqrt{\pi}\eta^{\frac{\alpha}{2}}\theta^{\alpha}\beta^{\frac{\eta}{2}}\Gamma\left(\frac{\eta}{2\alpha}\right)}\exp\left\{-\frac{v^{\alpha}}{\beta^{\alpha}}\left(1+\frac{t^{2\alpha}\beta^{\alpha}}{2\eta^{\alpha}\theta^{2\alpha}}\right)\right\}.$$

From this joint pdf, the pdf of T can be obtained as follows

$$(6)\quad \begin{aligned} f_T(t;\alpha,\beta,\theta,\eta) &= \int_0^{\infty} f_{T,V}(t,\ v)\,dv \\ &= \frac{\sqrt{2}\alpha t^{\alpha-1}}{\sqrt{\pi}\eta^{\alpha/2}\theta^{\alpha}\beta^{\eta/2}}\frac{\Gamma\left(\frac{\eta}{2\alpha}-\frac{1}{\alpha}+\frac{3}{2}\right)}{\Gamma\left(\frac{\eta}{2\alpha}\right)} \\ &\times\left[\frac{1}{\beta^{\alpha}}\left(1+\frac{t^{2\alpha}\beta^{\alpha}}{2\eta^{\alpha}\theta^{2\alpha}}\right)\right]^{-\frac{\eta}{2\alpha}-\frac{1}{2}}. \qquad \square \end{aligned}$$

We get the GHN distribution when η tends to infinity. If we set $\alpha = 1$ and $\beta = 2$, the pdf given in (4) becomes as

$$(7)\qquad f_T(t) = \frac{2\Gamma\left(\frac{\eta+1}{2}\right)}{\sqrt{\pi\eta}\theta\Gamma\left(\frac{\eta}{2}\right)}\left(1+\frac{t^2}{\eta\theta^2}\right)^{-\frac{\eta+1}{2}},$$

which is the pdf of HT disitribution given in [10], see also [9]. Further, if η tends to infinitiy we get the half normal distribution which is a special case of folded normal and truncated normal distribution [5].

Definition 2.1. *The random variable $T \in (0,\infty)$ is said to have a GHT distribution with the parameters $\alpha > 0$, $\beta > 0$, $\theta > 0$ and $\eta > 0$ ($T \sim GHT(\alpha,\ \beta,\ \theta,\ \eta)$) if its pdf has the form given in (4).*

Figure 1 shows the pdf plots of the GHT distribution for some values of the parameters. We observe from panel (a) Figure 1 that the location of mode and the peakedness of the density change for increasing α. From panel (b) Figure 1, we can see that when β is getting larger, the tail is getting thicker and the density becomes more peaked. On the other hand, we observe from panel (c) Figure 1 that the tail becomes thicker for decreasing η. Finally, we see from panel (d) Figure 1 that the parameter θ has influence on the kurtosis of the density. Overall, we can conclude that using the scale mixture approach thick tail distributions are produced, which may be useful alternatives in robust statistical analysis of life time data sets.

2.1 CDF and hazard function

Theorem 2.2. *i. For $t \in \mathbb{R}$, the cdf of the GHT distribution is*

$$(8)\quad \begin{aligned} F_T(t) &= \left(\frac{\eta}{2\alpha}-\frac{1}{\alpha}+\frac{1}{2}\right)\frac{\Gamma\left(\frac{\eta}{2\alpha}-\frac{1}{\alpha}+\frac{1}{2}\right)}{\Gamma\left(\frac{\eta}{2\alpha}+\frac{1}{2}\right)} \\ &-\frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{\eta}{2\alpha}-\frac{1}{\alpha}+\frac{3}{2}\right)}{\Gamma\left(\frac{\eta}{2\alpha}\right)}B\left(\frac{2\eta^{\alpha}\theta^{2\alpha}}{2\eta^{\alpha}\theta^{2\alpha}+\beta^{\alpha}t^{2\alpha}};\frac{\eta}{2\alpha},\frac{1}{2}\right), \end{aligned}$$

where $B(\cdot)$ shows the incomplete beta function.
ii. The hazard rate function of the GHT distribution is

$$(9)\quad \begin{aligned} h(t) &= \frac{\sqrt{2}t^{\alpha-1}\Gamma\left(\frac{\eta}{2\alpha}-\frac{1}{\alpha}+\frac{3}{2}\right)}{\sqrt{\pi}\eta^{\frac{\alpha}{2}}\beta^{\frac{\eta}{2}}\Gamma\left(\frac{\eta}{2\alpha}\right)}\left[\frac{1}{\beta^{\alpha}}\left(1+\frac{t^{2\alpha}\beta^{\alpha}}{2\eta^{\alpha}\theta^{2\alpha}}\right)\right]^{-\frac{\eta}{2\alpha}-\frac{1}{2}} \\ &\times\left(1-\left(\frac{\eta}{2\alpha}-\frac{1}{\alpha}+\frac{1}{2}\right)\frac{\Gamma\left(\frac{\eta}{2\alpha}-\frac{1}{\alpha}+\frac{1}{2}\right)}{\Gamma\left(\frac{\eta}{2\alpha}+\frac{1}{2}\right)}\right. \\ &\left.+\frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{\eta}{2\alpha}-\frac{1}{\alpha}+\frac{3}{2}\right)}{\Gamma\left(\frac{\eta}{2\alpha}\right)}B\left(\frac{2\eta^{\alpha}\theta^{2\alpha}}{2\eta^{\alpha}\theta^{2\alpha}+\beta^{\alpha}t^{2\alpha}};\frac{\eta}{2\alpha},\frac{1}{2}\right)\right)^{-1} \end{aligned}$$

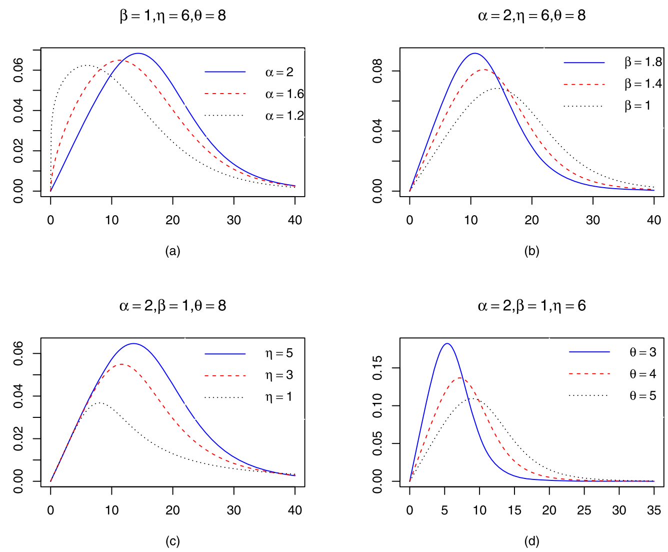

Figure 1. Examples of the GHT pdf for some values of parameters.

Proof. *i.* Using the definition of the cdf and the pdf given in (4), the cdf of T will be

$$\begin{aligned} F_T(t) &= \int_0^t f_X(x)\,dx \\ &= \int_0^t \frac{\sqrt{2}\alpha x^{\alpha-1}}{\sqrt{\pi}\eta^{\alpha/2}\theta^{\alpha}\beta^{\eta/2}}\frac{\Gamma\left(\frac{\eta}{2\alpha}-\frac{1}{\alpha}+\frac{3}{2}\right)}{\Gamma\left(\frac{\eta}{2\alpha}\right)} \\ &\times\left[\frac{1}{\beta^{\alpha}}\left(1+\frac{x^{2\alpha}\beta^{\alpha}}{2\eta^{\alpha}\theta^{2\alpha}}\right)\right]^{-\frac{\eta}{2\alpha}-\frac{1}{2}}dx \\ &= \frac{\sqrt{2}\alpha}{\sqrt{\pi}\eta^{\alpha/2}\theta^{\alpha}\beta^{\eta/2}}\frac{\Gamma\left(\frac{\eta}{2\alpha}-\frac{1}{\alpha}+\frac{3}{2}\right)}{\Gamma\left(\frac{\eta}{2\alpha}\right)} \\ &\times\int_0^t x^{\alpha-1}\left[\frac{1}{\beta^{\alpha}}\left(1+\frac{x^{2\alpha}\beta^{\alpha}}{2\eta^{\alpha}\theta^{2\alpha}}\right)\right]^{-\frac{\eta}{2\alpha}-\frac{1}{2}}dx. \end{aligned}$$

Using the transformation $w = \left(1+\frac{x^{2\alpha}\beta^{\alpha}}{2\eta^{\alpha}\theta^{2\alpha}}\right)^{-1}$, we obtain the following cdf

$$\begin{aligned} F_T(t) &= \left(\frac{\eta}{2\alpha}-\frac{1}{\alpha}+\frac{1}{2}\right)\frac{\Gamma\left(\frac{\eta}{2\alpha}-\frac{1}{\alpha}+\frac{1}{2}\right)}{\Gamma\left(\frac{\eta}{2\alpha}+\frac{1}{2}\right)} \\ &-\frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{\eta}{2\alpha}-\frac{1}{\alpha}+\frac{3}{2}\right)}{\Gamma\left(\frac{\eta}{2\alpha}\right)}B\left(\frac{2\eta^{\alpha}\theta^{2\alpha}}{2\eta^{\alpha}\theta^{2\alpha}+\beta^{\alpha}t^{2\alpha}};\frac{\eta}{2\alpha},\frac{1}{2}\right). \end{aligned}$$

ii. This part of the proof can be easily obtain using the definition of the hazard rate function and the pdf given in (4). □

2.2 Moments

The moments of the GHN distribution were derived by [3] which are given as follows. Let $X \sim GHN(\theta, \alpha)$, then k^{th} non-central moments for $k = 1, 2, \dots$ are given by

$$E\left[X^k\right] = \sqrt{\frac{2^{\frac{k}{\alpha}}}{\pi}}\Gamma\left(\frac{k+\alpha}{2\alpha}\right)\theta^k. \tag{10}$$

Using these moments, the moments of the GHT distributed random variable T can be obtained as follows.

Proposition 2.1. *Let $X \sim GHT(\alpha, \beta, \theta, \eta)$. For $k = 1, 2, \dots$, the k^{th} non-central moment of T is given by*

(11)
$$\mu_k = E\left[T^k\right] = \left(\frac{\eta}{\beta}\right)^{\frac{k}{2}} \frac{2^{\frac{k}{2\alpha}}}{\sqrt{\pi}}\theta^k \frac{\Gamma\left(\frac{k+\alpha}{2\alpha}\right)\Gamma\left(\frac{\eta-k}{2\alpha}\right)}{\Gamma\left(\frac{\eta}{2\alpha}\right)}, \eta > k.$$

Proof. Using the stochastic representation given in (3) and independence of X and V, we have

$$\begin{aligned}\mu_k &= E\left[T^k\right] = E\left[\left(\frac{X}{\sqrt{\frac{V}{\eta}}}\right)^k\right] \\ &= E\left[X^k \eta^{\frac{k}{2}} V^{-\frac{k}{2}}\right] = \eta^{\frac{k}{2}} E\left[X^k\right] E\left[V^{-\frac{k}{2}}\right].\end{aligned}$$

Then, using (10) and moments of the GG distribution the moment formula given in (11) can be obtained. □

Corollary 2.1. *Let $T \sim GHT(\alpha, \beta, \theta, \eta)$. The mean and the variance of T are given by*

$$E[T] = \frac{\sqrt{\pi}2^{\frac{1}{2\alpha}}\theta\Gamma\left(\frac{\alpha+1}{2\alpha}\right)\Gamma\left(\frac{\eta-1}{2\alpha}\right)}{\sqrt{\pi\beta}\Gamma\left(\frac{\eta}{2\alpha}\right)} \quad \eta > 1, \tag{12}$$

$$\begin{aligned}Var[T] = \frac{\eta 2^{\frac{1}{\alpha}}\theta^2}{\sqrt{\pi}\beta}\{&\frac{\Gamma\left(\frac{\alpha+2}{2\alpha}\right)\Gamma\left(\frac{\eta-2}{2\alpha}\right)}{\Gamma\left(\frac{\eta}{2\alpha}\right)} \\ &- \frac{\Gamma^2\left(\frac{\alpha+1}{2\alpha}\right)\Gamma^2\left(\frac{\eta-1}{2\alpha}\right)}{\sqrt{\pi}\Gamma^2\left(\frac{\eta}{2\alpha}\right)}\}, \eta > 2.\end{aligned} \tag{13}$$

3. MAXIMUM LIKELIHOOD ESTIMATION

Let $\boldsymbol{t} = (t_1, t_2, \dots, t_n)$ be a random sample from distribution with unknown parameters α, β, θ and η. The log-likelihood function is

$$\begin{aligned}\ell(\alpha, \beta, \theta, \eta; \boldsymbol{t}) &= \frac{n}{2}\log(2) + n\log(\alpha) \\ &+ n\log\left(\Gamma\left(\frac{\eta}{2\alpha} - \frac{1}{\alpha} + \frac{3}{2}\right)\right) - \frac{n\alpha}{2}\log(\eta) \\ &- n\alpha\log(\theta) - n\log\left(\Gamma\left(\frac{\eta}{2\alpha}\right)\right) \\ &+ (\alpha-1)\sum_{i=1}^n \log(t_i) + n\left(\frac{\eta+\alpha}{2}\right)\log(\beta) \\ &- \left(\frac{\eta}{2\alpha} + \frac{1}{2}\right)\sum_{i=1}^n \log\left(1 + \frac{t_i^{2\alpha}\beta^\alpha}{2\eta^\alpha\theta^{2\alpha}}\right).\end{aligned}$$

We can see that direct maximization of this function is not very tractable. On the other hand, the scale mixture representation of this distribution can allow us to use the EM algorithm [4] to obtain the ML estimators for the unknown parameters. Here, we will use the EM algorithm and it can be applied as follows.

Assume that the latent variable V cannot be observable. Let $\boldsymbol{v} = (v_1, v_2, \dots, v_n)$ be missing data and let $(\boldsymbol{t}, \boldsymbol{v})$ be the complete data. Now using the joint pdf of the $(\boldsymbol{t}, \boldsymbol{v})$ given in (5), the complete data log-likelihood function can be obtained as

$$\begin{aligned}\ell_c(\alpha, \beta, \theta, \eta; \boldsymbol{t}, \boldsymbol{v}) &= \frac{n}{2}\log\left(\frac{2}{\pi}\right) + 2n\log(\alpha) \\ &- \frac{n\alpha}{2}\log(\eta) - n\alpha\log(\theta) \\ &- \frac{n\eta}{2}\log(\beta) - n\log\left(\Gamma\left(\frac{\eta}{2\alpha}\right)\right) \\ &+ (\alpha-1)\sum_{i=1}^n \log(t_i) \\ &+ \left(\frac{\alpha}{2} + \frac{\eta}{2} - 1\right)\sum_{i=1}^n \log(v_i) \\ &- \frac{1}{\beta^\alpha}\sum_{i=1}^n v_i\left(1 + \frac{t_i^{2\alpha}\beta^\alpha}{2\eta^\alpha\theta^{2\alpha}}\right).\end{aligned} \tag{14}$$

To obtain the ML estimators for the parameters $(\alpha, \beta, \theta, \eta)$, we should maximize (14). However, since we cannot be able to observe V these estimators cannot be used. To overcome this latency, we have to replace the functions of V by their conditional expectations given the observed values t_i and the current values of parameters. Then, the conditional expectation of the complete data log-likelihood function will be

$$\begin{aligned}&Q(\alpha, \beta, \theta, \eta \mid \hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\eta}) = E(\ell_c(\alpha, \beta, \theta, \eta; \boldsymbol{t}, \boldsymbol{v}) \mid t_i, \hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\eta}) \\ &= \frac{n}{2}\log\left(\frac{2}{\pi}\right) + 2n\log(\alpha) - \frac{n\alpha}{2}\log(\eta) - n\alpha\log(\theta) \\ &- \frac{n\eta}{2}\log(\beta) - n\log\left(\Gamma\left(\frac{\eta}{2\alpha}\right)\right) + (\alpha-1)\sum_{i=1}^n \log(t_i) \\ &+ \left(\frac{\alpha}{2} + \frac{\eta}{2} - 1\right)\sum_{i=1}^n E\left(\log(V_i) \mid t_i, \hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\eta}\right) \\ &- \frac{1}{\beta^\alpha}\sum_{i=1}^n \left(1 + \frac{t_i^{2\alpha}\beta^\alpha}{2\eta^\alpha\theta^{2\alpha}}\right) E\left(V_i^\alpha \mid t_i, \hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\eta}\right),\end{aligned}$$

where $\left(\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\eta}\right)$ are the current estimates for the parameters. To find the conditional expectations

$E\left(\log\left(V_i\right) \mid t_i, \hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\eta}\right)$ and $E\left(V_i^{\alpha} \mid t_i, \hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\eta}\right)$ we need to find the conditional distribution of V given $T = t$. The pdf of this conditional distribution is

$$\begin{aligned} f(v \mid t) &= \frac{\alpha v^{\frac{\alpha}{2}+\frac{\eta}{2}-1}}{\Gamma\left(\frac{\eta}{2\alpha}-\frac{1}{\alpha}+\frac{3}{2}\right)}\left[\frac{1}{\beta^{\alpha}}\left(1+\frac{t_i^{2\alpha}\beta^{\alpha}}{2\eta^{\alpha}\theta^{2\alpha}}\right)\right]^{\frac{\eta}{2\alpha}+\frac{1}{2}} \\ &\times \exp\left\{-\frac{v^{\alpha}}{\beta^{\alpha}}\left(1+\frac{t^{2\alpha}\beta^{\alpha}}{2\eta^{\alpha}\theta^{2\alpha}}\right)\right\}. \end{aligned} \tag{15}$$

Proposition 3.1. *i. The conditional expectation of* V^{α} *given* $T = t$ *is*

$$E\left(V^{\alpha} \mid t\right) = \beta^{\alpha}\left(1+\frac{t^{2\alpha}\beta^{\alpha}}{2\eta^{\alpha}\theta^{2\alpha}}\right)^{-1}\frac{\Gamma\left(\frac{\eta}{2\alpha}+\frac{3}{2}\right)}{\Gamma\left(\frac{\eta}{2\alpha}-\frac{1}{\alpha}+\frac{3}{2}\right)}.$$

ii. The conditional expectation of $\log(V)$ *given* $T = t$ *is*

$$\begin{aligned} &E\left(\log(V) \mid t\right) \\ &= -\frac{1}{\alpha}\log\left(\frac{1}{\beta^{\alpha}}\left(1+\frac{t^{2\alpha}\beta^{\alpha}}{2\eta^{\alpha}\theta^{2\alpha}}\right)\right)\frac{\Gamma\left(\frac{\eta}{2\alpha}+\frac{1}{2}\right)}{\Gamma\left(\frac{\eta}{2\alpha}-\frac{1}{\alpha}+\frac{3}{2}\right)} \\ &+ \frac{1}{\alpha}\Psi\left(\frac{\eta}{2\alpha}+\frac{1}{2}\right)\frac{\Gamma\left(\frac{\eta}{2\alpha}+\frac{1}{2}\right)}{\Gamma\left(\frac{\eta}{2\alpha}-\frac{1}{\alpha}+\frac{3}{2}\right)}, \end{aligned}$$

where $\Psi(\cdot) = \frac{\Gamma'(\cdot)}{\Gamma(\cdot)}$ *is the digamma function.*

Proof. Using the conditional distribution given in (15) these expectations can be easily reached. □

The steps of the EM algorithm can be summarized as follows:

EM Algorithm:

1. Take initial parameter estimates $\left(\hat{\alpha}^{(0)}, \hat{\beta}^{(0)}, \hat{\theta}^{(0)}, \hat{\eta}^{(0)}\right)$ and a stopping rule ϵ.

2. E-step: Compute $\hat{u}_i^{(k)}$ and $\hat{w}_i^{(k)}$ for $i = 1, 2, \ldots, n$ using the following equations for $k = 0, 1, 2, \ldots$ iteration

$$\begin{aligned} \hat{u}_i^{(k)} &= E\left(\log\left(V_i\right) \mid t_i, \hat{\alpha}^{(k)}, \hat{\beta}^{(k)}, \hat{\theta}^{(k)}, \hat{\eta}^{(k)}\right) \\ &= \hat{\beta^{(k)}}^{\hat{\alpha}^{(k)}}\left(1+\frac{t^{2\hat{\alpha}^{(k)}}\hat{\beta}^{(k)\hat{\alpha}^{(k)}}}{2\hat{\eta}^{(k)\hat{\alpha}^{(k)}}\hat{\theta}^{(k)2\hat{\alpha}^{(k)}}}\right)^{-1} \\ &\times \frac{\Gamma\left(\frac{\hat{\eta}^{(k)}}{2\hat{\alpha}^{(k)}}+\frac{3}{2}\right)}{\Gamma\left(\frac{\hat{\eta}^{(k)}}{2\hat{\alpha}^{(k)}}-\frac{1}{\hat{\alpha}^{(k)}}+\frac{3}{2}\right)}, \\ \hat{w}_i^{(k)} &= E\left(V_i^{\alpha} \mid t_i, \hat{\alpha}^{(k)}, \hat{\beta}^{(k)}, \hat{\theta}^{(k)}, \hat{\eta}^{(k)}\right) \\ &= -\frac{1}{\hat{\alpha}^{(k)}}\log\left(\frac{1}{\hat{\beta}^{(k)\hat{\alpha}^{(k)}}}\left(1+\frac{t^{2\hat{\alpha}^{(k)}}\hat{\beta}^{(k)\hat{\alpha}^{(k)}}}{2\hat{\eta}^{(k)\hat{\alpha}^{(k)}}\hat{\theta}^{(k)2\hat{\alpha}^{(k)}}}\right)\right) \\ &\times \frac{\Gamma\left(\frac{\hat{\eta}^{(k)}}{2\hat{\alpha}^{(k)}}+\frac{1}{2}\right)}{\Gamma\left(\frac{\hat{\eta}^{(k)}}{2\hat{\alpha}^{(k)}}-\frac{1}{\hat{\alpha}^{(k)}}+\frac{3}{2}\right)} \\ &+ \frac{1}{\hat{\alpha}^{(k)}}\Psi\left(\frac{\hat{\eta}^{(k)}}{2\hat{\alpha}^{(k)}}+\frac{1}{2}\right)\frac{\Gamma\left(\frac{\hat{\eta}^{(k)}}{2\hat{\alpha}^{(k)}}+\frac{1}{2}\right)}{\Gamma\left(\frac{\hat{\eta}^{(k)}}{2\hat{\alpha}^{(k)}}-\frac{1}{\hat{\alpha}^{(k)}}+\frac{3}{2}\right)}. \end{aligned}$$

3. M-step: Maximize the following objective function with respect to the unknown parameters $(\alpha, \beta, \theta, \eta)$ to get the $(k+1)\,th$ parameter estimates

$$\begin{aligned} &Q\left(\alpha, \beta, \theta, \eta \mid \hat{\alpha}^{(k)}, \hat{\beta}^{(k)}, \hat{\theta}^{(k)}, \hat{\eta}^{(k)}\right) = \\ &= E\left(\ell_c\left(\alpha, \beta, \theta, \eta; t, v\right) \mid t_i, \hat{\alpha}^{(k)}, \hat{\beta}^{(k)}, \hat{\theta}^{(k)}, \hat{\eta}^{(k)}\right) \\ &= \frac{n}{2}\log\left(\frac{2}{\pi}\right) + 2n\log(\alpha) - \frac{n\alpha}{2}\log(\eta) - n\alpha\log(\theta) \\ &- \frac{n\eta}{2}\log(\beta) - n\log\left(\Gamma\left(\frac{\eta}{2\alpha}\right)\right) + (\alpha-1)\sum_{i=1}^{n}\log\left(t_i\right) \\ &+ \left(\frac{\alpha}{2}+\frac{\eta}{2}-1\right)\sum_{i=1}^{n}\left(1+\frac{t_i^{2\alpha}\beta^{\alpha}}{2\eta^{\alpha}\theta^{2\alpha}}\right)\hat{w}_i^{(k)}. \end{aligned}$$

The maximum of this function can be obtained by solving the following four equations simultaneously:

$$\begin{aligned} &i)\,\frac{2n}{\alpha} - \frac{n}{2}\log(\eta) - n\log(\theta) + \frac{n\eta}{2\alpha^2}\Psi\left(\frac{\eta}{2\alpha}\right) + \sum_{i=1}^{n}\log\left(t_i\right) \\ &\quad + \frac{1}{\beta^{\alpha}}\log(\beta)\sum_{i=1}^{n}\left(1+\frac{t_i^{2\alpha}\beta^{\alpha}}{2\eta^{\alpha}\theta^{2\alpha}}\right)\hat{w}_i^{(k)} \\ &\quad - \frac{1}{2\eta^{\alpha}\theta^{2\alpha}}\sum_{i=1}^{n}t_i^{2\alpha}\log\left(\beta t_i^2\right)\hat{w}_i^{(k)} \\ &\quad + \frac{\beta^{\alpha}\log\left(\eta\theta^2\right)}{4\eta^{\alpha}\theta^{2\alpha}}\sum_{i=1}^{n}t_i^{2\alpha}\hat{w}_i^{(k)} = 0, \\ &ii) - \frac{n\eta}{2\beta} + \frac{\alpha}{\beta^{(\alpha+1)}}\sum_{i=1}^{n}\left(1+\frac{t_i^{2\alpha}\beta^{\alpha}}{2\eta^{\alpha}\theta^{2\alpha}}\right)\hat{w}_i^{(k)} - \frac{\alpha}{2\beta\eta^{\alpha}\theta^{2\alpha}} \\ &\qquad \times \sum_{i=1}^{n}t_i^{2\alpha}\hat{w}_i^{(k)} = 0, \\ &iii) - \frac{\alpha n}{2\eta} - \frac{n}{2}\log(\beta) - \frac{n}{2\alpha}\Psi\left(\frac{\eta}{2\alpha}\right) + \frac{1}{2}\sum_{i=1}^{n}\hat{u}_i^{(k)} \\ &\qquad + \frac{\alpha}{2\eta^{(\alpha+1)}\theta^{2\alpha}}\sum_{i=1}^{n}t_i^{2\alpha}\hat{w}_i^{(k)} = 0, \\ &iv) - \frac{\alpha n}{\theta} + \frac{\alpha}{\eta^{\alpha}\theta^{(2\alpha+1)}}\sum_{i=1}^{n}\hat{w}_i^{(k)}t_i^{2\alpha} = 0. \end{aligned}$$

From the solution of these equations, we will get $\left(\hat{\alpha}^{(k+1)}, \hat{\beta}^{(k+1)}, \hat{\theta}^{(k+1)}, \hat{\eta}^{(k+1)}\right)$.

4. Repeat E and M steps until convergence criterion $\left\|\left(\hat{\alpha}^{(k+1)}-\hat{\alpha}^{(k)}, \hat{\beta}^{(k+1)}-\hat{\beta}^{(k)}, \hat{\theta}^{(k+1)}-\hat{\theta}^{(k)}, \hat{\eta}^{(k+1)}-\hat{\eta}^{(k)}\right)\right\| < \epsilon$ is satisfied.

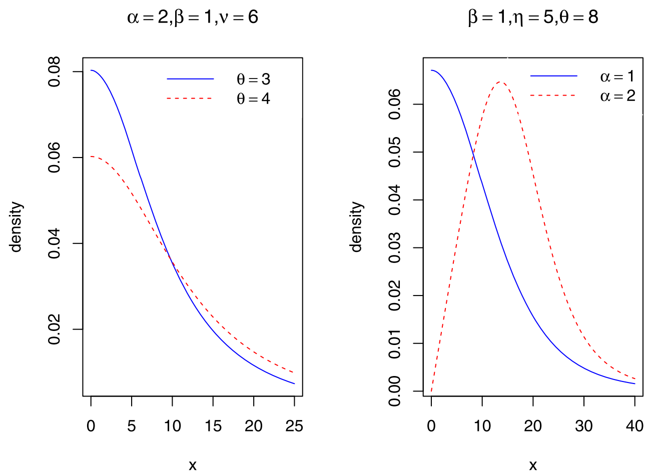

Figure 2. The density plots of the GHT distributions used in the simulation study.

4. SIMULATION STUDY

In this section, we will give a small simulation study to show the performances of ML estimators for GHT distribution. The data are randomly generated from GHT distribution using the scale mixture representation given in (3). We first generate data from from GHN distribution using its cdf [3], and generate data from GG distribution. Finally, we use the equation (3) to generate data from GHT distribution. Several different values of α, β, θ and η are taken. We compute estimates, biases and mean squared error (MSE) using the following formulas

$$bias\left(\hat{\alpha}\right) = \bar{\alpha} - \alpha, \quad bias\left(\hat{\beta}\right) = \bar{\beta} - \beta,$$

$$bias\left(\hat{\theta}\right) = \bar{\theta} - \theta, \quad bias\left(\hat{\eta}\right) = \bar{\eta} - \eta,$$

$$MSE\left(\hat{\alpha}\right) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\alpha}_i - \alpha\right)^2,$$

$$MSE\left(\hat{\beta}\right) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\beta}_i - \beta\right)^2,$$

$$MSE\left(\hat{\theta}\right) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta\right)^2,$$

$$MSE\left(\hat{\eta}\right) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\eta}_i - \eta\right)^2,$$

where $\bar{\alpha} = \frac{1}{N}\sum_{i=1}^{N}\hat{\alpha}_i$, $\bar{\beta} = \frac{1}{N}\sum_{i=1}^{N}\hat{\beta}_i$, $\bar{\theta} = \frac{1}{N}\sum_{i=1}^{N}\hat{\theta}_i$, $\bar{\eta} = \frac{1}{N}\sum_{i=1}^{N}\hat{\eta}_i$ and $N = 100$. We take the sample sizes as $n = 25, 50, 100$ and the parameter values $(\alpha, \beta, \theta, \eta) = (1, 1, 1.5, 8), (1, 1, 1.5, 6), (1, 1, 5, 8), (2, 1, 5, 8)$. Figure 2 displays the pdf plots of the distributions that we generate data. In the simulation study, we use the method of moment (MOM) estimators of GHT distribution as starting value for the EM algorithm given in Section 3 which is suggested by one of the referee. The simulation study and real data example are performed using MATLAB R2013a for the computation and four estimating equations are simultaneously solved by using *fsolve* function.

Table 1 summarizes the simulation results for the sample sizes $25, 50$ and 100. In the table, we give the estimates of parameters, bias and MSE values of the parameter estimates and true parameter values. We observe from the simulation results that the proposed EM algorithm is working accurately to obtain estimates for all parameters.

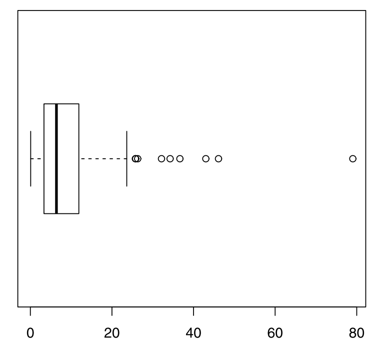

Figure 3. Boxplot of the remission times of bladder cancer patients.

5. REAL DATA EXAMPLE

In this section, we will analyze the data set used by [6]. The same data set has been also used by [2]. In their study, they introduce Lindley-exponential distribution as a more flexible model for modelling lifetime data set. The data set consists of remission times (in months) of a random sample from 128 bladder cancer patients.

In Figure 3, we show the boxplot of the cancer data set. From this figure, we can reveal that the point 79.05 may be possible outlier for this data set. In this real data example, we will compare the performances of the GHT, GHN and HT distributions for modeling this data set with the cases without and with outlier. We assume that all the parameters are unknown for all distributions. We use the ML estimation method to obtain the estimates for the parameters of the GHN and HT distributions and we use the EM algorithm given in Section 3 to obtain the estimates for the parameters of the GHT distribution. To compare the performances of the distributions, we use the values of the Akaike information criterion (AIC) [1] and the Bayesian information criterion (BIC) [7] with the following formula

$$-2\ell_{max} + mc_n,$$

where ℓ_{max} shows the maximized log-likelihood, m is the number of free parameters to be estimated in the model and c_n is the penalty term. Here, we set $c_n = 2$ for AIC and $c_n = \log(n)$ for BIC.

Table 2 gives the estimates, the values of the AIC and the BIC for the data set without the point 79.05 for the GHT,

Table 1. *Estimates of parameters, bias and MSE for different sample sizes*

n		MLE α	β	η	θ
		$\alpha=1,\ \beta=1,\ \eta=1.5,\ \theta=8$			
25	Estimate	1.0076	1.2009	1.7624	8.4514
	Bias	0.0076	0.2009	0.2624	0.4514
	MSE	0.1231	3.1284	0.0838	10.9877
		$\alpha=1,\ \beta=1,\ \eta=1.5,\ \theta=6$			
	Estimate	0.6296	1.1836	1.8722	6.3515
	Bias	−0.3704	0.1836	0.3722	0.3515
	MSE	0.2046	1.9785	0.1556	5.0932
		$\alpha=1,\ \beta=1,\ \eta=5,\ \theta=8$			
	Estimate	0.9659	1.0644	5.0448	7.2055
	Bias	−0.0341	0.0644	0.0448	−0.7945
	MSE	0.0298	2.0380	0.3908	10.8055
		$\alpha=2,\ \beta=1,\ \eta=5,\ \theta=8$			
	Estimate	1.1172	0.7130	4.9993	8.2982
	Bias	−0.8828	−0.2870	−0.0007	0.2982
	MSE	0.9057	0.7913	0.0790	3.5795
		$\alpha=1,\ \beta=1,\ \eta=1.5,\ \theta=8$			
50	Estimate	1.1052	0.9090	1.7457	7.8656
	Bias	0.1052	−0.0910	0.2457	−0.1344
	MSE	0.1070	0.0257	0.0768	0.3995
		$\alpha=1,\ \beta=1,\ \eta=1.5,\ \theta=6$			
	Estimate	0.7129	0.9599	1.8039	5.9828
	Bias	-0.2871	−0.0401	0.3039	−0.0172
	MSE	0.1881	0.0243	0.1173	0.1416
		$\alpha=1,\ \beta=1,\ \eta=5,\ \theta=8$			
	Estimate	0.9880	1.0584	4.9756	7.0633
	Bias	−0.0120	0.0584	−0.0244	−0.9367
	MSE	0.0234	1.4666	0.2361	9.8821
		$\alpha=2,\ \beta=1,\ \eta=5,\ \theta=8$			
	Estimate	1.1379	0.6785	5.0081	8.2368
	Bias	−0.8621	−0.3215	0.0081	0.2368
	MSE	0.8586	0.2561	0.0611	2.5755
		$\alpha=1,\ \beta=1,\ \eta=1.5,\ \theta=8$			
100	Estimate	1.1978	0.9060	1.7280	7.9984
	Bias	0.1978	−0.0940	0.2280	−0.0016
	MSE	0.0827	0.0150	0.0557	0.0280
		$\alpha=1,\ \beta=1,\ \eta=1.5,\ \theta=6$			
	Estimate	0.7855	0.9604	1.7727	6.0509
	Bias	−0.2145	−0.0396	0.2727	0.0509
	MSE	0.1838	0.0114	0.0926	0.0251
		$\alpha=1,\ \beta=1,\ \eta=5,\ \theta=8$			
	Estimate	0.9931	0.9176	4.9535	7.2117
	Bias	−0.0069	−0.0824	−0.0465	−0.7883
	MSE	0.0111	0.9289	0.1832	8.2417
		$\alpha=2,\ \beta=1,\ \eta=5,\ \theta=8$			
	Estimate	1.2400	0.6383	5.0388	7.8891
	Bias	−0.7600	−0.3617	0.0388	−0.1109
	MSE	0.6772	0.1393	0.0540	1.4889

Table 2. ML estimates and some information criterion for fitting GHT, GHN and HT distributions to the remission times of bladder cancer patients without outlier (79.05)

	Model		
	GHN	HT	GHT
	Estimate	Estimate	Estimate
α	0.8364	-	0.9549
β	-	-	1.0397
η	-	3.4174	1.6915
θ	11.0941	8.3961	7.9730
ℓ_{max}	-450.4970	**-401.7152**	-409.7782
AIC	904.9940	**807.4304**	827.5564
BIC	910.6824	**813.1188**	838.9332

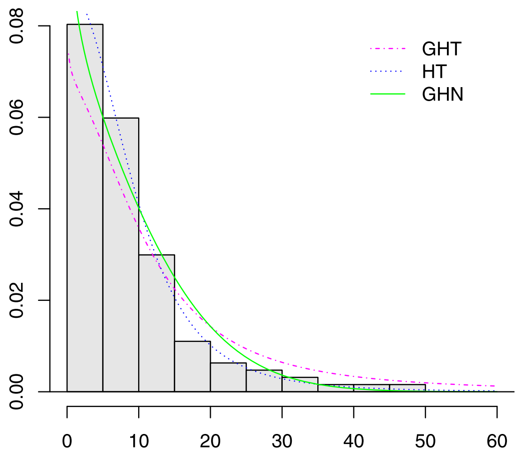

Figure 4. Histogram with the fitted densities obtained from ML estimates of GHT, HT and GHN distributions of the remission times of bladder cancer patients without outlier (79.05).

GHN and HT distributions. Also, we give the histogram of the data set without the point 79.05 with the fitted densities obtained from the three considered distributions in Figure 4. We can see from the results that the HT distribution performs better than the GHN and GHT distributions in terms of AIC and BIC values.

To see the applicability of the proposed model for the outlier case, we will analyze the full data set for all distributions. We give the estimates and the values of AIC and BIC in Table 3. Also, we display the histogram of the full data set with the fitted densities obtained from the GHT, GHN and HT distributions in Figure 5. We can observe that since the values of the AIC and BIC for the GHT distribution is smaller than the GHN and HT distributions, the GHT distribution provides better fit than the GHN and HT distributions when the data set has outlier.

6. CONCLUSIONS

In this paper, we have proposed the new distribution and called it "Generalized Half-t Distribution". The new distribution has been defined as a scale mixture of GHN and GG distribution. We have explored some properties of the newly proposed distribution. The pdf, cdf, hazard function and moments of this distribution have been given in detail. We have used the EM algorithm to estimate the parameters of the GHT distribution. We have given a small simulation study and a real data example to illustrate the modeling performance of the proposed distribution. From the simulation results, we have observed that the parameters can be accurately estimated. According to the real data example, the new distribution can be used as a robust alternative distribution to the GHN and HT distributions to model data sets that may have outliers.

Table 3. ML estimates and some information criterion for fitting GHT, GHN and HT distributions to the remission times of bladder cancer patients

	Model		
	GHN	HT	GHT
	Estimate	Estimate	Estimate
α	0.7598	-	0.9387
β	-	-	1.1845
η	-	2.7494	1.5166
θ	11.6398	8.1452	8.0454
ℓ_{max}	-465.0387	-411.0771	**-403.9355**
AIC	934.0774	826.1542	**815.8711**
BIC	939.7815	831.8583	**827.2792**

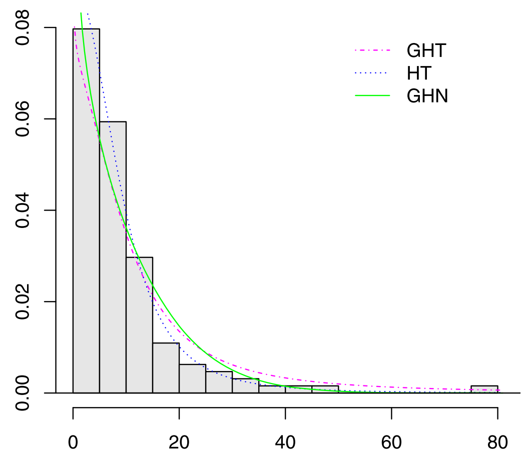

Figure 5. Histogram with the fitted densities obtained from ML estimates of GHT, HT and GHN distributions of the remission times of bladder cancer patients.

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