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The beta transmuted-H family for lifetime data

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We introduce a new family of continuous distributions called the *beta transmuted-H* family which extends the transmuted family pioneered by Shaw and Buckley [34]. Some of its mathematical properties including explicit expressions for the ordinary moments, quantiles, generating functions and order statistics are derived. Some special models of the new family are provided. The maximum likelihood method is used for estimating the model parameters, and the finite sample performance of the estimators is assessed by simulation. The importance and flexibility of the proposed family are illustrated by applications to two real data sets.

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1. INTRODUCTION

Recently, there has been a great interest in developing more flexible distributions through extending the classical distributions by introducing additional shape parameters to the baseline model. Many generalized families of distributions have been proposed and studied over the last two decades for modeling data in many applied areas such as economics, engineering, biological studies, environmental sciences, medical sciences and finance. For example, Zografos and Balakrishnan [37] proposed the gamma-generated family, Nofal et al. [31] defined and studied the generalized transmuted-G family, Yousof et al. [36] proposed the transmuted exponentiated generalized-G class and Afify et al. [2] proposed the Kumaraswamy transmuted-G family.

The generated families have attracted many researchers and statisticians to develop new models because of the computational and analytical facilities available in most symbolic computation software platforms. Several mathematical properties of the extended distributions may be easily explored using mixture forms of exponentiated-G (exp-G) distributions.

Consider a baseline cumulative distribution function (cdf) $H(x;\boldsymbol{\phi})$ with corresponding probability density function (pdf) $h(x;\boldsymbol{\phi})$ and parameter vector $\boldsymbol{\phi}$. Then, the cdf of the *transmuted-H* (T-H) family of distributions (for $x>0$) is

$$G(x;\lambda,\boldsymbol{\phi}) = H(x;\boldsymbol{\phi})\left[1+\lambda-\lambda H(x;\boldsymbol{\phi})\right]. \tag{1}$$

The corresponding pdf is

$$g(x;\lambda,\boldsymbol{\phi}) = h(x;\boldsymbol{\phi})\left[1+\lambda-2\lambda H(x;\boldsymbol{\phi})\right], \tag{2}$$

where $|\lambda|\le 1$. It is clear that the T-H family is a mixture of the baseline and exp-H distributions, the last one with power parameter equal to two. Further, if $\lambda=0$, (2) gives the baseline distribution. More details can be found in Shaw and Buckley [34].

In this paper, we define and study a new family of distributions by adding two extra shape parameters in (1) to provide more flexibility to the generated family. Using the beta-generalized (B-G) family due to Eugene et al. [14], we construct a new family called the *beta transmuted-H* (BT-H) family and give a description of some of its mathematical properties. The BT-H family is shown to provide better fits than at least five other families each having the same number of parameters. We hope that the new family will attract wider applications in reliability, engineering and other areas of research.

For an arbitrary baseline cdf $G(x)$ the B-G family due to Eugene et al. [14] has the cdf and pdf given (for $x>0$) by

$$F(x;a,b) = I_{G(x)}(a,b) = \frac{1}{B(a,b)}\int_0^{G(x)} t^{a-1}(1-t)^{b-1}\,dt$$

and

$$f(x;a,b) = \frac{1}{B(a,b)} g(x)\, G(x)^{a-1}\left\{1-G(x)\right\}^{b-1},$$

respectively, where $g(x)=dG(x)/dx$, a and b are two additional positive shape parameters, $I_y(a,b)=B_y(a,b)/B(a,b)$ is the incomplete beta function ratio, $B_y(a,b)=\int_0^y t^{a-1}(1-t)^{b-1}\,dt$ is the incomplete beta function, $B(a,b)=\Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function and $\Gamma(\cdot)$ is the gamma function. Clearly, for $a=b=1$, we obtain the baseline distribution. The additional parameters a and b govern skewness and tail weight of the generated distribution. An attractive feature of this family is that a and b can afford greater control over the weights in both tails and in the center of the distribution.

To this end, we generalize the T-H family by incorporating two additional shape parameters to yield a more flexible generator. The cdf of the BT-H family is defined (for $x>0$) by

$$F(x) = I_{H(x;\boldsymbol{\phi})[1+\lambda-\lambda H(x;\boldsymbol{\phi})]}(a,b). \tag{3}$$

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Table 1. Sub-models of the BT-H family

Reduced model	a	b	λ	Authors
B-H family	a	b	0	Eugene et al. [14]
T-H family	1	1	λ	Shaw and Buckley [34]
exp-H family	a	1	0	Gupta et al. [18]
GT-H family	1	b	λ	New
ET-H family	a	1	λ	New
$H(x;\phi)$	1	1	0	–

Table 2. The pdfs and cdfs of baseline distributions

Distribution	$h(x)$	$H(x)$
W $(x>0)$	$\beta\alpha^{\beta}x^{\beta-1}\ e^{-(\alpha x)^{\beta}}$	$1-e^{-(\alpha x)^{\beta}}$
Pa $(x\geq\theta)$	$\alpha\theta^{\alpha}/x^{\alpha+1}$	$1-\left(\frac{\theta}{x}\right)^{\alpha}$
Fr $(x>0)$	$\beta\alpha^{\beta}x^{-(\beta+1)}\ e^{-\left(\frac{\alpha}{x}\right)^{\beta}}$	$e^{-\left(\frac{\alpha}{x}\right)^{\beta}}$
Go $(x>0)$	$\beta e^{\alpha x}e^{-\frac{\beta}{\alpha}\left(e^{\alpha x}-1\right)}$	$1-e^{-\frac{\beta}{\alpha}\left(e^{\alpha x}-1\right)}$
Li $(x>0)$	$\frac{\alpha^{2}}{1+\alpha}(1+x)e^{-\alpha x}$	$1-\frac{1+\alpha+\alpha x}{1+\alpha}e^{-\alpha x}$

The pdf corresponding to (3) is

$$f(x)=\left\{1+\lambda-2\lambda H(x;\phi)\right\} \cdot\left\{(1+\lambda)H(x;\phi)-\lambda H(x;\phi)^{2}\right\}^{a-1} \cdot\frac{h(x;\phi)}{B(a,b)}\left\{1-\left[(1+\lambda)H(x;\phi)-\lambda H(x;\phi)^{2}\right]\right\}^{b-1}, \quad (4)$$

where $a>0$, $b>0$ and $|\lambda|\leq 1$. A random variable X having the pdf (4) shall be denoted by $X\sim$BT-H(a,b,λ,ϕ). The quantile function (qf) of X, say $Q(u)=F^{-1}(u)$, can be obtained by inverting (3) numerically. Some special cases of the new family are listed in Table 1.

The remainder of this paper is outlined as follows. Five special models of the BT-H family are presented in Section 2 and some plots of their pdfs are provided. In Section 3, we derive a useful representation for the BT-H pdf and obtain some mathematical properties of the proposed family including ordinary moments, quantiles, generating functions and order statistics. Maximum likelihood estimation of the model parameters is investigated in Section 4. The finite sample performance of the estimators is assessed by simulation in Section 5. In Section 6, we provide applications to two real data sets to illustrate the importance of the new family. Finally, some concluding remarks are presented in Section 7.

2. SPECIAL MODELS

In this section, we provide five special models of the BT-H family. The pdf (4) will be most tractable when $h(x;\phi)$ and $H(x;\phi)$ have simple analytic expressions. These special models generalize some well-known distributions in the literature. These special models correspond to the baseline Weibull (W), Pareto (Pa), Fréchet (Fr), Gompertz (Go) and Lindley (Li) distributions. The pdf and cdf of these baseline distributions are listed in Table 2.

The parameters of the above pdfs belong to the set of positive real numbers.

2.1 The BT-W distribution

The BT-W pdf is given (for $x>0$) by

$$f(x)=\frac{1}{B(a,b)}\beta\alpha^{\beta}x^{\beta-1}\ e^{-(\alpha x)^{\beta}}\left[1-\lambda+2\lambda e^{-(\alpha x)^{\beta}}\right] \cdot\left\{\left[1-e^{-(\alpha x)^{\beta}}\right]\left[1+\lambda e^{-(\alpha x)^{\beta}}\right]\right\}^{a-1} \cdot\left\{1-\left[1-e^{-(\alpha x)^{\beta}}\right]\left[1+\lambda e^{-(\alpha x)^{\beta}}\right]\right\}^{b-1},$$

where $\alpha>0$, $\beta>0$, $a>0$, $b>0$ and $|\lambda|\leq 1$. The BT-W distribution includes the transmuted Weibull (TW) distribution (Aryal and Tsokos [9]) when $a=b=1$. For $\lambda=0$ the BT-W distribution (Pal and Tiensuwan [32]) reduces to the BW distribution (Lee et al. [21]). For $\lambda=0$ and $b=1$, we obtain the exponentiated Weibull (EW) distribution (Mudholkar and Srivastava [27]). For $\beta=2$, we obtain the BT-Rayleigh (BT-R) distribution. For $\beta=1$, we have the BT-exponential (BT-E) distribution. For $a=b=1$ and $\beta=2$, we obtain the T-Rayleigh (T-R) distribution (Merovci [23]). Plots of the BT-W pdf for some parameter values are displayed in Figure 1.

2.2 The BT-Pa distribution

The BT-Pa pdf is given (for $x\geq\theta$) by

$$f(x)=\frac{\alpha\theta^{\alpha}}{B(a,b)}x^{-\alpha-1}\left[1-\lambda+2\lambda\left(\frac{\theta}{x}\right)^{\alpha}\right] \cdot\left\{\left[1-\left(\frac{\theta}{x}\right)^{\alpha}\right]\left[1+\lambda\left(\frac{\theta}{x}\right)^{\alpha}\right]\right\}^{a-1} \cdot\left\{1-\left[1-\left(\frac{\theta}{x}\right)^{\alpha}\right]\left[1+\lambda\left(\frac{\theta}{x}\right)^{\alpha}\right]\right\}^{b-1},$$

where $\alpha>0$, $\theta>0$, $a>0$, $b>0$ and $|\lambda|\leq 1$. The BT-Pa distribution includes the transmuted Pareto (T-Pa) distribution (Merovcia and Puka [25]) when $a=b=1$. For $\lambda=0$ the BT-Pa distribution reduces to the B-Pa distribution (Akinsete et al. [6]). For $\lambda=0$ and $b=1$, we obtain the exponentiated Pareto (E-Pa) distribution (Nadarajah [28]). Plots of the BT-Pa pdf are displayed in Figure 2 for some parameter values.

2.3 The BT-Fr distribution

The BT-Fr pdf is given (for $x>0$) by

$$f(x)=\frac{\beta\alpha^{\beta}x^{-\beta-1}}{B(a,b)}e^{-a\left(\frac{\alpha}{x}\right)^{\beta}}\left[1+\lambda-2\lambda e^{-\left(\frac{\alpha}{x}\right)^{\beta}}\right]$$

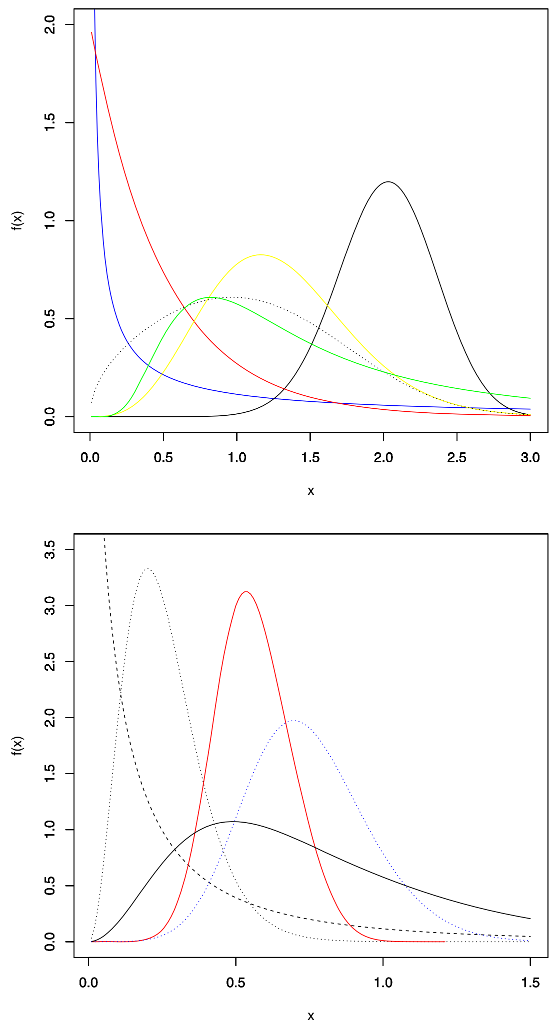

Figure 1. The BT-W pdfs: $\alpha = \beta = a = b = \lambda = 0.5$ *(blue line, left);* $\alpha = b = 0.5$, $\beta = 5$, $a = 2$, $\lambda = 1$ *(black line, left);* $\alpha = \beta = a = b = \lambda = 1$ *(red line, left);* $\alpha = 0.5$, $\beta = a = b = 2$, $\lambda = 0.8$ *(yellow line, left);* $\alpha = a = 0.5$, $\beta = 3$, $b = 1$, $\lambda = 0.9$ *(dotted line, left);* $\alpha = 2$, $\beta = 0.9$, $a = 9$, $b = 0.5$, $\lambda = 0.8$ *(green line, left);* $\alpha = a = 2$, $b = 2$, $\beta = 0.5$, $\lambda = 0.4$ *(dashed line, right);* $\alpha = a = 2$, $\beta = 1.5$, $b = 3$, $\lambda = 0.5$ *(dotted line, right);* $\alpha = a = 2$, $\beta = 4$, $b = 0.5$, $\lambda = 0.6$ *(red line, right);* $\alpha = a = 2$, $\beta = 1.5$, $b = \lambda = 0.5$ *(black line, right);* $\alpha = a = 2$, $\beta = 2.5$, $b = 0.5$, $\lambda = -0.6$ *(blue line, right).*

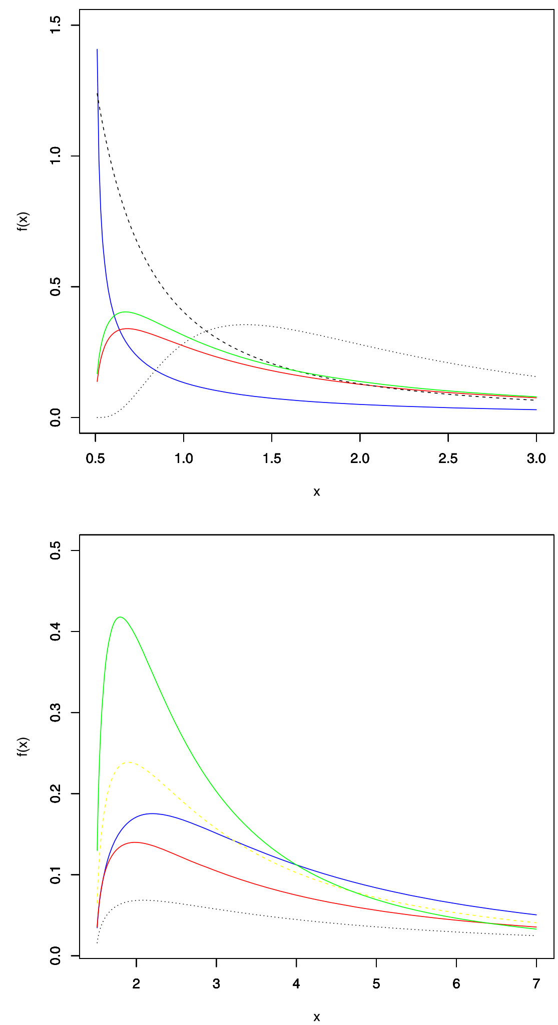

Figure 2. The BT-Pa pdfs: $\alpha = 0.2$, $\theta = 0.5$, $a = b = 0.5$, $\lambda = -0.5$ *(blue line, left);* $\alpha = 0.2$, $\theta = 0.5$, $a = 5$, $b = 7$, $\lambda = 0.7$ *(dotted line, left);* $\alpha = 0.2$, $\theta = 0.5$, $a = b = 1.5$, $\lambda = 0.7$ *(red line, left);* $\alpha = 0.2$, $\theta = 0.5$, $a = 1$, $b = 2$, $\lambda = 0.6$ *(dashed line, left);* $\alpha = 0.2$, $\theta = 0.5$, $a = 1.5$, $b = 2$, $\lambda = 0.5$ *(green line, left);* $\alpha = 5$, $\theta = a = 1.5$, $b = 2.5$, $\lambda = -0.5$, *(dotted line, right);* $\alpha = 5$, $\theta = a = 1.5$, $b = 0.5$, $\lambda = 0.2$ *(black line, right);* $\alpha = 5$, $\theta = a = 1.5$, $b = 0.8$, $\lambda = 0.4$ *(red line, right);* $\alpha = 5$, $\theta = a = 1.5$, $b = \lambda = 0.9$ *(dashed line, right);* $\alpha = 5$, $\theta = a = 1.5$, $b = 2$, $\lambda = 0.5$ *(green line, right).*

$$\cdot \left[1 + \lambda - \lambda e^{-\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{a-1}$$
$$\cdot \left\{1 - e^{-\left(\frac{\alpha}{x}\right)^{\beta}} \left[1 + \lambda - \lambda e^{-\left(\frac{\alpha}{x}\right)^{\beta}}\right]\right\}^{b-1},$$

where $\alpha > 0$, $\beta > 0$, $a > 0$, $b > 0$ and $|\lambda| \leq 1$. The BT-Fr distribution includes the transmuted Fréchet (T-Fr) distribution (Mahmoud and Mandouh [22]) when $a = b = 1$. For $\lambda = 0$ the BT-Fr distribution reduces to the B-Fr distribution (Nadarajah and Gupta [29]). For $\lambda = 0$ and $b = 1$, we obtain the exponentiated Fréchet (E-Fr) distribution (Nadarajah and Kotz [30]). We display some possible shapes of the BT-Fr pdf in Figure 3.

2.4 The BT-Go distribution

The BT-Go pdf is given (for $x > 0$) by

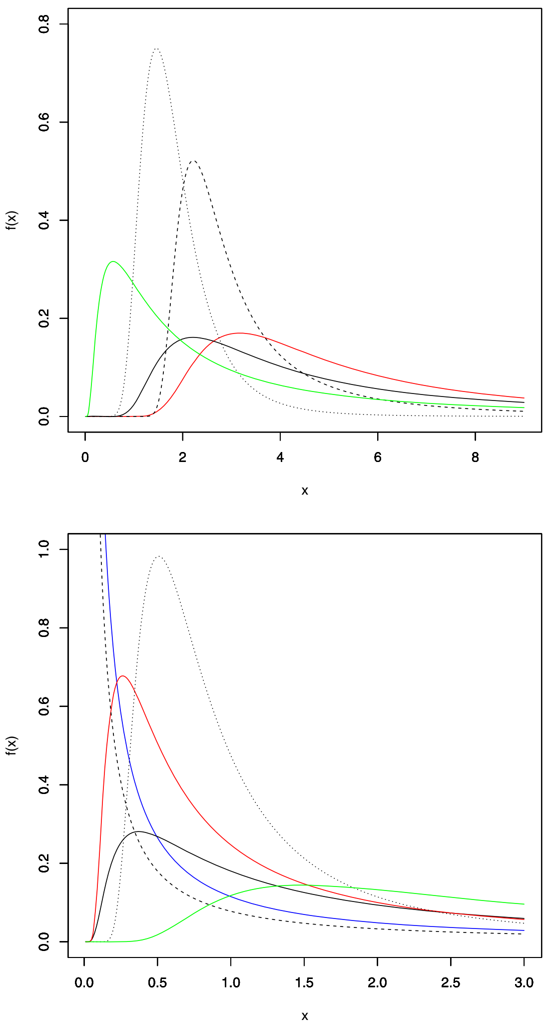

Figure 3. The BT-Fr pdfs: $\alpha = a = 2$, $\beta = 2.5$, $b = 0.5$, $\lambda = -0.6$ *(red line, left);* $\alpha = a = 2$, $\beta = 1.5$, $b = \lambda = 0.5$ *(black line, left);* $\alpha = a = 2$, $\beta = 4$, $b = 0.5$, $\lambda = 0.6$ *(dashed line, left);* $\alpha = a = 2$, $\beta = 1.5$, $b = 3$, $\lambda = 0.5$ *(dotted line, left);* $\alpha = a = 2$, $\beta = 0.5$, $b = 2$, $\lambda = 0.4$ *(green line, left);* $\alpha = 0.5$, $\beta = \lambda = a = b = 0.5$ *(blue line, right);* $\alpha = 0.5$, $\beta = 0.7$, $a = 2$, $b = 0.5$, $\lambda = 0.4$ *(black line, right);* $\alpha = 0.5$, $\beta = a = 1$, $b = 0.5$, $\lambda = 0.2$ *(red line, right);* $\alpha = 0.5$, $\beta = 1.5$, $a = 2$, $b = 0.6$, $\lambda = 0.8$ *(dotted line, right);* $\alpha = 0.5$, $\beta = 0.3$, $a = 0.5$, $b = 1$, $\lambda = 0.3$ *(dashed line, right);* $\alpha = 0.5$, $\beta = 0.9$, $a = 9$, $b = 0.5$, $\lambda = 0.8$ *(green line, right).*

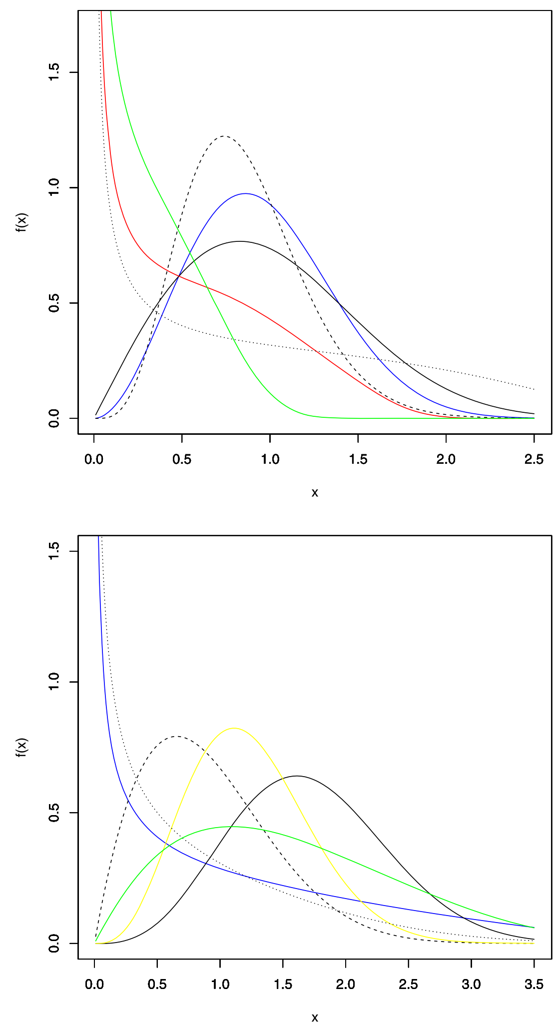

Figure 4. The GT-Go pdfs: $\beta = 0.5$, $\alpha = 0.9$, $a = 3$, $b = 2$, $\lambda = 0.4$ *(blue line, left);* $\beta = 0.5$, $\alpha = 0.5$, $a = 1$, $b = 3$, $\lambda = -1$ *(black line, left);* $\beta = 0.5$, $\alpha = 2$, $a = 0.4$, $b = 0.5$, $\lambda = 0.6$ *(red line, left);* $\beta = 0.5$, $\alpha = 1$, $a = b = \lambda = 0.4$ *(dotted line, left);* $\beta = 0.5$, $\alpha = 0.4$, $a = 5$, $b = 3.5$, $\lambda = 0.9$ *(dashed line, left);* $\beta = 0.5$, $\alpha = 2$, $a = 0.5$, $b = 2.5$, $\lambda = -0.3$ *(green line, left);* $\alpha = 0.5$, $\beta = a = b = \lambda = 0.5$ *(blue line, right);* $\alpha = 0.5$, $\beta = 0.2$, $a = 4$, $b = 3$, $\lambda = 0.7$ *(black line, right);* $\alpha = 0.5$, $\beta = 0.2$, $a = 2$, $b = 5$, $\lambda = 0.5$ *(dashed line, right);* $\alpha = 0.5$, $\beta = 0.3$, $a = 4$, $b = 3$, $\lambda = 0.8$ *(yellow line, right);* $\alpha = 0.5$, $\beta = 0.3$, $a = 0.5$, $b = 1$, $\lambda = 0.9$ *(dotted line, right);* $\alpha = 0.5$, $\beta = 0.8$, $a = 2$, $b = 0.5$, $\lambda = 0.4$ *(green line, right).*

$$
\begin{aligned}
f(x) = {} & \frac{\beta e^{\alpha x}}{B(a,b)} e^{-\frac{\beta}{\alpha}\left(e^{\alpha x}-1\right)} \left[1 - \lambda + 2\lambda e^{-\frac{\beta}{\alpha}\left(e^{\alpha x}-1\right)}\right] \\
& \cdot \left\{\left[1 - e^{-\frac{\beta}{\alpha}\left(e^{\alpha x}-1\right)}\right]\left[1 + \lambda e^{-\frac{\beta}{\alpha}\left(e^{\alpha x}-1\right)}\right]\right\}^{a-1} \\
& \cdot \left\{1 - \left[1 - e^{-\frac{\beta}{\alpha}\left(e^{\alpha x}-1\right)}\right]\left[1 + \lambda e^{-\frac{\beta}{\alpha}\left(e^{\alpha x}-1\right)}\right]\right\}^{b-1},
\end{aligned}
$$

where $\alpha > 0$, $\beta > 0$, $a > 0$, $b > 0$ and $|\lambda| \leq 1$. The BT-Go distribution includes the transmuted Gompertz (T-Go) distribution (Abdul-Moniem et al. [4]) when $a = b = 1$. For $\lambda = 0$ the BT-Go distribution reduces to the B-Go distribution (Jafari et al. [19]). For $\lambda = 0$ and $b = 1$, we obtain the exponentiated Gompertz (E-Go) distribution. For $a = 1$ and $\lambda = 0$, we obtain the generalized Gompertz (G-Go) distribution (El-Gohary et al. [13]). Figure 4 plots the BT-Go pdf for selected parameter values.

2.5 The BT-Li distribution

The BT-Li pdf is given (for $x>0$) by

$$f(x)=\frac{\alpha^2(1+x)e^{-\alpha x}}{B\left(a,b\right)\left(1+\alpha\right)}\left(1-\lambda+\frac{1+\alpha+\alpha x}{1+\alpha}2\lambda e^{-\alpha x}\right)$$
$$\cdot\left(1-\frac{1+\alpha+\alpha x}{1+\alpha}e^{-\alpha x}\right)^{a-1}$$
$$\cdot\left(1+\frac{1+\alpha+\alpha x}{1+\alpha}\lambda e^{-\alpha x}\right)^{a-1}$$
$$\cdot\left[1-\left(1-\frac{1+\alpha+\alpha x}{1+\alpha}e^{-\alpha x}\right)\left(1+\frac{1+\alpha+\alpha x}{1+\alpha}\lambda e^{-\alpha x}\right)\right]^{b-1},$$

where α, β, a and b are positive parameters and $|\lambda|\le 1$. The BT-Li distribution includes the transmuted Lindley (T-Li) distribution (Merovci [24]) when $a=b=1$. For $\lambda=0$ the BT-Li distribution reduces to the B-Li distribution (Merovci and Sharma [26]). For $\lambda=0$ and $b=1$, we obtain the exponentiated Lindley (E-Li) distribution. Plots of the BT-Li pdf are displayed in Figure 5 for some parameter values.

3. MATHEMATICAL PROPERTIES

The formulae derived throughout the paper can be easily handled in most symbolic computation software platforms such as Maple, Mathematica and Matlab because of their ability to deal with analytic expressions of formidable size and complexity. Established explicit expressions to calculate statistical measures can be more efficient than computing them directly by numerical integration.

3.1 Mixture representation

In this section, we derive a useful representation for the BT-H pdf. The derived representation is crucial for the derivation of mathematical properties in Sections 3.2 to 3.4. It allows moments, generating functions, order statistic properties, etc to be expressed as mixtures.

Consider the power series

$$(5)\qquad (1-z)^{b-1}=\sum_{j=0}^{\infty}\frac{(-1)^j\ \Gamma\left(b\right)}{j!\ \Gamma\left(b-j\right)}\ z^j,$$

which holds for $|z|<1$ and $b>0$ real non-integer.

The pdf in (4) can be rewritten as

$$f(x)=\frac{1}{B\left(a,b\right)}h\left(x\right)\left[1+\lambda-2\lambda H\left(x\right)\right]$$
$$\cdot\left[\left(1+\lambda\right)H\left(x\right)-\lambda H\left(x\right)^2\right]^{a-1}$$
$$\cdot\left\{1-\left[\left(1+\lambda\right)H\left(x\right)-\lambda H\left(x\right)^2\right]\right\}^{b-1}.$$

Consider $A=\left\{1-\left[\left(1+\lambda\right)H\left(x\right)-\lambda H\left(x\right)^2\right]\right\}^{b-1}$.

Applying the power series (5) to the quantity A, we obtain

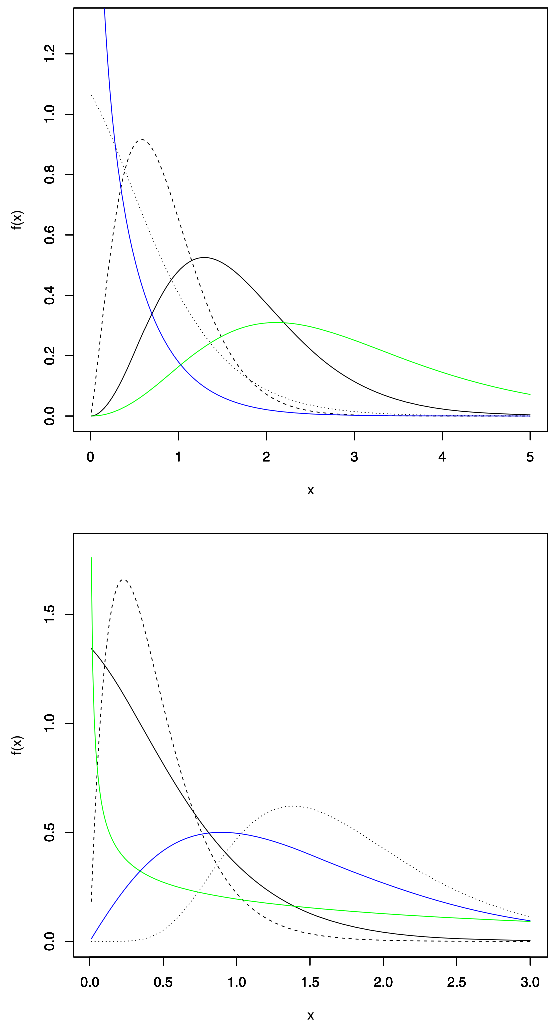

Figure 5. The BT-Li pdfs: $\alpha=0.5$, $\lambda=0.3$, $a=3$, $b=5$ *(black line, left);* $\alpha=0.5$, $\lambda=0.6$, $a=1$, $b=4$ *(dotted line, left);* $\alpha=0.5$, $\lambda=0.7$, $a=2.3$, $b=7$ *(dashed line, left);* $\alpha=0.5$, $\lambda=0.8$, $a=3$, $b=1.5$ *(green line, left);* $\alpha=0.5$, $\lambda=a=0.5$, $b=5$ *(blue line, left);* $\alpha=1.5$, $a=0.5$, $b=2$, $\lambda=-1$ *(black line, right);* $\alpha=2$, $a=9$, $b=0.9$, $\lambda=0.2$ *(dotted line, right);* $\alpha=1.5$, $a=2$, $b=3$, $\lambda=0.4$ *(dashed line, right);* $\alpha=0.9$, $a=\lambda=0.5$, $b=0.3$ *(green line, right);* $\alpha=0.7$, $a=b=2$, $\lambda=0.5$ *(blue line, right).*

$$f(x)=\frac{1}{B\left(a,b\right)}\underbrace{h\left(x\right)\left[1+\lambda-2\lambda H\left(x\right)\right]}_{g(x)}$$
$$\cdot\sum_{k=0}^{\infty}\frac{\left(-1\right)^k\Gamma\left(k\right)}{k!\Gamma\left(b-k\right)}$$
$$\cdot\underbrace{\left[\left(1+\lambda\right)H\left(x\right)-\lambda H\left(x\right)^2\right]^{a+k-1}}_{G(x)^{a+k-1}}.$$

Further, we can write the last equation as

$$(6) \qquad f(x)=\sum_{j=0}^{\infty}\frac{(-1)^{k}\Gamma(k)}{B(a,b)\,k!\Gamma(b-k)}\;g(x)\,G(x)^{a+k-1}.$$

Finally, the pdf (6) can be expressed as a mixture of exp-G pdfs

$$(7) \qquad f(x)=\sum_{k=0}^{\infty}t_k\;\pi_{a+k}(x),$$

where $\pi_\alpha(x)=\alpha g(x)\,G(x)^{\alpha-1}$ is the exp-G pdf with power parameter $\alpha>0$ and

$$t_k=\frac{(-1)^{k}\Gamma(k)}{B(a,b)\,k!\Gamma(b-k)\,(a+k)}.$$

Thus, several mathematical properties of the BT-H family can be obtained simply from those properties of the exp-G family. Equation (7) is the main result of this section.

The cdf of the BT-H family can also be expressed as a mixture of exp-G cdfs. By integrating (7), we obtain the mixture representation

$$F(x)=\sum_{k=0}^{\infty}t_k\;\Pi_{a+k}(x),$$

where $\Pi_{a+k}(x)$ is the cdf of the exp-G family with power parameter $a+k$.

3.2 Moments

The rth moment of X, say μ_r', follows from (7) as

$$\mu_r'=E(X^r)=\sum_{k=0}^{\infty}t_k\;E\left(Y_{a+k}^r\right).$$

Henceforth, Y_{a+k} denotes the exp-G random variable with power parameter $a+k$.

The nth central moment of X, say M_n, is given by

$$\begin{aligned} M_n &= E(X-\mu_1')^n=\sum_{r=0}^{n}\binom{n}{r}(-\mu_1')^{n-r}E(X^r) \\ &= \sum_{r=0}^{n}\sum_{k=0}^{\infty}(-1)^{n-r}\,t_k\binom{n}{r}\mu_1'^{(n-r)}E\left(Y_{a+k}^r\right). \end{aligned}$$

The cumulants, κ_n, of X follow recursively from

$$\kappa_n=\mu_n'-\sum_{r=0}^{n-1}\binom{n-1}{r-1}\kappa_r\mu_{n-r}',$$

where $\kappa_1=\mu_1'$, $\kappa_2=\mu_2'-\mu_1'^2$, $\kappa_3=\mu_3'-3\mu_2'\mu_1'+\mu_1'^3$, etc. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships.

3.3 Quantile and generating functions

The qf of X, where $X\ \sim$BT-H$(a,b,\lambda,\boldsymbol{\phi})$, is easily simulated by inverting (3) as follows: if T is a beta random variable with positive parameters a and b, then

$$(8) \qquad X=H^{-1}\left\{\frac{\lambda+1-\sqrt{(\lambda+1)^2+4\lambda T}}{2\lambda};\boldsymbol{\phi}\right\}$$

is a BT-H variate.

Now, we provide two formulae for the moment generating function (mgf) $M_X(t)=E\left(e^{t\,X}\right)$ of X. Clearly, the first one can be derived from (7) as

$$M_X(t)=\sum_{k=0}^{\infty}t_k\;M_{a+k}(t),$$

where $M_{a+k}(t)$ is the mgf of Y_{a+k}. Hence, $M_X(t)$ can be determined from the exp-G generating function.

A second formula for $M_X(t)$ follows from (7) as

$$M_X(t)=\sum_{k=0}^{\infty}t_{a+k}\;\tau(t,a+k-1),$$

where $\tau(t,a+k-1)=\int_0^1\exp\left[t\;Q_H(u)\right]\;u^{a+k-1}du$ and $Q_H(u)$ is the qf corresponding to $H(x;\boldsymbol{\phi})$, i.e., $Q_H(u)=H^{-1}(u;\boldsymbol{\phi})$.

3.4 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Let $X_1,\ldots,X_n$ be a random sample from the BT-H family of distributions. The pdf of ith order statistic, say $X_{i:n}$, can be written as

$$(9) \qquad f_{i:n}(x)=\frac{f(x)}{\mathrm{B}(i,n-i+1)}\cdot\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}F(x)^{j+i-1}.$$

Then

$$\begin{aligned} F(x)^{j+i-1} &= \left(\sum_{k=0}^{\infty}t_k\;\Pi_{a+k}(x)\right)^{j+i-1} \\ (10) \qquad &= \left\{\sum_{k=0}^{\infty}t_k\left[H(x)^{\frac{a}{k}+1}\right]^k\right\}^{i+j-1}. \end{aligned}$$

Here and henceforth, we use an equation given in page 17 of Gradshteyn and Ryzhik [16] for a power series raised to a positive integer n:

$$(11) \qquad \left(\sum_{r=0}^{\infty}b_r\;u^r\right)^n=\sum_{r=0}^{\infty}C_{n,r}\;u^r,$$

where the coefficients $C_{n,r}$ are easily determined from the recurrence equation

$$C_{n,r} = (rb_0)^{-1} \sum_{m=1}^{r} \left[m\,(n+1) - r\right](b_m)\,(C_{n,r-m})\,,$$

where $C_{n,0} = b_0^n$. The coefficient $C_{n,r}$ can be calculated from $C_{n,0}, \ldots, C_{n,r-1}$ and hence from the quantities $b_0, \ldots, b_r$.

Using (10) and (11), we obtain

$$\begin{aligned} F(x)^{j+i-1} &= \left\{ \sum_{k=0}^{\infty} t_k \left[H(x)^{\frac{a}{k}+1}\right]^k \right\}^{j+i-1} \\ &= \sum_{k=0}^{\infty} C_{k,i+j-1} \left[H(x)^{\frac{a}{k}+1}\right]^k. \end{aligned} \tag{12}$$

Substituting (7) and (12) into (9) and using a power series expansion, the pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \frac{\sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j}}{\mathrm{B}(i, n-i+1)} \sum_{k=0}^{\infty} C_{k,i+j-1}\ \pi_{a+k}(x)\,,$$

where $\pi_{a+k}(x)$ is the exp-G pdf with power parameter $a+k$.

It follows that the pdf of BT-H order statistics is a mixture of exp-G pdfs. Hence, the properties of $X_{i:n}$ follow from properties of Y_{a+k}. For example, the moments of $X_{i:n}$ can be expressed as

$$\begin{aligned} E\left(X_{i:n}^q\right) &= \sum_{k=0}^{\infty} \sum_{j=0}^{n-i} \frac{(-1)^j}{\mathrm{B}(i, n-i+1)} \\ &\cdot \binom{n-i}{j} C_{k,i+j-1} E\left(Y_{a+k}^q\right). \end{aligned} \tag{13}$$

The L-moments are analogous to the ordinary moments but can be estimated by linear combinations of order statistics. They exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers. Based upon the moments in (13), we can derive explicit expressions for the L-moments of X as infinite weighted linear combinations of the means of suitable BT-H order statistics. They are linear functions of expected order statistics defined by

$$\lambda_r = \frac{1}{r} \sum_{d=0}^{r-1} (-1)^d \binom{r-1}{d} E\left(X_{r-d:r}\right)$$

for $r \geq 1$.

4. MAXIMUM LIKELIHOOD ESTIMATION

Several approaches for parameter estimation exist in the literature but maximum likelihood method is the most commonly employed. The maximum likelihood estimators (MLEs) enjoy desirable properties and can be used when constructing confidence intervals and also in test statistics. The normal approximation for these estimators in large sample theory is easily handled either analytically or numerically. So, we consider estimation of the unknown parameters for the BT-H family only by maximum likelihood. Here, we determine the MLEs of the parameters for complete samples only. Let $x_1, \ldots, x_n$ be a random sample from the BT-H family with parameters λ, a, b and $\boldsymbol{\phi}$. Let $\boldsymbol{\Theta} = (a, b, \lambda, \boldsymbol{\phi}^\intercal)^\intercal$ be a $(p \cdot 1)$ parameter vector. Then, the log-likelihood function for $\boldsymbol{\Theta}$, say $\ell = \ell(\boldsymbol{\Theta})$, is

$$\begin{aligned} \ell &= -n \log\left[B(a,b)\right] + \sum_{i=1}^{n} \log h(x_i; \boldsymbol{\phi}) + \sum_{i=1}^{n} \log p_i \\ &\quad + (a-1) \sum_{i=1}^{n} \log q_i + (b-1) \sum_{i=1}^{n} \log(1 - q_i)\,, \end{aligned} \tag{14}$$

where $p_i = 1 + \lambda - 2\lambda H(x_i; \boldsymbol{\phi})$, $q_i = z_i H(x_i; \boldsymbol{\phi})$ and $z_i = 1 + \lambda - \lambda H(x_i; \boldsymbol{\phi})$.

The score vector components, say $\mathbf{U}(\boldsymbol{\Theta}) = \frac{\partial \ell}{\partial \boldsymbol{\Theta}} = \left(\frac{\partial \ell}{\partial a}, \frac{\partial \ell}{\partial b}, \frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial \phi_k}\right)^\intercal = (U_a, U_b, U_\lambda, U_{\phi_k})^\intercal$, are given by

$$U_a = -n\left\{\psi(a) - \psi(a+b)\right\} + \sum_{i=1}^{n} \log q_i,$$

$$U_b = -n\left\{\psi(b) - \psi(a+b)\right\} + \sum_{i=1}^{n} \log(1 - q_i)\,,$$

$$\begin{aligned} U_\lambda &= \sum_{i=1}^{n} \frac{1}{p_i}\left[1 - 2H(x_i; \boldsymbol{\phi})\right] \\ &\quad + (a-1) \sum_{i=1}^{n} \frac{1}{q_i} \left[H(x_i; \boldsymbol{\phi}) - H(x_i; \boldsymbol{\phi})^2\right] \\ &\quad - (b-1) \sum_{i=1}^{n} \frac{1}{1 - q_i} \left[H(x_i; \boldsymbol{\phi}) - H(x_i; \boldsymbol{\phi})^2\right] \end{aligned}$$

and

$$\begin{aligned} U_{\phi_k} &= (a-1) \sum_{i=1}^{n} \frac{1}{q_i} H'(x_i; \boldsymbol{\phi})\left[z_i - \lambda H(x_i; \boldsymbol{\phi})\right] \\ &\quad -2\lambda \sum_{i=1}^{n} \frac{H'(x_i; \boldsymbol{\phi})}{p_i} \\ &\quad + \sum_{i=1}^{n} \frac{h'(x_i; \boldsymbol{\phi})}{h(x_i; \boldsymbol{\phi})} \\ &\quad -(b-1) \sum_{i=1}^{n} \frac{H'(x_i; \boldsymbol{\phi})\left[z_i - \lambda H(x_i; \boldsymbol{\phi})\right]}{(1 - q_i)}, \end{aligned}$$

where $\psi(a)$ is the digamma function, $h'(x_i; \boldsymbol{\phi}) = \partial h(x_i; \boldsymbol{\phi}) / \partial \phi_k$ and $H'(x_i; \boldsymbol{\phi}) = \partial H(x_i; \boldsymbol{\phi}) / \partial \phi_k$. Setting the nonlinear system of equations $U_a = U_b = U_\lambda = U_{\phi_k} = \mathbf{0}$ and solving them simultaneously yields the MLE $\widehat{\boldsymbol{\Theta}} = \left(\widehat{a}, \widehat{b}, \widehat{\lambda}, \widehat{\boldsymbol{\phi}}^\intercal\right)^\intercal$ of $\boldsymbol{\Theta} = (a, b, \lambda, \boldsymbol{\phi}^\intercal)^\intercal$. These equations cannot be solved analytically and statistical software can be used to solve them numerically using iterative methods such as Newton-Raphson type algorithms.

The MLEs can also be obtained by maximizing (14) directly by using `R` (`optim` function), `SAS` (`PROC NLMIXED`), `Ox` program (sub-routine `MaxBFGS`) or a `MATHCAD` program. In Sections 5 and 6, we used the `optim` function in `R`. We maximized (14) using a wide range of starting values. The starting values were taken in a fine scale. For the BT-Go distribution, for example, they were taken to correspond to all combinations of $\alpha = 1, 2, \ldots, 10$, $\beta = 1, 2, \ldots, 10$, $a = 1, 2, \ldots, 10$, $b = 1, 2, \ldots, 10$ and $\lambda = -0.9, -0.7, \ldots, 0.9$. For the BT-Li distribution, for example, the starting values were taken to correspond to all combinations of $\alpha = 1, 2, \ldots, 10$, $a = 1, 2, \ldots, 10$, $b = 1, 2, \ldots, 10$ and $\lambda = -0.9, -0.7, \ldots, 0.9$. The call to `optim` converged about 98 percent of the time. When the calls to `optim` did converge, the maximum likelihood solution was unique. The unique solution was verified by using the `PROC NLMIXED` function in `SAS`. None of the unique solutions corresponded to boundaries of the parameter spaces.

We experimented maximization of (14) for a wide range of choices for H that are smooth (smooth in the sense of continuity and differentiability) and for a wide range of starting values. The reported observations held for each choice. That is, `optim` converged about 98 percent of the time, the maximum likelihood solution was unique when `optim` did converge and none of the unique solutions corresponded to boundaries of the parameter spaces. Generally, the likelihood surface was smooth whenever H was smooth.

For interval estimation of the model parameters, we require the observed information matrix

$$J(\boldsymbol{\Theta}) = -\left(\begin{array}{ccccc} U_{aa} & U_{ab} & U_{a\lambda} & \mid & U_{a\boldsymbol{\phi}}^{\intercal} \\ U_{ba} & U_{bb} & U_{b\lambda} & \mid & J_{b\boldsymbol{\phi}}^{\intercal} \\ J_{\lambda a} & U_{\lambda b} & U_{\lambda\lambda} & \mid & U_{\lambda\boldsymbol{\phi}}^{\intercal} \\ -- & -- & -- & -- & -- \\ U_{a\boldsymbol{\phi}} & J_{b\boldsymbol{\phi}} & U_{\lambda\boldsymbol{\phi}} & \mid & U_{\varphi\boldsymbol{\phi}} \end{array}\right).$$

Explicit expressions for the elements of this matrix are available from the corresponding author.

Under standard regularity conditions as $n \to \infty$, the distribution of $\widehat{\boldsymbol{\Theta}}$ approximates to a multivariate normal $N_p\left(\mathbf{0}, J\left(\widehat{\boldsymbol{\Theta}}\right)^{-1}\right)$ distribution to construct approximate confidence intervals for the parameters. Here, $J\left(\widehat{\boldsymbol{\Theta}}\right)$ is the total observed information matrix evaluated at $\widehat{\boldsymbol{\Theta}}$. The method of re-sampling bootstrap can be used for correcting the biases of the MLEs of the model parameters. Interval estimates may also be obtained using the bootstrap percentile method. Likelihood ratio tests can be performed for the proposed family of distributions in the usual way.

5. SIMULATION STUDY

Here, we assess the finite sample behaviors of the MLEs for the five-parameter BT-Go and four-parameter BT-Li distributions, the distributions used in the data application section. The assessments were based on simulation studies.

5.1 BT-Go distribution

The assessment of the finite sample behavior of the MLEs for this distribution was based on the following:

1. use the inversion method to generate ten thousand samples of size n from the BT-Go distribution, i.e., generate values of

$$X = H^{-1}\left\{\frac{\lambda + 1 - \sqrt{(\lambda+1)^2 + 4\lambda T}}{2\lambda}; \boldsymbol{\phi}\right\},$$

   where T is a beta variate with parameters (a, b) and

$$H^{-1}(u) = \frac{1}{\alpha}\log\left[1 - \frac{\alpha}{\beta}\log(1-u)\right].$$

2. compute the MLEs for the ten thousand samples, say $\left(\widehat{\alpha}_i, \widehat{\beta}_i, \widehat{a}_i, \widehat{b}_i, \widehat{\lambda}_i\right)$ for $i = 1, 2, \ldots, 10000$.
3. compute the standard errors of the MLEs for the ten thousand samples, say $\left(s_{\widehat{\alpha}_i}, s_{\widehat{\beta}_i}, s_{\widehat{a}_i}, s_{\widehat{b}_i}, s_{\widehat{\lambda}_i}\right)$ for $i = 1, 2, \ldots, 10000$. The standard errors were computed by inverting the observed information matrix.
4. compute the biases and mean squared errors given by

$$\mathrm{bias}_h(n) = \frac{1}{10000}\sum_{i=1}^{10000}\left(\widehat{h}_i - h\right),$$

$$\mathrm{MSE}_h(n) = \frac{1}{10000}\sum_{i=1}^{10000}\left(\widehat{h}_i - h\right)^2$$

   for $h = \alpha, \beta, a, b, \lambda$.
5. compute the coverage probabilities and coverage lengths given by

$$\mathrm{CP}_h(n) = \frac{1}{10000}\sum_{i=1}^{10000} I\left\{\widehat{h}_i - 1.959964 s_{\widehat{h}_i} < h < \widehat{h}_i + 1.959964 s_{\widehat{h}_i}\right\},$$

$$\mathrm{CL}_h(n) = \frac{3.919928}{10000}\sum_{i=1}^{10000} s_{\widehat{h}_i}$$

   for $h = \alpha, \beta, a, b, \lambda$, where $I\{\cdot\}$ denotes the indicator function.

We repeated these steps for $n = 10, 11, \ldots, 200$ with $\alpha = 1$, $\beta = 2$, $a = 1$, $b = 1$ and $\lambda = 1$, so computing $\mathrm{bias}_h(n)$, $\mathrm{MSE}_h(n)$, $\mathrm{CP}_h(n)$ and $\mathrm{CL}_h(n)$ for $h = \alpha, \beta, a, b, \lambda$ and $n = 10, 11, \ldots, 200$.

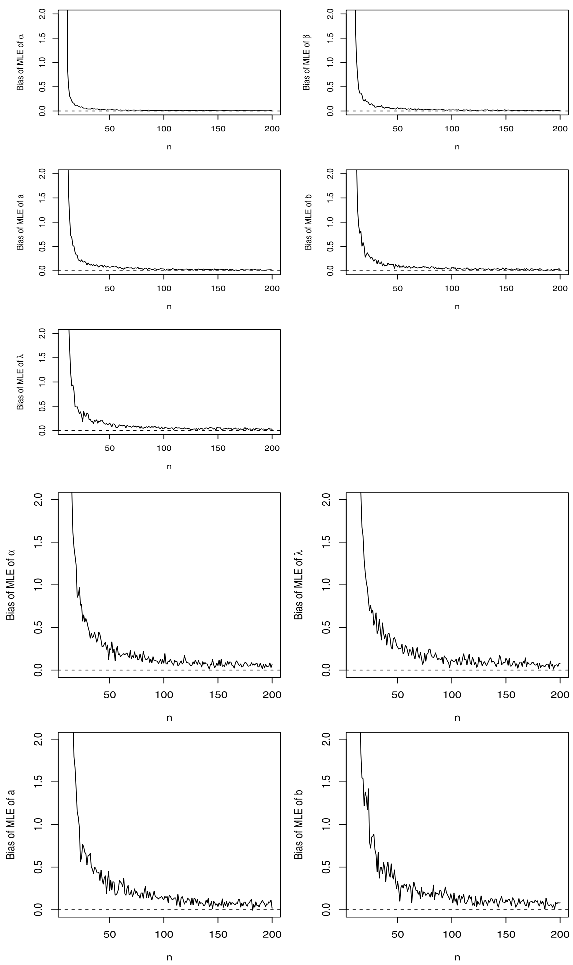

Figure 6. Biases of α, β, a, b and λ versus $n = 10, 11, \ldots, 200$ for the BT-Go distribution (left); Biases of α, λ, a and b versus $n = 10, 11, \ldots, 200$ for the BT-Li distribution (right).

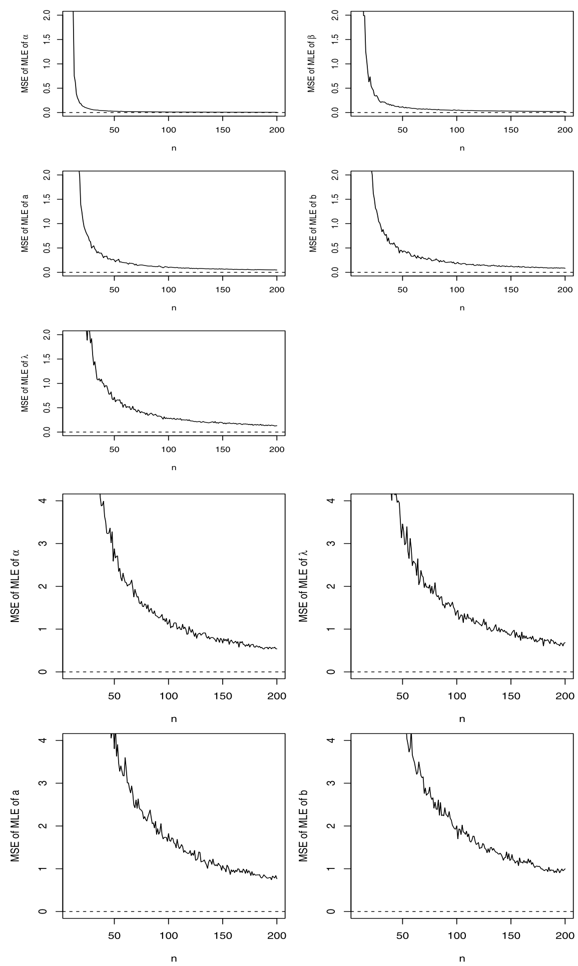

Figure 7. Mean squared errors of α, β, a, b and λ versus $n = 10, 11, \ldots, 200$ for the BT-Go distribution (left); Mean squared errors of α, λ, a and b versus $n = 10, 11, \ldots, 200$ for the BT-Li distribution (right).

Figure 6 shows how the five biases vary with respect to n. Figure 7 shows how the five mean squared errors vary with respect to n. Figure 8 shows how the five coverage probabilities vary with respect to n. Figure 9 shows how the five coverage lengths vary with respect to n. The broken line in Figure 6 corresponds to the biases being zero. The broken line in Figure 7 corresponds to the mean squared errors being zero. The broken line in Figure 8 corresponds to the nominal coverage probabilities of 0.95. The broken line in Figure 9 corresponds to the coverage lengths being zero.

The following observations can be made from Figures 6 to 9: the biases for each parameter are generally positive and decrease to zero as $n \to \infty$; the mean squared errors for each parameter decrease to zero as $n \to \infty$; the coverage probabilities for each parameter approach the nominal level as $n \to \infty$; the coverage lengths for each parameter decrease to zero as $n \to \infty$.

5.2 BT-Li distribution

The assessment of the finite sample behavior of the MLEs for this distribution was based on the following:

1. use the inversion method to generate ten thousand samples of size n from the BT-Li distribution, i.e., generate values of

$$X = H^{-1}\left\{\frac{\lambda + 1 - \sqrt{(\lambda+1)^2 + 4\lambda T}}{2\lambda}; \phi\right\},$$

where T is a beta variate with parameters (a, b) and

$$H^{-1}(u) = W\left(-(1+\alpha)e^{-1-\alpha}(1-u)\right),$$

where $W(\cdot)$ denotes the Lambert W function, see Corless et al. [12] for detailed properties.

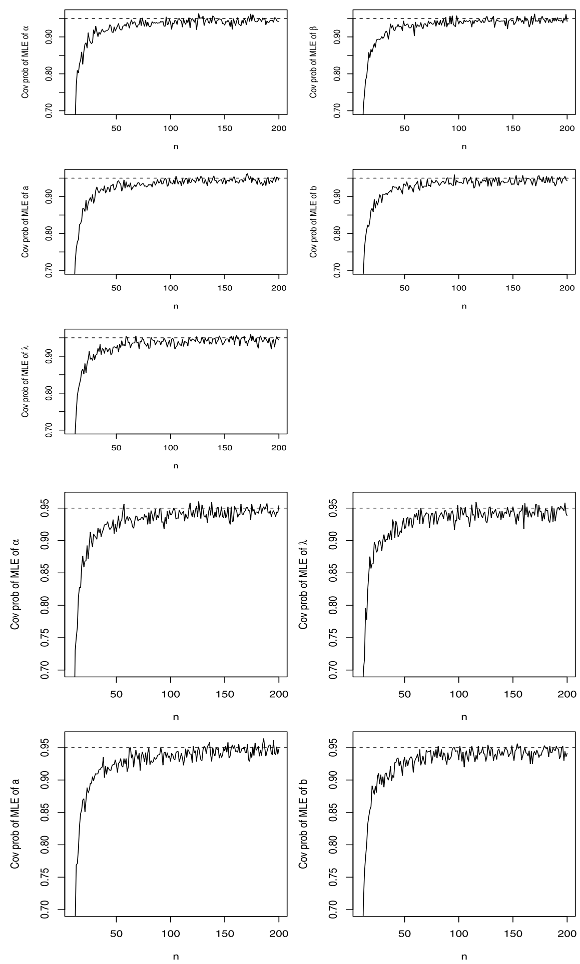

Figure 8. Coverage probabilities of α, β, a, b and λ versus $n = 10, 11, \ldots, 200$ for the BT-Go distribution (left); Coverage probabilities of α, λ, a and b versus $n = 10, 11, \ldots, 200$ for the BT-Li distribution (right).

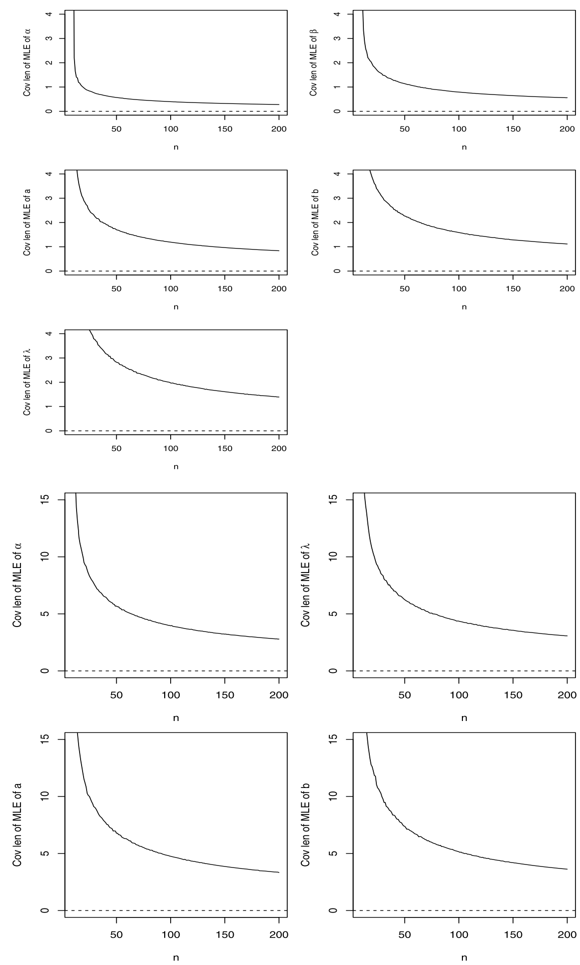

Figure 9. Coverage lengths of α, β, a, b and λ versus $n = 10, 11, \ldots, 200$ for the BT-Go distribution (left); Coverage lengths of α, λ, a and b versus $n = 10, 11, \ldots, 200$ for the BT-Li distribution (right).

2. compute the MLEs for the ten thousand samples, say $\left(\widehat{\alpha}_i, \widehat{a}_i, \widehat{b}_i, \widehat{\lambda}_i\right)$ for $i = 1, 2, \ldots, 10000$.
3. compute the standard errors of the MLEs for the ten thousand samples, say $\left(s_{\widehat{\alpha}_i}, s_{\widehat{a}_i}, s_{\widehat{b}_i}, s_{\widehat{\lambda}_i}\right)$ for $i = 1, 2, \ldots, 10000$.
4. compute the biases and mean squared errors given by

$$\mathrm{bias}_h(n) = \frac{1}{10000} \sum_{i=1}^{10000} \left(\widehat{h}_i - h\right),$$

$$\mathrm{MSE}_h(n) = \frac{1}{10000} \sum_{i=1}^{10000} \left(\widehat{h}_i - h\right)^2$$

for $h = \alpha, a, b, \lambda$.
5. compute the coverage probabilities and coverage lengths given by

$$\mathrm{CP}_h(n) = \frac{1}{10000} \sum_{i=1}^{10000} I\Big\{\widehat{h}_i - 1.959964 s_{\widehat{h}_i} < h < \widehat{h}_i + 1.959964 s_{\widehat{h}_i}\Big\},$$

$$\mathrm{CL}_h(n) = \frac{3.919928}{10000} \sum_{i=1}^{10000} s_{\widehat{h}_i}$$

for $h = \alpha, a, b, \lambda$.

We repeated these steps for $n = 10, 11, \ldots, 200$ with $\alpha = 1$, $a = 1$, $b = 1$ and $\lambda = 1$, so computing $\mathrm{bias}_h(n)$, $\mathrm{MSE}_h(n)$, $\mathrm{CP}_h(n)$ and $\mathrm{CL}_h(n)$ for $h = \alpha, a, b, \lambda$ and $n = 10, 11, \ldots, 200$.

Figure 6 also shows how the four biases vary with respect to n. Figure 7 also shows how the four mean squared errors vary with respect to n. Figure 8 also shows how the four coverage probabilities vary with respect to n. Figure 9 also shows how the four coverage lengths vary with respect to n.

The observations from these figures are the same as those for the BT-Go distribution. That is, the biases for each parameter are generally positive and decrease to zero as $n \to \infty$; the mean squared errors for each parameter decrease to zero as $n \to \infty$; the coverage probabilities for each parameter approach the nominal level as $n \to \infty$; the coverage lengths for each parameter decrease to zero as $n \to \infty$.

We have presented results for only one choice for $(\alpha, \beta, a, b, \lambda)$ for the BT-Go distribution and one choice for (α, λ, a, b) for the BT-Li distribution. But the results were similar for a wide range of other values of the parameters and for other BT-H distributions. In particular, the biases for each parameter always approached zero as $n \to \infty$, the mean squared errors for each parameter always decreased to zero as $n \to \infty$, the coverage probabilities for each parameter always approached the nominal level as $n \to \infty$ and the coverage lengths for each parameter always decreased to zero as $n \to \infty$.

6. DATA APPLICATIONS

Here, we provide applications to two real data sets to illustrate the importance and potentiality of the BT-Go and BT-Li distributions presented in Section 2. The goodness-of-fit statistics for these distributions and other competitive distributions are compared and the MLEs of their parameters are provided.

6.1 Waiting times in a queue

The first real data set (Ghitany et al. [15]) consists of 100 observations on waiting time (in minutes) before the customer received service in a bank. The data are: 0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 2.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23, 27, 31.6, 33.1, 38.5. We compare the fits of the BT-Go distribution with other competitive distributions, namely: the T-Go, G-Go beta Burr XII (BBXII) (Paranaba et al. [33]), McDonald Weibull (McW) (Cordeiro et al. [11]), modified beta Weibull (MBW) (Khan [20]), transmuted exponentiated modified Weibull (TEMW) (Ashour and Eltehiwy [10]), transmuted Weibull Lomax (TWL) (Afify et al. [5]) and generalized transmuted Weibull (GTW) (Nofal et al. [31]) distributions. The pdfs of these distributions are (for $x > 0$):

BBXII: $f(x) = \frac{\delta\beta}{B(a,b)} \alpha^{-\beta} x^{\beta-1} \left[1 + \left(\frac{x}{\alpha}\right)^{\beta}\right]^{-(\delta b+1)}$
$\left\{1 - \left[1 + \left(\frac{x}{\alpha}\right)^{\beta}\right]^{-\delta}\right\}^{a-1}$;

McW: $f(x) = \frac{\beta c \alpha^{\beta}}{B(a/c,b)}\ x^{\beta-1}\ \mathrm{e}^{-(\alpha x)^{\beta}} \left[1 - \mathrm{e}^{-(\alpha x)^{\beta}}\right]^{a-1}$
$\left\{1 - \left[1 - \mathrm{e}^{-(\alpha x)^{\beta}}\right]^{c}\right\}^{b-1}$;

MBW: $f(x) = \frac{\beta\alpha^{-\beta} c^{a}}{B(a/c,b)}\ x^{\beta-1}\ \mathrm{e}^{-b\left(\frac{x}{\alpha}\right)^{\beta}} \left[1 - \mathrm{e}^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right]^{a-1}$
$\left\{1 - (1-c)\left[1 - \mathrm{e}^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right]^{c}\right\}^{-a-b}$;

TEMW: $f(x) = \delta\left(\alpha + \gamma\beta x^{\beta-1}\right) \mathrm{e}^{-\left(\alpha x + \gamma x^{\beta}\right)}$
$\left[1 - \mathrm{e}^{-\left(\alpha x + \gamma x^{\beta}\right)}\right]^{\delta-1}$
$\left\{1 + \lambda - 2\lambda\left[1 - \mathrm{e}^{-\left(\alpha x + \gamma x^{\beta}\right)}\right]^{\delta}\right\}$;

TWL: $f(x) = \frac{ab\alpha}{\beta}\left(1 + \frac{x}{\beta}\right)^{b\alpha-1} \mathrm{e}^{-a\left[\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right]^{b}}$
$\left[1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}\right]^{b-1}$
$\left\{1 - \lambda + 2\lambda \mathrm{e}^{-a\left[\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right]^{b}}\right\}$;

GTW: $f(x) = \beta\alpha^{\beta} x^{\beta-1} \mathrm{e}^{-(\alpha x)^{\beta}} \left[1 - \mathrm{e}^{-(\alpha x)^{\beta}}\right]^{a-1}$
$\left\{a(1+\lambda) - \lambda(a+b)\left[1 - \mathrm{e}^{-(\alpha x)^{\beta}}\right]^{b}\right\}$.

The parameters of the above pdfs are all positive real numbers except for the TEMW, TWL and GTW distributions for which $|\lambda| \le 1$.

6.2 Relief times of twenty patients

The second data set (Gross and Clark [17], page 105) on the relief times of twenty patients receiving an analgesic is: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2. For these data, we compare the fits of the BT-Li distribution with the B-Li, T-Li, G-Li, transmuted complementary Weibull geometric (TCWG) (Afify et al. [3]), McDonald log-logistic (McLL) (Tahir et al. [35]), beta Weibull (BW) (Lee et al. [21]), exponentiated transmuted generalized Rayleigh (ETGR) (Afify et al. [4]), Kumaraswamy-transmuted exponentiated modified Weibull (KwTEMW) (Al-Babtain et al. [7]) and the new modified Weibull (NMW) (Almalki and Yuan [8]) distributions. The pdfs of these distributions are (for $x > 0$):

TCWG: $f(x) = \alpha\beta\gamma(\gamma y)^{\beta-1}\ \mathrm{e}^{-(\gamma y)^{\beta}}$
$\left[\alpha(1-\lambda) - (\alpha - \alpha\lambda - \lambda - 1)\ \mathrm{e}^{-(\gamma y)^{\beta}}\right]$
$\left[\alpha + (1-\alpha)\ \mathrm{e}^{-(\gamma y)^{\beta}}\right]^{-3}$;

McLL: $f(x) = \frac{\alpha c}{B(a/c,b)\beta}\left(\frac{x}{\beta}\right)^{a\alpha-1} \left[1 + \left(\frac{x}{\beta}\right)^{\alpha}\right]^{-a-1}$
$\left(1 - \left\{1 - \left[1 + \left(\frac{x}{\beta}\right)^{\alpha}\right]^{-1}\right\}^{c}\right)$;

BW: $f(x) = \frac{\beta\alpha^{\beta}}{B(a,b)} x^{\beta-1} \mathrm{e}^{-b(\alpha x)^{\beta}}$
$\left[1 - \mathrm{e}^{-(\alpha x)^{\beta}}\right]^{a-1}$;

ETGR: $f(x) = 2\alpha\delta\beta^{2}\ x\ \mathrm{e}^{-(\beta x)^{2}} \left[1 - \mathrm{e}^{-(\beta x)^{2}}\right]^{\alpha\delta-1}$
$\left\{1 + \lambda - 2\lambda\left[1 - \mathrm{e}^{-(\beta x)^{2}}\right]^{\alpha}\right\}\left\{1 + \lambda - \lambda\left[1 - \mathrm{e}^{-(\beta x)^{2}}\right]^{\alpha}\right\}^{\delta-1}$;

KwTEMW: $f(x) = ab\delta \mathrm{e}^{-\left(\alpha x + \gamma x^{\beta}\right)} \left(\alpha + \gamma\beta x^{\beta-1}\right)$
$\left[1 - \mathrm{e}^{-\left(\alpha x + \gamma x^{\beta}\right)}\right]^{a\delta-1}$

Table 3. Goodness-of-fit statistics for waiting times data

Model	$-2\widehat{\ell}$	AIC	$CAIC$	$HQIC$	BIC	W^*	A^*
BT-Go	634.007	644.007	644.645	649.279	657.033	0.01824	0.12988
McW	634.042	644.042	644.68	649.314	657.068	0.01835	0.13181
BBXII	634.63	644.63	645.268	649.902	657.656	0.01902	0.13501
TWL	634.725	644.725	645.364	649.997	657.751	0.01936	0.13445
TEMW	635.289	645.289	645.927	650.561	658.315	0.03366	0.216
MBW	637.248	647.248	647.886	652.519	660.273	0.04528	0.34588
GTW	641.837	651.837	652.475	657.109	664.863	0.0916	0.66481
G-Go	647.928	653.928	654.178	657.091	661.744	0.15536	1.18407
T-Go	648.335	654.335	654.585	657.499	662.151	0.1545	1.18453

Table 4. MLEs and their standard errors for waiting times data

Model	Estimates (Standard errors)				
G-Go(α,β,b)	0.0408	0.0636	1.1198		
	(0.012)	(0.192)	(3.377)		
T-Go(α,β,λ)	0.047	0.0603	0.1833		
	(0.007687)	(0.013)	(0.299)		
BT-Go$(\alpha,\beta,a,b,\lambda)$	0.0021	0.0892	2.3033	1.5698	0.3812
	(0.016)	(0.073)	(0.499)	(2.615)	(0.832)
McW(α,β,a,b,c)	1.9083	0.2639	16.5138	10.9511	24.6904
	(186.163)	(4.54)	(574.036)	(249.095)	(924.94)
BBXII$(\alpha,\beta,a,b,\delta)$	18.5011	1.4159	1.4773	2.4928	1.6005
	(14.266)	(0.766)	(1.271)	(12.649)	(7.612)
TWL$(\alpha,\beta,a,b,\lambda)$	0.2331	7.4315	14.1387	1.4984	-0.8317
	(0.258)	(11.9819)	(39.457)	(0.633)	(0.247)
TEMW$(\alpha,\beta,\gamma,\delta,\lambda)$	0.1695	0.6532	0.0004	1.5409	-0.6625
	(0.017)	(0.4)	(0.009455)	(0.431)	(0.375)
MBW(α,β,a,b,c)	1.8449	0.4477	20.6663	2.0835	2.0677
	(3.842)	(0.405)	(14.989)	(3.203)	(3.369)
GTW$(\alpha,\beta,a,b,\lambda)$	5.532	0.3669	33.8286	0.0118	0.0533
	(6.208)	(0.062)	(22.276)	(0.425)	(0.915)

$$\left\{1+\lambda-2\lambda\left[1-\mathrm{e}^{-\left(\alpha x+\gamma x^{\beta}\right)}\right]^{\delta}\right\}$$
$$\left\{1+\lambda-\lambda\left[1-\mathrm{e}^{-\left(\alpha x+\gamma x^{\beta}\right)}\right]^{\delta}\right\}^{a-1}$$
$$\left(1-\left[1-\mathrm{e}^{-\left(\alpha x+\gamma x^{\beta}\right)}\right]^{a\delta}\right.$$
$$\left.\left\{1+\lambda-\lambda\left[1-\mathrm{e}^{-\left(\alpha x+\gamma x^{\beta}\right)}\right]^{\delta}\right\}^{a}\right)^{b-1};$$

NMW: $f(x)=\left[\alpha\theta x^{\theta-1}+\gamma\left(\beta+\delta x\right)x^{\beta-1}\mathrm{e}^{\delta x}\right]$ $\mathrm{e}^{-\left(\alpha x^{\theta}+\gamma x^{\beta}\mathrm{e}^{\delta x}\right)}$.

The parameters of the above pdfs are all positive real numbers except for the TCWG, ETGR and KwTEMW distributions for which $|\lambda|\leq 1$.

In order to compare the fitted distributions, we consider some goodness-of-fit measures including the Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion ($HQIC$), consistent Akaike information criterion ($CAIC$), $-2\widehat{\ell}$, Anderson-Darling statistic (A^*) and Cramér-von Mises statistic (W^*), where $\widehat{\ell}$ denotes the maximized log-likelihood. Generally, the smaller these statistics are, the better the fit.

Tables 3 and 5 list the values of $-2\widehat{\ell}$, AIC, BIC, $HQIC$, $CAIC$, W^* and A^* whereas the MLEs of the model parameters and their corresponding standard errors are given in Tables 4 and 6.

Table 3 compares the fits of the BT-Go distribution with the T-Go, G-Go, McW, BBXII, TEMW, TWL, MBW and GTW distributions. The figures in these tables show that the BT-Go distribution has the lowest values for the $-2\widehat{\ell}$, AIC, BIC, $HQIC$, $CAIC$, W^* and A^* statistics (for waiting times data) among the fitted distributions. So, the BT-Go distribution could be chosen as the best model. Table 5 compares the fits of the BT-Li distribution with the B-Li, T-Li, G-Li, TCWG, McLL, BW, ETGR, KwTEMW and NMW distributions. The BT-Li distribution has the lowest values for goodness-of-fit statistics (for the relief times data) among all fitted distributions. So, the BT-Li distribution can be chosen as the best model. It is clear from Tables 3 and 5 that the BT-Go and BT-Li distributions provide the best fits. The histograms of the fitted distri-

Table 5. Goodness-of-fit statistics for the relief times data

Model	$-2\widehat{\ell}$	AIC	$CAIC$	$HQIC$	BIC	W^*	A^*
BT-Li	32.849	40.849	43.515	41.626	44.832	0.06125	0.35992
TCWG	33.607	41.607	44.274	42.385	45.590	0.07252	0.43603
McLL	33.854	43.854	48.833	44.826	48.14	0.07904	0.46199
BW	34.396	42.396	45.063	43.174	46.379	0.0873	0.51316
B-Li	36.977	42.977	44.477	43.56	45.964	0.13201	0.77319
ETGR	36.856	44.856	47.523	45.633	48.839	0.13629	0.79291
KwTEMW	37.203	51.203	60.537	52.564	58.173	0.13212	0.77604
G-Li	39.029	43.0289	43.735	43.418	45.0204	0.11962	1.12444
NMW	41.173	51.173	55.459	52.145	56.151	0.17585	1.0678
T-Li	61.729	65.729	66.435	66.118	67.721	0.53509	10.74948

Table 6. MLEs and their standard errors for the relief times data

Model	Estimates (standard errors)				
T-Li(α, λ)	0.6653	0.3587			
	(0.048)	(0.332)			
G-Li(α, b)	1.5895	5.4124			
	(0.251)	(2.121)			
B-Li(α, a, b)	0.2427	5.6308	35.4058		
	(0.012)	(0.002792)	(0.133)		
BT-Li(α, λ, a, b)	2.175	0.4893	30.8824	1.057	
	(0.297)	(0.338)	(32.65)	(0.5568)	
TCWG($\alpha, \beta, \gamma, \lambda$)	43.6627	5.1271	0.2823	-0.2713	
	(45.459)	(0.814)	(0.042)	(0.656)	
BW(α, β, a, b)	0.8314	0.6126	29.9468	11.6319	
	(0.954)	(0.34)	(40.413)	(21.9)	
ETGR($\alpha, \beta, \lambda, \delta$)	0.1033	0.6917	-0.342	23.5392	
	(0.436)	(0.086)	(1.971)	(105.137)	
McLL(α, β, a, b, c)	0.8811	2.0703	19.2254	32.0332	1.9263
	(0.109)	(3.693)	(22.341)	(43.077)	(5.165)
NMW($\alpha, \beta, \gamma, \delta, \theta$)	0.1215	2.7837	$8.2272 \cdot 10^{-5}$	0.0003	2.7871
	(0.056)	(20.37)	($1.512 \cdot 10^{-3}$)	(0.025)	(0.428)
KwTEMW	0.462	$9.157 \cdot 10^{-3}$	$1.248 \cdot 10^{-7}$	2.247	0.33
($\alpha, \beta, \gamma, \delta, \lambda, a, b$)	(0.197)	(0.379)	($6.059 \cdot 10^{-5}$)	(2.398)	(0.419)
	2.508	8.929			
	(2.877)	(8.447)			

butions for the BT-Go and BT-Li models are displayed in Figure 10. The plots support the results obtained from Tables 3 and 5. Figures 11 and 12 display the fitted cdf's and the QQ plots for both models. It is evident from these plots that the two models provide close fit to the two data sets.

If a and b are integers then the distribution given by (3)–(4) is the distribution of the ath order statistic for a random sample of size $a + b - 1$ from the transmuted H distribution. So, the fitted BT-Go and BT-Li distributions can be interpreted approximately as follows:

- Suppose there are $a + b - 1 \approx 3$ customers in a bank with each having waiting time distributed according to a transmuted Gompertz distribution with mean 8.9 and variance 89.5. Then the fitted BT-Go distribution is the distribution of the second longest waiting time.
- Suppose there are $a + b - 1 \approx 32$ patients receiving an analgesic with each having relief time distributed according to a transmuted Lindley distribution with mean 0.5 and variance 0.2. Then the fitted BT-Li distribution is the distribution of the second longest relief time.

The moments of order statistics given by (13) can be used to give future predictions. Using the estimates of the fitted BT-Go distribution, we obtain the longest waiting time when the bank has $200, 300, \ldots, 1000$ customers as 57.6, 159.3, 205.2, 240.6, 433.1, 501.8, 541.7, 668.7, 940.2, these were obtained by setting $n = i = 200, 300, \ldots, 1000$ and $q = 1$ in (13). Using the estimates of the fitted BT-Li distribution, we obtain the longest relief time when the clinic has $30, 40, \ldots, 100$ patients as 4.7, 5.2, 5.9, 6.7, 7.3, 9.6, 10.2,

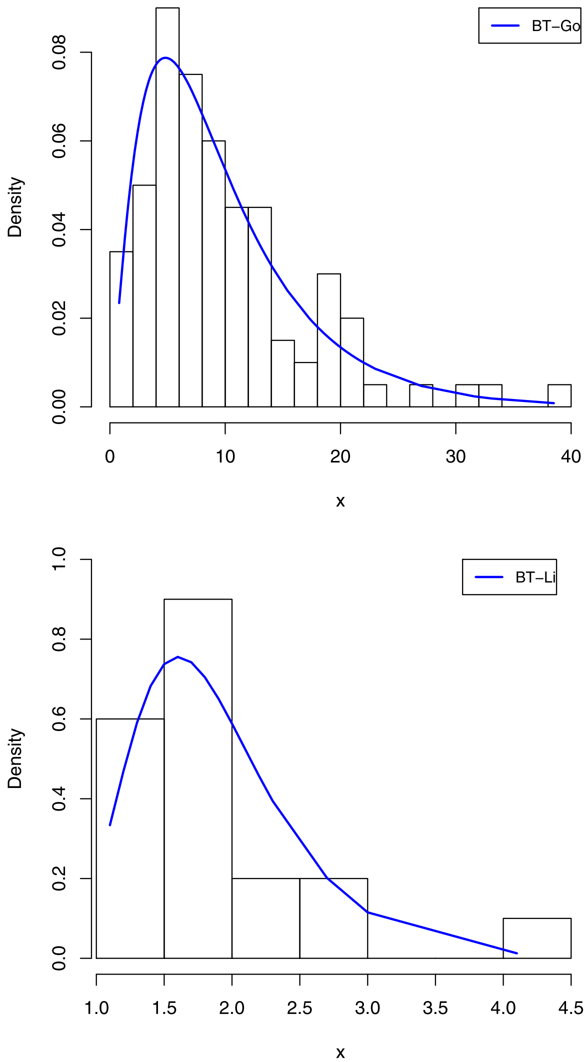

Figure 10. Fitted pdfs of the BT-Go and BT-Li distributions.

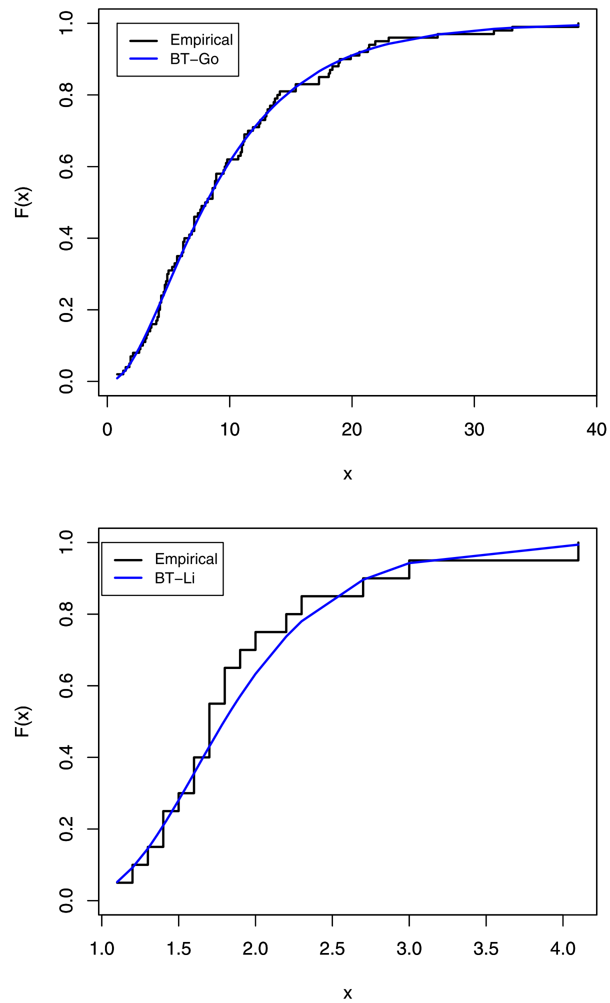

Figure 11. Fitted cdfs of the BT-Go and BT-Li distributions.

11.2, these were obtained by setting $n = i = 30, 40, \ldots, 100$ and $q = 1$ in (13).

The strengths of the proposed distributions evident from the two data applications are: their ability to provide better fits (to the waiting time data) than six other distributions each having the same number of parameters; their ability to provide better fits (to the relief times data) than five other distributions each having the same number of parameters; their ability to provide better fits (to the relief times data) than a distribution having one more parameter.

7. CONCLUSIONS

There is great interest among statisticians and practitioners in the past decade to generate new generalized families from classic ones. We have presented a new *beta transmuted*-H (BT-H) family of distributions, which extends the transmuted-H family by adding two extra shape parameters. Many well-known distributions emerge as special cases of the BT-H family. The mathematical properties of the new family including explicit expansions for the ordinary moments, quantiles, generating functions, and order statistics have been provided. The model parameters have been estimated by the maximum likelihood estimation method and the observed information matrix has been determined. It has been shown, by means of two real data sets, that special cases of the BT-H family can provide better fits than at least five other families each having the same number of parameters.

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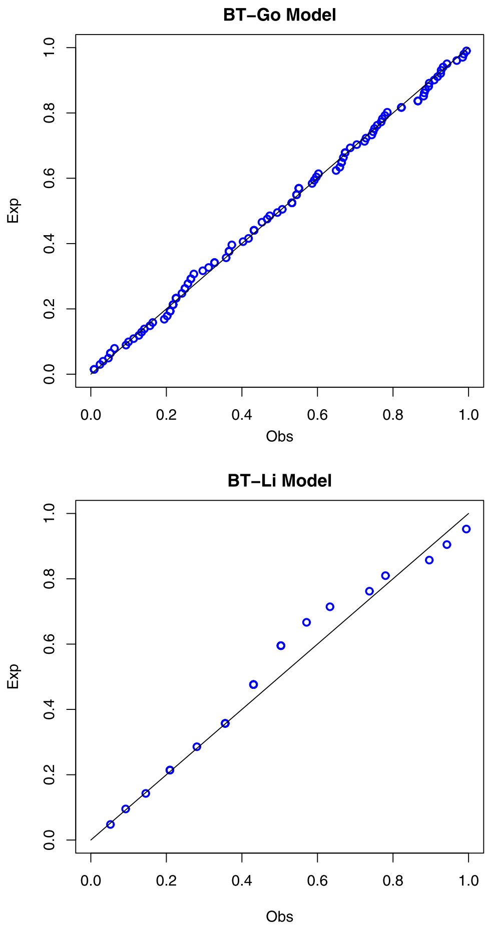

Figure 12. QQ plots of the BT-Go and BT-Li distributions.

REFERENCES

[1] Abdul-Moniem, I. B. and Seham, M. (2015). Transmuted Gompertz distribution. *Computational and Applied Mathematics* **1** 88–96.

[2] Afify, A. Z., Cordeiro, G. M., Yousof, H. M., Alzaatreh, A. and Nofal, Z. M. (2016). The Kumaraswamy transmuted-G family of distributions: Properties and applications. *Journal of Data Science* **14** 245–270.

[3] Afify, A. Z., Nofal, Z. M. and Butt, N. S. (2014). Transmuted complementary Weibull geometric distribution. *Pakistan Journal of Statistics and Operations Research* **10** 435–454. MR3302433

[4] Afify, A. Z., Nofal, Z. M. and Ebraheim, A. N. (2015). Exponentiated transmuted generalized Rayleigh distribution: A new four parameter Rayleigh distribution. *Pakistan Journal of Statistics and Operations Research* **11** 115–134.

[5] Afify, A. Z., Nofal, Z. M., Yousof, H. M., El Gebaly, Y. M. and Butt, N. S. (2015). The transmuted Weibull Lomax distribution: Properties and application. *Pakistan Journal of Statistics and Operations Research* **11** 135–152.

[6] Akinsete, A., Famoye, F. and Lee, C. (2008) The beta-Pareto distribution. *Statistics* **42** 547–563. MR2465134

[7] Al-Babtain, A., Fattah, A. A., Ahmed, A. N. and Merovci, F. (2015). The Kumaraswamy-transmuted exponentiated modified Weibull distribution. *Communications in Statistics-Simulation and Computation*, forthcoming.

[8] Almalki, S. J. and Yuan, J. (2013). A new modified Weibull distribution. *Reliability Engineering and System Safety* **111** 164–170.

[9] Aryal, G. R. and Tsokos, C. P. (2011). Transmuted Weibull distribution: A generalization of the Weibull probability distribution. *European Journal of Pure and Applied Mathematics* **4** 89–102. MR2802943

[10] Ashour, S. K. and Eltehiwy, M. A. (2013). Transmuted exponentiated modified Weibull distribution. *International Journal of Basic and Applied Sciences* **2** 258–269.

[11] Cordeiro, G. M., Hashimoto, E. M. and Ortega, E. (2014). The McDonald Weibull model. *Statistics* **48** 256–278. MR3175769

[12] Corless, R. M., Gonnet, G. H., Hare, D. E. G., Jeffrey, D. J. and Knuth, D. E. (1996). On the Lambert W function. *Advances in Computational Mathematics* **5** 329–359. MR1414285

[13] El-Gohary, A., Alshamrani, A. Al-Otaibi, A. N. (2013). The generalized Gompertz distribution. *Applied Mathematical Modelling* **37** 13–24. MR2994162

[14] Eugene, N., Lee, C. and Famoye, F. (2002). Beta-normal distribution and its applications. *Communications in Statistics-Theory Methods* **31** 497–512. MR1902307

[15] Ghitany, M. E., Atieh B. and Nadarajah, S. (2008). Lindley distribution and its application. *Mathematics and Computers in Simulation* **78** 493–506. MR2424558

[16] Gradshteyn, I. S. and Ryzhik, I. M. (2007). *Table of Integrals, Series, and Products*, seventh edition. Academic Press, New York. MR2360010

[17] Gross, A. J. and Clark, V. A. (1975). *Survival Distributions: Reliability Applications in the Biomedical Sciences.* John Wiley and Sons, New York.

[18] Gupta, R. C., Gupta, P. L. and Gupta, R. D. (1998). Modeling failure time data by Lehman alternatives. *Communications in Statistics-Theory and Methods* **27** 887–904. MR1613497

[19] Jafari, A. A., Tahmasebi, S. and Alizadeh, M. (2014). The beta-Gompertz distribution. *Revista Colombiana de Estadistica* **37** 141–158. MR3242217

[20] Khan, M. N. (2015). The modified beta Weibull distribution with applications. *Hacettepe Journal of Mathematics and Statistics* **44** 1553–1568. MR3495010

[21] Lee, C., Famoye, F. and Olumolade, O. (2007). Beta-Weibull distribution: Some properties and applications to censored data. *Journal of Modern Applied Statistical Methods* **6** 173–186.

[22] Mahmoud, M. R. and Mandouh, R. M. (2013). On the transmuted Fréchet distribution. *Journal of Applied Sciences Research* **9** 5553–5561.

[23] Merovci, F. (2013). Transmuted Rayleigh distribution. *Austrian Journal of Statistics* **42** 21–31.

[24] Merovci, F. (2013). Transmuted Lindley distribution. *International Journal of Open Problems in Computer Science and Mathematics* **6** 63–72.

[25] Merovci, F. and Puka, L. (2014). Transmuted Pareto distribution. *ProbStat Forum* **7** 1–11. MR3203046

[26] Merovci, F. and Sharma, V. K. (2014). The beta Lindley distribution: Properties and Applications. *Journal of Applied Mathematics* **51** 1–10. MR3253614

[27] Mudholkar, G. S. and Srivastava, D. K. (1993). Exponentiated Weibull family for analyzing bathtub failure rate data. *IEEE Transactions on Reliability* **42** 299–302.

[28] Nadarajah, S. (2005). Exponentiated Pareto distributions. *Statistics* **39** 255–260. MR2153940

[29] Nadarajah, S. and Gupta, A. K. (2004). The beta Fréchet distribution. *Far East Journal of Theoretical Statistics* **14** 15–24. MR2108090

[30] Nadarajah, S. and Kotz, S. (2003). The exponentiated Fréchet distribution. *Interstat Electron* 1–7.

[31] Nofal, Z. M., Afify, A. Z., Yousof, H. M. and Cordeiro, G. M. (2015). The generalized transmuted-G family of distributions. *Communications in Statistics-Theory and Methods*, forthcoming.

[32] Pal, M. and Tiensuwan. M. (2014). The beta transmuted Weibull distribution. *Austrian Journal of Statistics* **43** 133–149.

[33] Paranaba, P., Ortega, E., Cordeiro, G. M. and Pescim, R. (2011). The beta Burr XII distribution with application to lifetime

data. *Computational Statistics and Data Analysis* **55** 1118–1136. MR2736499
[34] Shaw, W. T. and Buckley, I. R. C. (2007). The alchemy of probability distributions: Beyond Gram-Charlier expansions and a skew-kurtotic-normal distribution from a rank transmutation map. arXiv preprint arXiv:0901.0434.
[35] Tahir, M. H., Mansoor, M., Zubair, M. and Hamedani, G. (2014). McDonald log-logistic distribution with an application to breast cancer data. *Journal of Statistical Theory and Applications* **13** 65–82. MR3197531
[36] Yousof, H. M., Afify, A. Z., Alizadeh, M., Butt, N. S., Hamedani, G. G. and Ali, M. M. (2015). The transmuted exponentiated generalized-G family of distributions. *Pakistan Journal of Statistics and Operations Research* **11** 441–464. MR3447541
[37] Zografos, K. and Balakrishnan, N. (2009). On families of beta and generalized gamma-generated distributions and associated inference. *Statistical Methodology* **6** 344–362. MR2751078

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