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Optimal progressive Type-I interval censored scheme under step-stress life testing

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The parametric estimation and optimal censoring scheme are considered under the progressive multi-stage Type-I censoring scheme as well as step-stress accelerated lifetime model. Nonparametric estimators, using the information of the observable numbers of failures and numbers of censored units at the censoring times, are used to derive estimates of the reliability function at the censoring times. Then two parametric estimators, the maximum likelihood and the minimum-distance, are used to estimate the unknown Euclidean parameters of a parametric model. We use D-optimality criterion to determine an optimal sequential step-stress plan under progressive Type-I censoring. Simulation studies are also conducted to assess the finite performance of our estimators.

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1. INTRODUCTION

In medicine tests, the patients may be inspected only at some specific times, with the potential allowance of removing surviving patients during the life test, the exact failure times cannot be observed. The information available is only the records of the failure numbers and the numbers of censored in some predetermined time intervals. Which results in a life test with the progressively type-I interval censoring. One have to develop life test plans under the progressively type-I interval censoring with intermittent inspections, especially under dynamic environment, to give more accurate results.

In this paper, we consider the parametric estimation and optimal censoring scheme, under the combination of the step-stress model and progressive Type-I interval censoring scheme, an issue commonly discussed in life-testing experiments. Finding the "optimal" censoring scheme has attracted considerable attention in the applied statistics literatures, medicine studies and in industrial practical studies ([1]). Considerable number of references has been developed in past decades, for example, [2, 3, 4, 5], and [6] give elaborate discussions in this direction. When it is necessary to reduce the cost and/or the duration of a life-testing experiment, one may choose to terminate the experiment early which results in the so-called censored sampling plan or censored sampling scheme. Many types of censoring have been discussed in the literatures, with the most common censoring schemes being Type-I right censoring and Type-II right censoring. Generalizations of these censoring schemes to progressive Type-I and Type-II right censoring have also been discussed; see, for example, [1] and [7]. When the event is only known to have occurred in an interval between two monitoring time points, this type of data is referred to as interval censored data. Applications of interval censored data can be found in social demographical studies, industrial reliability and clinical studies. Progressively censored samples are observed when at various stages of an experiment, some of the surviving units are removed from the experiment. The remaining units are then observed until either a failure or a subsequent stage of censoring. Several progressive censoring schemes have been extensively studied, and been used in reliability analysis, product testing, and animal carcinogenicity experiments.

When the information of the failure is difficult to be obtained under standard environment, such as when dealing with a unit of high reliability, an accelerated lifetime method can be used to shorter the lifetime of the tested unit. In step-stress accelerated lifetime model, the stress is assumed to be increased step by step until the failure of the tested unit holds. Step-stress accelerated lifetime methods are discussed for instance in [8, 9, 10, 11, 12, 13].

The structure of the paper is as follows. In Section 2, we give the framework of the model we considered. Then in Section 3, minimum distance estimation and maximum likelihood estimation are discussed as well as their asymptotic properties. Some numerical results based on Monte-Carlo are presented in Section 4. The optimal sequential step-stress planning under Type-I progressive censoring is studied in Section 5.

2. MODEL AND ASSUMPTIONS

We consider a progressively Type-I censored sample defined by the progressive censoring scheme $R_1, \cdots, R_{m-1}$ (or $\alpha_1, \cdots, \alpha_{m-1}$ in $[0, 1)$; see [14] or [15]) and pre-fixed censoring times $T_1, \cdots, T_m$. Suppose that n independent units are

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placed simultaneously on stress x_1 at time 0, and run until T_1 at which time the stress changes to x_2, N_1 units fail in the time interval $(0, T_1]$, and R_1 $(= \alpha_1(n-N_1))$ units are selected at random and removed (censored) from the life-test at time T_1. The test continued for the $n - N_1 - R_1$ units until T_2, when the stress changes to x_3, N_2 units fail in the time interval $(T_1, T_2]$, and R_2 $(= \alpha_2(n - N_1 - R_1 - N_2))$ units are selected at random and removed (censored) from the life-test at time T_2, and so on. The life-test ends at time T_m (at the latest), which means that all surviving units at time T_m are censored by this time.

Assumption 2.1.

(1) The stress level increases gradually, i.e., $x_1 \le x_2 \le \cdots \le x_m$.
(2) The failure probability is defined by the cumulative exposure model.
(3) For any stress level x_i*, the lifetime follows an exponential distribution with mean* μ_i*:*

$$\mu_i = \exp(a + bx_i). \tag{1}$$

To specify the parameters, let $\overline{F}_t(\theta)$ denotes the reliability function at time t under the stress $\theta = (x_1, x_2, \ldots, x_m)$ (or $\theta = (\mu_1, \mu_2, \ldots, \mu_m)$). Then under the Assumption 2.1, the reliability function is (see [16])

$$\overline{F}_t(\theta) = \begin{cases} \exp(-\frac{t}{\mu_1}) & 0 \le t \le T_1 \\ \exp(-\frac{t-T_1}{\mu_2} - \frac{T_1}{\mu_1}) & T_1 \le t \le T_2 \\ \vdots & \\ \exp(-\frac{t-T_{m-1}}{\mu_m} - \frac{T_{m-1}-T_{m-2}}{\mu_{m-1}} - \cdots - \frac{T_1}{\mu_1}) & T_{m-1} \le t \le T_m \end{cases}, \tag{2}$$

and $\overline{F}_t(\theta) \equiv \overline{F}_{T_m}(\theta)$ for $t > T_m$, where μ_i are defined as Eq. (1).

The special case of equi-spaced stress level $x_i = x_0 + id$ is extensively discussed in, for example [13, 17, 18], where $d > 0$ is the amount of stress cumulated at each stage, the relationship between mean lifetimes of the i-th and the $(i+1)$-th stages is $\mu_{i+1} = \rho\mu_i,\ i = 1, 2, ..., m-1$ and $\mu_1 = \mu$, where $0 < \rho < 1$ is the ratio of mean lifetimes between two successive stages.

Lemme 2.2. *Suppose that* $Y_1 \sim F = 1 - \overline{F}$*, and* G *is an arbitrary invertible distribution function, then* $G^{-1}(F(Y_1)) \sim G$.

Remark 2.3. *Denote* Y_1 *the random lifetime under cumulative stress* $x_1, \ldots, x_k (k = 1, \cdots, m)$*, i.e.,* $Y_1 \sim \exp(-\frac{t-T_{k-1}}{\mu_k} - \frac{T_{k-1}}{\mu_{k-1}} - \cdots - \frac{T_1}{\mu_1}),\ \ t \ge T_{k-1}$*, then the random lifetime* Y_2 *of the survival units after* T_k*, with cumulative stress* $x_1, \ldots, x_{k+1}$*, can be defined by* $Y_2 = T_k + \frac{\mu_{k+1}}{\mu_k}(Y_1 - T_k),\ t \ge T_k$.

Remark 2.4. *The reliability functions at censoring times* T_i *can be defined recursively, i.e.*

$$\begin{aligned} \overline{F}_{T_i}(\theta) &= \overline{F}_{T_{i-1}}(\theta) \exp\left(-\frac{T_i - T_{i-1}}{\mu_i}\right) \\ &= \prod_{k=1}^{i} \exp\left(-\frac{T_k - T_{k-1}}{\mu_k}\right) \end{aligned}$$

with the convention that $T_0 = 0$. For convenience, the reliability function $F_{T_i}(\theta)$ is denoted by $F_i(\theta)$ hereafter.

3. PARAMETRIC ESTIMATION

In this Section, we consider a parametric model defined by the Euclidean parameter $\theta = (a,\ b) \in \mathbb{R}^2$. We then propose a simple estimator of θ and give its asymptotic properties as well.

3.1 Nonparametric estimation of the reliability function

In the case of $m = 1$, corresponding to a single-stage Type-I censoring sampling plan, the variable N_1 has a binomial distribution with parameters n and $F_{T_1}(\mu_1)$. Then, $(n - N_1)/n$ is an estimator of $\overline{F}_{T_1}(\mu_1)$.

In the case of $m = 2$, the random variable N_2, if being not null, contains the information that can be accounted for. Indeed, conditional on $(N_1, R_1) = (n_1, r_1)$, the lifetimes of the remaining $n - n_1 - r_1$ units under test follow left-truncated distribution with density (see [2])

$$\frac{f_t(\mu_1, \mu_2)}{\overline{F}_{T_1}(\mu_1)} 1(t \ge T_1).$$

Then, conditionally on $(N_1, R_1) = (n_1, r_1)$ and $n_1 + r_1 < n$, the random variable N_2 has a binomial distribution with parameters $n - n_1 - r_1$ and $(F_{T_2}(\mu_1, \mu_2) - F_{T_1}(\mu_1))/\overline{F}_{T_1}(\mu_1)$. Consequently, $1 - p = \overline{F}_{T_2}(\mu_1, \mu_2)/\overline{F}_{T_1}(\mu_1)$ is approximated by $(n - n_1 - r_1 - n_2)/(n - n_1 - r_1)$, and since $\overline{F}_{T_1}(\mu_1)$ is approximated by $(n - n_1)/n$, we can approximate $\overline{F}_{T_2}(\mu_1, \mu_2)$ by

$$\frac{(n - n_1)(n - n_1 - r_1 - n_2)}{n(n - n_1 - r_1)},$$

For $m \ge 2$, let us introduce the following notations, for $j = 1, \cdots, m$:

$$\begin{aligned} V_j^- &= \#\{i \in \{1, \cdots, n\} : X_i \ge T_j\}, \\ V_j^+ &= \#\{i \in \{1, \cdots, n\} : X_i \ > \ T_j\}. \end{aligned} \tag{3}$$

Here, V_j^- $(1 \le j \le m)$ denotes the number of items still functioning just before time T_j, and V_j^+ $(0 \le j \le m)$ denotes the number of items still functioning at time T_j. By induction we show that $\overline{F}_{T_i}(\theta)$ is approximated by

$$\prod_{j=1}^{i} \frac{V_j^-}{V_{j-1}^+},$$

with the convention that $0/0 = 0$.

Remark 3.1. *Estimators above only take into account partial information from the progressively censored sample. Indeed, to estimate the reliability function $\overline{F}(.)$, we only need to know the N_i's and the R_i's. In fact,*

$$V_j^- = n - \sum_{k=1}^{j} N_k - \sum_{k=1}^{j-1} R_k,$$

and

$$V_j^+ = n - \sum_{k=1}^{j} N_k - \sum_{k=1}^{j} R_k.$$

It shows that estimating $\overline{F}$ at the censoring times do not require to know the actual lifetimes of the failed units.

Remark 3.2. *The expectation of N_i and $\widetilde{R}_i = \alpha_i V_i^-$ can be expressed as follows:*

$$E(N_i) = n(\overline{F}_{i-1}(\theta) - \overline{F}_i(\theta)) \prod_{j=1}^{i-1}(1-\alpha_j),$$

and

$$E(\widetilde{R}_i) = n\overline{F}_i(\theta)\alpha_i \prod_{j=1}^{i-1}(1-\alpha_j).$$

3.2 Minimum distance estimator

We propose to estimate $\theta = (\mu_1, \dots, \mu_m)$ by minimizing the square of the Euclidean distance between $(\overline{F}_1(\theta), \cdots, \overline{F}_m(\theta))$ and its nonparametric estimate. The estimator is therefore defined by

$$\hat{\theta} = \arg\min_{\theta\in\Theta} \sum_{i=1}^{m} \left(\overline{F}_i(\theta) - \prod_{j=1}^{i} \frac{V_j^-}{V_{j-1}^+} \right)^2, \tag{4}$$

where V_j^+ and V_j^- is defined by (3).

Define

$$\phi(\theta) = \sum_{i=1}^{m} (\overline{F}_i(\theta) - \overline{F}_i(\theta_0))^2.$$

We have $\phi(\theta_0) = 0$.

In [7], the authors have proved the following asymptotic result.

Theorem 3.3. *Let $\hat{\theta}$ be the estimator of θ defined by (4). If ϕ is a contrast function, i.e., $\phi(\theta) = 0$ if and only if $\theta = \theta_0$, then we have $\hat{\theta} \xrightarrow{P} \theta_0$ as $n \to +\infty$.*

Theorem 3.4. *Let $\hat{\theta}$ be the estimator of θ defined by (4). Assume that ϕ is a contrast function, then we have*

$$\sqrt{n}(\hat{\theta} - \theta_0) \rightsquigarrow \mathcal{N}(0, MB\Sigma B^T M^T), \tag{5}$$

where,

(i) Σ is a $m \times m$ matrix whose (i,j)-th entry is given by

$$\sigma_{ij} = \begin{cases} \frac{(\overline{F}_{i-1}(\theta) - \overline{F}_i(\theta))\overline{F}_i(\theta)}{\overline{F}^3_{i-1}(\theta)\prod_{j=1}^{i-1}(1-\alpha_j)}, & i = j \\ 0, & i \neq j \end{cases};$$

(ii)

$$B = -\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{\overline{F}_2(\theta)}{\overline{F}_1(\theta)} & \overline{F}_1(\theta) & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \frac{\overline{F}_{m-1}(\theta)}{\overline{F}_1(\theta)} & \frac{\overline{F}_{m-1}(\theta)\overline{F}_1(\theta)}{\overline{F}_2(\theta)} & \frac{\overline{F}_{m-1}(\theta)\overline{F}_2(\theta)}{\overline{F}_3(\theta)} & \cdots & 0 \\ \frac{\overline{F}_m(\theta)}{\overline{F}_1(\theta)} & \frac{\overline{F}_m(\theta)\overline{F}_1(\theta)}{\overline{F}_2(\theta)} & \frac{\overline{F}_m(\theta)\overline{F}_2(\theta)}{\overline{F}_3(\theta)} & \cdots & \overline{F}_{m-1}(\theta) \end{pmatrix};$$

(iii)

$$\begin{aligned} M &= -2\left(\frac{\partial^2\phi(\theta)}{\partial\theta\partial\theta^T}\right)^{-1} \times Q(\theta) \\ &= -2\left(\frac{\partial\overline{F}_i(\theta)}{\partial\theta}\frac{\partial\overline{F}_i(\theta)}{\partial\theta^T}\right)^{-1} \times Q(\theta) \end{aligned}$$

with $Q(\theta) = (q_{ij}) = \left(\frac{\partial\overline{F}_j(\theta)}{\partial\theta_i}\right)$ and

$$\begin{aligned} \frac{\partial\overline{F}_i(\theta)}{\partial\theta} &= \begin{pmatrix} \frac{\partial\overline{F}_i(a,b)}{\partial a} \\ \frac{\partial\overline{F}_i(a,b)}{\partial b} \end{pmatrix} \\ &= \begin{pmatrix} \overline{F}_i(\theta)\left(\frac{T_i - T_{i-1}}{\mu_i} + \frac{T_{i-1}-T_{i-2}}{\mu_{i-1}} + \cdots + \frac{T_1}{\mu_1}\right) \\ \overline{F}_i(\theta)\left(\frac{T_i - T_{i-1}}{\mu_i}x_i + \frac{T_{i-1}-T_{i-2}}{\mu_{i-1}}x_{i-1} + \cdots + \frac{T_1}{\mu_1}x_1\right) \end{pmatrix}. \end{aligned} \tag{6}$$

3.3 Maximum likelihood estimator

Also we can estimate θ by maximizing the likelihood function. The likelihood function for a Type-I progressive group-censoring sample is

$$L(\theta) \propto \prod_{i=1}^{m} (\overline{F}_{i-1}(\theta) - \overline{F}_i(\theta))^{N_i} (\overline{F}_i(\theta))^{R_i}. \tag{7}$$

The estimator is therefore defined by

$$\hat{\theta} = \arg\max_{\theta\in\Theta} \log L(\theta). \tag{8}$$

The score function is

$$\begin{aligned} U(\theta) &= \frac{\partial \log L(\theta)}{\partial\theta} \\ &= \sum_{i=1}^{m} \left(N_i \frac{\frac{\partial\overline{F}_{i-1}(\theta)}{\partial\theta} - \frac{\partial\overline{F}_i(\theta)}{\partial\theta}}{\overline{F}_{i-1}(\theta) - \overline{F}_i(\theta)} + R_i \frac{\partial\overline{F}_i(\theta)/\partial\theta}{\overline{F}_i(\theta)} \right) \end{aligned}$$

and the estimator $\hat{\theta}$ can be derived by solving $U(\theta) = 0$. For a regular parametric model, we can show that

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} \mathcal{N}(0, I^{-1}(\theta_0)), \tag{9}$$

where

$$I(\theta_0) = \sum_{i=1}^{m} \left(\frac{A_i(\theta_0) \prod_{j=1}^{i-1} (1-\alpha_j)}{\overline{F}_{i-1}(\theta_0) - \overline{F}_i(\theta_0)} - \frac{B_i(\theta_0)\alpha_i \prod_{j=1}^{i-1}(1-\alpha_j)}{\overline{F}_i(\theta_0)} \right)$$

with

$$A_i(\theta) = \left(\frac{\partial \overline{F}_{i-1}(\theta)}{\partial \theta} - \frac{\partial \overline{F}_i(\theta)}{\partial \theta}\right)\left(\frac{\partial \overline{F}_{i-1}(\theta)}{\partial \theta} - \frac{\partial \overline{F}_i(\theta)}{\partial \theta}\right)^T - \left(\frac{\partial^2 \overline{F}_{i-1}(\theta)}{\partial \theta \partial \theta^T} - \frac{\partial^2 \overline{F}_i(\theta)}{\partial \theta \partial \theta^T}\right)(\overline{F}_{i-1}(\theta)) - \overline{F}_i(\theta)),$$

and

$$B_i(\theta) = \left(\frac{\partial \overline{F}_i(\theta)}{\partial \theta}\right)\left(\frac{\partial \overline{F}_i(\theta)}{\partial \theta}\right)^T - \left(\frac{\partial^2 \overline{F}_i(\theta)}{\partial \theta \partial \theta^T}\right)\overline{F}_i(\theta).$$

where $\frac{\partial \overline{F}_i(\theta)}{\partial \theta}$ are defined as in Eq. (6) and

$$\frac{\partial^2 \overline{F}_i(\theta)}{\partial \theta \partial \theta^T} = \begin{pmatrix} \frac{\partial^2 \overline{F}_i(a,b)}{\partial a^2} & \frac{\partial^2 \overline{F}_i(\theta)}{\partial a \partial b} \\ \frac{\partial^2 \overline{F}_i(\theta)}{\partial a \partial b} & \frac{\partial^2 \overline{F}_i(a,b)}{\partial a^2} \end{pmatrix}$$

with

$$\begin{aligned} \frac{\partial^2 \bar{F}_i(a,b)}{\partial a^2} &= \bar{F}_i(\theta)(\ln(\bar{F}_i(\theta)))^2 + \bar{F}_i(\theta)\ln(\bar{F}_i(\theta)); \\ \frac{\partial^2 \bar{F}_i(a,b)}{\partial a \partial b} &= -\frac{\partial \bar{F}_i(\theta)}{\partial b}(\ln(\bar{F}_i(\theta)) + 1); \\ \frac{\partial^2 \bar{F}_i(a,b)}{\partial b^2} &= \bar{F}_i(\theta)\left(\left(\sum_{k=1}^{i} \frac{T_k - T_{k-1}}{\mu_k} x_k\right)^2 \right. \\ &\quad \left. - \sum_{k=1}^{i}\left(\frac{T_k - T_{k-1}}{\mu_k} x_k^2\right)\right). \end{aligned}$$

4. SEQUENTIAL STEP STRESS PLANNING UNDER TYPE-I PROGRESSIVE CENSORING

Based on the variance-covariance matrix derived above, we compute the expected asymptotic variance-covariance matrix of the NP-estimators, and then determine an optimal progressive group-censoring plan based on a D-optimality criterion.

Given $T_1, \cdots, T_i$, assume that $\hat{\theta}^{(i)}$ is an estimator of θ satisfying

$$\sqrt{n}(\hat{\theta}^{(i)} - \theta) \rightsquigarrow \mathcal{N}(0, \Omega(T_1, \cdots, T_i, \theta)). \tag{10}$$

Since the volume of the asymptotic joint confidence region of θ is proportional to the determinant of the asymptotic variance-covariance matrix of $\hat{\theta}^{(i)}$, we can find the optimal T_{i+1} by minimizing the determinant $|\Omega(T_1, \cdots, T_i, T, \hat{\theta}^{(i)})|$. This is the so-called D-optimality criterion. Therefore, we have

$$T_{i+1} = \arg\min_{T > T_i} |\Omega(T_1, \cdots, T_i, T, \hat{\theta}^{(i)})|. \tag{11}$$

4.1 Optimal progressive group-censoring plans using the maximum likelihood estimator

We look for the optimal progressive censoring plans using the maximum likelihood estimator under group-censored samples. We denote by $\Omega(T_k, \theta)$ the asymptotic variance of the k-stage estimator $\theta^{(k)} (k = 1, \cdots, m)$. Therefore, the k-th optimal censoring time can be determined as

$$T_k = \arg\max_{T_k > T_{k-1}} \sum_{i=1}^{k} \left(\frac{A_i(\hat{\theta}^{(k-1)}) \prod_{j=1}^{i-1}(1-\alpha_j)}{\overline{F}_{i-1}(\hat{\theta}^{(k-1)}) - \overline{F}_i(\hat{\theta}^{(k-1)})} + \frac{B_i(\hat{\theta}^{(k-1)})\alpha_i \prod_{j=1}^{i-1}(1-\alpha_j)}{\overline{F}_i(\hat{\theta}^{(k-1)})} \right). \tag{12}$$

4.2 Optimal progressive group-censoring plans using the minimum-distance estimator

In Theorem 3.4, it has been proved that the k-stage MDE estimator $\hat{\theta}^{(k)}$ satisfies

$$\sqrt{n}(\hat{\theta}^{(k)} - \theta) \rightsquigarrow \mathcal{N}(0, MB\Sigma B^T M^T),$$

where the asymptotic variance-covariance matrix $MB\Sigma B^T M^T$, which depends on $(T_1, \cdots, T_k, \theta)$, is written as $\Omega(T_1, \cdots, T_k, \theta)$. Therefore, given $T_1, \cdots, T_{k-1}$ and $\hat{\theta}^{(k-1)}$, we define the k-th optimized censoring time T_k as

$$T_k = \arg\min_{T > T_{k-1}} |\Omega(T_1, \cdots, T_{k-1}, T, \hat{\theta}^{(k-1)})|. \tag{13}$$

Therefore, an optimal Type-I multi-stage censoring scheme can be determined in the following manner. Assume that the initial sample size is n and set the censoring proportions to $\alpha_1, \cdots, \alpha_m$. The first censoring time is fixed to be T_1. We can then determine the optimal progressive censoring scheme by repeatedly using formule (12) or (13).

5. NUMERICAL RESULTS

5.1 Parametric estimation

We consider an exponential-based distribution as in (2) with parameters $a = 3$ and $b = -0.5$, $m = 4$ and $X = (x_1, \ldots, x_4) = (1, 2, 3, 5)$. We choose $T = (T_1, \ldots, T_4) = (10, 25, 35, 40)$. In this numerical study, we assume that the proportions to be removed at different stages are all equal.

Table 1. Minimum distance estimation and maximum likelihood estimation of the parameters $a = 3$ and $b = -0.5$ for an exponential distribution with step-stress for $T = (10, 25, 35, 40)$ and $X = (1, 2, 3, 5)$: mean and standard deviation (within parentheses)

	MDE		MLE	
sample size n	$\hat{a}$	$\hat{b}$	$\hat{a}$	$\hat{b}$
200	3.0202 (0.2345)	- 0.5201 (0.1663)	3.0343 (0.2103)	- 0.5305 (0.1454)
400	3.0110 (0.1629)	- 0.5116 (0.1186)	3.0217 (0.1464)	- 0.5197 (0.1031)
600	3.0106 (0.1360)	- 0.5082 (0.0958)	3.0186 (0.1229)	- 0.5144 (0.0840)
800	3.0105 (0.1232)	- 0.5091 (0.0869)	3.0139 (0.1110)	- 0.5115 (0.0754)
1000	3.0037 (0.1092)	- 0.5049 (0.0776)	3.0081 (0.0989)	- 0.5083 (0.0681)

Table 2. Minimum distance estimation and maximum likelihood estimation of the parameters $\mu = 100$ and $\rho = 0.5$ for an exponential distribution with step-stress for $T_i = 25i(i = 1, \ldots, 4)$: mean and standard deviation (within parentheses)

	MDE		MLE	
sample size n	$\hat{\rho}$	$\hat{\mu}$	$\hat{\rho}$	$\hat{\mu}$
200	0.4968 (0.0494)	101.7079 (13.6014)	0.4965 (0.0442)	101.5679 (13.1272)
400	0.4982 (0.0350)	101.0312 (9.6512)	0.4975 (0.0318)	101.0855 (9.3548)
600	0.4993 (0.0286)	100.7005 (7.9331)	0.4995 (0.0264)	100.6195 (7.7722)
800	0.4995 (0.0245)	100.1890 (6.7914)	0.4995 (0.0219)	100.1443 (6.5673)
1000	0.4996 (0.0214)	100.1731 (5.8564)	0.4996 (0.0194)	100.1288 (5.6924)

Table 3. Minimum distance estimation and maximum likelihood estimation of the parameters $\mu = 100$ and $\rho = 0.5$ for an exponential distribution with step-stress for $T = (50, 100, 125, 150)$: mean and standard deviation (within parentheses)

	MDE		MLE	
sample size n	$\hat{\rho}$	$\hat{\mu}$	$\hat{\rho}$	$\hat{\mu}$
200	0.5015 (0.0562)	100.9777 (10.9522)	0.5004 (0.0500)	101.0081 (10.7284)
400	0.4984 (0.0425)	100.2709 (7.6124)	0.4979 (0.0373)	100.2765 (7.4124)
600	0.4980 (0.0331)	100.5626 (6.1301)	0.4978 (0.0295)	100.5741 (6.0588)
800	0.4986 (0.0293)	100.2538 (5.3538)	0.4989 (0.0262)	100.2421 (5.2423)
1000	0.4993 (0.0254)	100.2420 (4.9341)	0.4993 (0.0224)	100.2095 (4.8395)

That is, $\alpha_1 = \alpha_2 = \cdots = \alpha_{m-1} = 0.2$ for $i = 1, \ldots, 4$. Then, at time T_i $(i = 1, \ldots, 4)$, 20% of units still functioning are removed from the sample, which means that $R_i = \lfloor \alpha_i V_i^- \rfloor$, and the stress changes to a higher level. The study finishes at time T_4, meaning that all surviving units at time T_4 are censored.

The estimator $\hat{\theta} = (\hat{a}, \hat{b})$ is defined by (4) or (8). The performance of the estimator is illustrated for various sample sizes in Table 1. For each sample size the mean and the standard deviation have been obtained from $N = 1000$ simulated samples.

We also explore a special case for an exponential-based distributed sample with mean $\mu_1 = \mu$ and

$$\mu_i = \rho \mu_{i-1} \quad \text{for } 2 \leq i \leq 4. \tag{14}$$

We chose $T_i = 25i(1 \leq i \leq 4)$ and $T = (50, 100, 125, 150)$ and $\alpha_i = 0.2$ for $i = 1, \ldots, 4$. The performances of the estimators are illustrated for various sample sizes in Table 2 and Table 3.

All the results in Tables 1–3 show that the performances of the MDE and the MLE are close. As expected the MLE has a behavior slightly better than the MDE. In addition we remark that multiplying the sample size by four reduces the standard deviation by half.

5.2 Optimal sequential step stress planning

Suppose that the lifetimes of the units under test are exponential distributed with mean μ and scale parameter ρ. At the censoring time T_i, a proportion α_i of surviving units are randomly selected and removed (censored) from the experiment. For each simulated sample, the censoring times $T_2, \cdots, T_m$ are determined by D-optimality criteria.

We simulated $N = 1000$ samples and reported the empirical behavior of the estimated T_i's for various choices of the μ and ρ.

Table 4 and Table 5 summarize the empirical behavior of the optimal sequential progressive censoring plans when the MDE and MLE method are used with various parameters.

From Tables 4 and 5 we can conclude that:

Table 4. Optimal censoring planning for an exponential distribution with step-stress w.r.t parameter $(a,\ b)$*:* $T_1 = 20$

		MDE			MLE		
a	b	T_2	T_3	T_4	T_2	T_3	T_4
$a = 3$	-0.1	46.20	62.40	71.62	55.69	90.73	119.86
	-0.25	39.41	49.72	54.06	46.35	68.68	82.46
	-0.5	31.77	36.63	38.45	35.88	46.42	50.37
$a = 4$	-0.1	91.24	136.31	162.49	118.75	213.41	291.16
	-0.25	72.77	101.36	113.60	93.01	153.48	190.34
	-0.5	52.01	65.42	68.88	64.11	92.74	103.36
$a = 5$	-0.1	213.64	344.15	425.77	290.73	542.53	748.51
	-0.25	163.45	245.55	282.93	220.39	381.59	479.26
	-0.5	107.01	145.07	199.15	141.34	217.96	311.89

Table 5. Optimal censoring planning for an exponential distribution with step-stress w.r.t parameter $(\mu,\ \rho)$*:* $T_1 = 50$

		MDE			MLE		
μ	ρ	T_2	T_3	T_4	T_2	T_3	T_4
$\mu = 100$	0.2	81.87	86.30	86.84	81.87	88.24	89.52
	0.4	113.74	132.51	137.72	113.74	139.23	149.43
	0.5	129.68	159.01	165.90	129.68	169.52	189.44
$\mu = 300$	0.2	145.62	160.65	162.98	145.62	164.74	168.56
	0.4	241.23	297.62	312.63	241.23	317.72	348.32
	0.5	289.04	382.97	419.45	289.04	408.56	468.32
$\mu = 500$	0.2	209.36	237.34	242.05	209.36	241.23	247.60
	0.4	368.72	480.63	518.39	368.72	496.21	547.21
	0.5	448.40	651.63	740.28	448.40	647.60	747.20

(1) For a fixed μ (resp. a), a lower value ρ (resp. b) corresponds to a higher stress level (more severe condition) and results in an acceleration of failures, and finally gives a shorter interval of censoring time.
(2) Higher censoring-stage results in a quicker censoring ($\Delta T_k = T_k - T_{k-1}$ decreases as k increases), because of the data scarcity and more severe stress.
(3) The larger the value of the μ (resp. b) is, which means longer lifetime of the test unit in standard environment, the larger the interval of censoring is.

6. CONCLUSIONS

In this paper, we discuss parametric and optimal sequential censoring plan under the combination of progressive multi-stage Type-I interval censoring scheme and step-stress accelerated lifetime model. MLE and MDE are used to estimate the unknown Euclidean parameter of the cumulative exposure model. Then the D-optimality criterion is used to determine an optimal sequential progressive censoring plan under step-stress accelerated life test.

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