Statistics and Its Interface Volume 10 (2017) 451–470

# Bayesian forecasting of Value-at-Risk based on variant smooth transition heteroskedastic models

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To allow for a higher degree of flexibility in model parameters, we propose a general and time-varying nonlinear smooth transition (ST) heteroskedastic model with a second-order logistic function of varying speed in the mean and variance. This paper evaluates the performance of Value-at-Risk (VaR) measures in a class of risk models, specifically focusing on three distinct ST functions with GARCH structures: first- and second-order logistic functions, and the exponential function. The likelihood function is non-differentiable in terms of the threshold values and delay parameter. We employ Bayesian Markov chain Monte Carlo sampling methods to update the estimates and quantile forecasts. The proposed methods are illustrated using simulated data and an empirical study. We estimate VaR forecasts for the proposed models alongside some competing asymmetric models with skew and fat-tailed error probability distributions, including realized volatility models. To evaluate the accuracy of VaR estimates, we implement two loss functions and three backtests. The results show that at the 1% level the ST model with a second-order logistic function and skew Student's t error is a worthy choice, when compared to a range of existing alternatives.

Keywords and phrases: Asymmetric Laplace, Nonlinear time series model, Second-order logistic function, Markov chain Monte Carlo methods, Value-at-Risk, Volatility forecasting, Realized volatility models.

## 1. INTRODUCTION

Financial risk management is widely used in financial institutions in order to control risk exposures, such as credit risk, operation risk, and volatility. The Basel Capital Accord of 1996 played a significant role as it permitted banks to use their "appropriate model" to compute their regulatory capital requirements. Since then, Value-at-risk (VaR) has been one of the extensively used measures of market risk, which is designed to forecast the worst expected loss over a given time interval under normal market conditions, at a given confidence level $\alpha$ (Jorion 1997). The Basel II Accord, initially published in June 2004, is intended to create international standards for banking regulators to better control risk exposures. In theory, Basel II set up risk and capital management requirements designed to ensure that Authorized Deposit-taking Institutions (ADIs) have the ability to maintain solvency.

There are many VaR estimation methods in the literature that can be classified into three categories. They include non-parametric methods, for example, historical simulation (HS) (using past or in-sample quantiles); semi-parametric methods, for example, the extreme value theory, the dynamic quantile regression CAViaR model (Engle & Manganelli, 2004), and the threshold CAViaR model (Gerlach, Chen, & Chan, 2011, Chen et al., 2012); and parametric statistical approaches that fully specify model dynamics and distribution assumptions, for example, the autoregressive conditional heteroskedasticity (ARCH) model proposed by Engle (1982) and its generalized version by Bollerslev (1986), popularly known as the GARCH model. It is well known that the GARCH model cannot capture the asymmetric response of volatility, which is a phenomenon discovered by Black (1976), Rabemananjara and Zakoïan (1993) and Zakoïan (1994), among others.

Chan and Tong (1986) introduce a smooth transition (ST) autoregressive model to allow for model parameter changes through a smooth transition, which has gained popularity via Granger and Teräsvirta (1993) and Teräsvirta (1994). Their first-order logistic function gives a continuous value between zero and one. Jansen and Teräsvirta (1996) appear to be the first to discuss the second-order logistic function in ST models. Van Dijk, Teräsvirta, and Franses (2002) investigate the second-order logistic function with a slight difference in format from that of Jansen and Teräsvirta (1996), but these two papers only focus on a transition in describing the mean equation. Most financial time series exhibit asymmetric behavior in the mean and in the volatility as well. While the smooth transition GARCH models have extensively examined conditional volatility, their use in modeling conditional variance is limited to a two-regime study undertaken by Anderson, Nam, and Vahid (1999).

In this paper we propose to implement a specification of the second-order logistic function in van Dijk, Teräsvirta, and Franses (2002), a more general and time-varying ST-GARCH model, to allow for an ST function with varying speed in the mean and variance. The literature has described the second-order logistic function, but it is not often used

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in practice due to the difficulty in parameter estimation. The problem of estimating second-order ST-GARCH models has become quite a challenge. Specifically, the likelihood function is non-differentiable in regards to the threshold values and delay parameter. Our paper examines whether such double asymmetry might be better modeled by an ST function in both the mean and volatility equations for VaR forecasting and volatility estimation. Based on Markov chain Monte Carlo (MCMC) methods, we employ a Bayesian approach and allow a simultaneous inference for all unknown parameters, while the parameter constraints simply and properly form part of the prior distribution, and the problems with estimating threshold limits and delay lags disappear. To our knowledge, this study is the first in the literature to make Bayesian inferences and quantile forecasting for the ST-GARCH model with a second-order logistic function.

The use of our proposed Bayesian forecasting of nonlinear ST models to deal with some complex derivatives and to calculate their corresponding VaR formulae is of practical importance and theoretical interests. Bayesian MCMC methods offer many advantages in estimation, inference, and forecasting, including: (i) accounting for parameter uncertainty in both probabilistic and point forecasting; (ii) allowing simultaneous inference for all unknown parameters; (iii) efficient and flexible handling of complex models and nonstandard parameters; and (iv) parameter constraints simply and properly form part of the prior distribution.

We employ three distinct ST functions with autoregressive conditional heteroskedastic models for VaR forecasting purposes: the first- and second-order logistic functions, and the exponential functions. Gerlach and Chen (2008) incorporate the first-order ST functions into GARCH models to allow for smooth nonlinearity in the mean and asymmetry of the volatility. Chen et al. (2010) employ an exponential function to capture size asymmetry in the mean and volatility. Compared to existing models, our proposed model conveys that observations in the extremes can have a dissimilar effect and an ST function with varying speed in the mean and variance. Moreover, the ST-GARCH model, with the second-order logistic function, can be viewed as three regimes interpreted as follows: the first regime relates to extremely low negative shocks ("bad news"), the middle regime represents low absolute returns ("tranquil periods"), and finally, the third regime relates to high positive shocks ("good news").

As discussed in Andersen et al. (2001), the theory of quadratic variation indicates that, under suitable conditions, realized volatility is an unbiased and highly efficient estimator of return volatility. We also deal with the inclusion of realized measures of volatility in a GARCH modelling set-up. The realized GARCH (RV) model of Hansen, Huang and Shek (2012) provides an excellent framework for the joint modelling of returns and realized measures of volatility.

This paper focuses on parametric models and Monte Carlo simulation to forecast VaR. We consider popular variants and extensions of the GARCH model family as follows: RiskMetrics; GARCH; asymmetric GJR-GARCH (Glosten, Jaganathan, Runkle 1993); ST-GARCH with three distinct ST functions; and threshold nonlinear GARCH (TGARCH; Chen and So 2006). Each model includes a specification for the volatility dynamics, and most consider three specifications for the conditional asset return distribution: Gaussian, Student's t, and the skew Student's t of Hansen (1994). This paper extensively examines the VaR forecast performance over 12 risk models and two HS methods during two out-of-sample periods: the two-year post-global financial crisis period and the three-year global financial crisis period. To shed light on the advantage of Bayesian updating forecasting, this study examines a sample of ten European stock markets, seven Asian stock markets, one North American market, and one South American market, for a total of 19 stock markets over the post-global financial crisis period. We focus on Japan and U.S. stock markets for the three-year global financial crisis period, including the RV model.

When forecasting VaR thresholds, our aim is to find the optimal combination of volatility dynamics and error distribution in relation to the observed violation rates and two loss functions in Lopez (1999 a, b) for both out-of-sample periods. We further consider three backtesting methods for evaluating and testing the accuracy of VaR models. We also investigate the accuracy of volatility forecasts for all models under two volatility proxies with two loss functions.

This paper is organized as follows. Section 2 illustrates the ST models with different ST functions. Section 3 demonstrates the Bayesian set-up and details of parameter inferences. Section 4 describes the process of VaR forecasting. Section 5 presents a simulation study of a double ST-GARCH model with the second-order logistic function showing the estimation performance. We further extend this class of ST-GARCH models to incorporate a different effect (smooth transition function) for the mean and variance. Section 6 lists empirical results, focusing on the forecasts of VaR and volatility and furthermore showing the forecast accuracy for all models under two volatility proxies. Section 7 provides concluding remarks.

## 2. THE SMOOTH TRANSITION HETEROSKEDASTIC MODEL

We consider a general double ST GARCH model to capture mean and volatility asymmetry in financial markets. We use a ST function that ensures the mean and volatility parameters are smooth functions of past news or volatility. We then extend our model to different ST functions with varying speed in the mean and volatility in Section 3.

Suppose that $\{y_t\}$ is the observation data. We present the ST-GARCH model as below:

$$
\begin{aligned}
(1) \quad y_t &= \mu_t^{(1)} + F(z_{t-d};\gamma,c)\mu_t^{(2)} + a_t \\
a_t &= \sqrt{h_t}\epsilon_t, \qquad \epsilon_t \overset{\text{i.i.d.}}{\sim} D(0,1), \\
h_t &= h_t^{(1)} + F(z_{t-d};\gamma,c)h_t^{(2)} \\
\mu_t^{(l)} &= \phi_0^{(l)} + \sum_{i=1}^{p}\phi_i^{(l)} y_{t-i} \\
h_t^{(l)} &= \alpha_0^{(l)} + \sum_{i=1}^{g}\alpha_i^{(l)} a_{t-i}^2 + \sum_{i=1}^{q}\beta_i^{(l)} h_{t-i}, \quad l=1,2,
\end{aligned}
$$

where $\mu_t^{(l)}$ and $h_t^{(l)}$ are the respective conditional mean and volatility at regime $l$; $z_t$ is the threshold variable; $d$ is the delay lag; and $D(0,1)$ is an error distribution with mean 0 and variance 1. The representation of the proposed ST GARCH model in (1) highlights the model's basic characteristic, which is that at any given point in time $t$, $y_t$ is determined as a weighted average of two AR models and $h_t$ is determined as a weighted average of two GARCH models, where the weights assigned to the two models depend on the value taken by the transition function $F(z_{t-d};\gamma,c)$. The parameter $\gamma$ determines the smoothness of the change in the value of the $F(z_{t-d};\gamma,c)$ function and the smoothness of the transition from one regime to the other.

We consider three types of ST functions in this work. Different choices for the transition function lead to different types of regime-switching behaviour. A popular choice for $F(z_{t-d};\gamma,c)$ is the first-order logistic function:

$$
(2) \qquad F(z_{t-d};\gamma,c) = \frac{1}{1+\exp\left\{\frac{-\gamma(z_{t-d}-c)}{s_z}\right\}},
$$

where $s_z$ is the sample standard deviation of $z_t$. This type of regime-switching can be convenient for modelling, for example, asymmetry in stock markets, to distinguish bad news and good news. The first-order logistic function is an odd function and captures sign asymmetry; in other words, the asymmetric responses to positive and negative values of $z_{t-d}-c$. Teräsvirta and Anderson (1992) and Teräsvirta (1994) apply the STAR model with a first-order logistic ST function to financial data, finding evidence of sign asymmetry in the mean.

We next consider a specification of the second-order logistic function in van Dijk, Teräsvirta, and Franses (2002).

$$
(3) \quad F(z_{t-d};\gamma,\boldsymbol{c}) = \frac{1}{1+\exp\left\{\frac{-\gamma(z_{t-d}-c_1)(z_{t-d}-c_2)}{s_z}\right\}}, \; c_1 < c_2,
$$

where now $\boldsymbol{c} = (c_1;c_2)'$, as proposed by Jansen and Teräsvirta (1996). Figure 1 shows some examples for the second-order ST function for various values of the smoothness parameter $\gamma$ when $c_1 = -1.5$ and $c_2 = 1.5$. We observe that smaller values of $\gamma$ cause smoother, slower transitions, while $\gamma \geq 10$ is effectively a sharp or abrupt transition. When $\gamma = 20$, the transition function starts at 1, then decreases to zero during the range of $(c_1, c_2)$, and then increases back to one again.

When $\gamma \to 0$, both logistic functions are equal to a constant (equal to 0.5) and the ST-GARCH model reduces to a linear GARCH model. If we impose the restriction that $\gamma$ in the transition function is infinitely large and $c_1 \neq c_2$, then our model becomes the three-regime threshold GARCH model of Chen, Gerlach, and Lin (2010). Hence, the ST-GARCH model with the particular transition function nests a restricted three-regime GARCH model, with a restriction for the outer regimes to be identical. For the third-type function, we consider the exponential function:

$$
(4) \qquad F(z_{t-d};\gamma,c) = 1 - \exp\left\{\frac{-\gamma(z_{t-d}-c)^2}{s_z}\right\}.
$$

The behavior of $y_t$ depends on the size of the deviation from $z_t$. The exponential ST (EST) is an even function, which captures size asymmetry, or asymmetric responses to the magnitude of $z_{t-d}-c$. See Granger and Teräsvirta (1993) and Teräsvirta (1994) for applications of STAR with the EST model. Chen et al. (2010) apply ST-GARCH with the EST function to daily stock markets, finding evidence of size asymmetry in mean and volatility, while the favorite transition variable is the intra-day range. A limitation of the exponential function (4) is that for either $\gamma \to 0$ or $\gamma \to \infty$, the function collapses to a constant (equal to 0 and 1, respectively). Hence, the model becomes linear in both cases and the exponential STAR model does not nest a self-exciting TAR model as a special case (see van Dijk, Teräsvirta, and Franses 2002 for details).

For the ST-GARCH(1,1) model in (1), we impose the restrictions that three coefficients $\alpha_0^{(1)}$, $\alpha_1^{(1)}$, and $\beta_1^{(1)}$ are positive, and also that $\alpha_0^{(1)} + \alpha_0^{(2)} > 0$, $\alpha_1^{(1)} + \alpha_1^{(2)} > 0$, and $\beta_1^{(1)} + \beta_1^{(2)} > 0$, to ensure that the conditional variance is always positive. For extensions to incorporate longer lag-lengths, the non-negativeness of the conditional variance is:

$$
(5) \qquad \begin{aligned}
&\alpha_0^{(1)} > 0, \alpha_i^{(1)} > 0, \beta_i^{(1)} > 0 \\
&\sum_{i=1}^{g}\left(\alpha_i^{(1)} + \alpha_i^{(2)}\right) > 0 \;\; \sum_{j=1}^{q}\left(\beta_j^{(1)} + \beta_j^{(2)}\right) > 0.
\end{aligned}
$$

The ST-GARCH(1,1) model in (1) will be covariance-stationary if $0 < \alpha_1^{(1)} + 0.5\alpha_1^{(2)} + \beta_1^{(1)} + 0.5\beta_1^{(2)} < 1$, but this condition is sufficient, rather than necessary, for stationarity (Anderson, Nam, and Vahid 1999). For extensions to incorporate longer lag-lengths, the covariance-stationary restriction is:

$$
(6) \quad \sum_{i=1}^{g}\left(\alpha_i^{(1)} + 0.5\alpha_i^{(2)}\right) + \sum_{j=1}^{q}\left(\beta_j^{(1)} + 0.5\beta_j^{(2)}\right) < 1.
$$

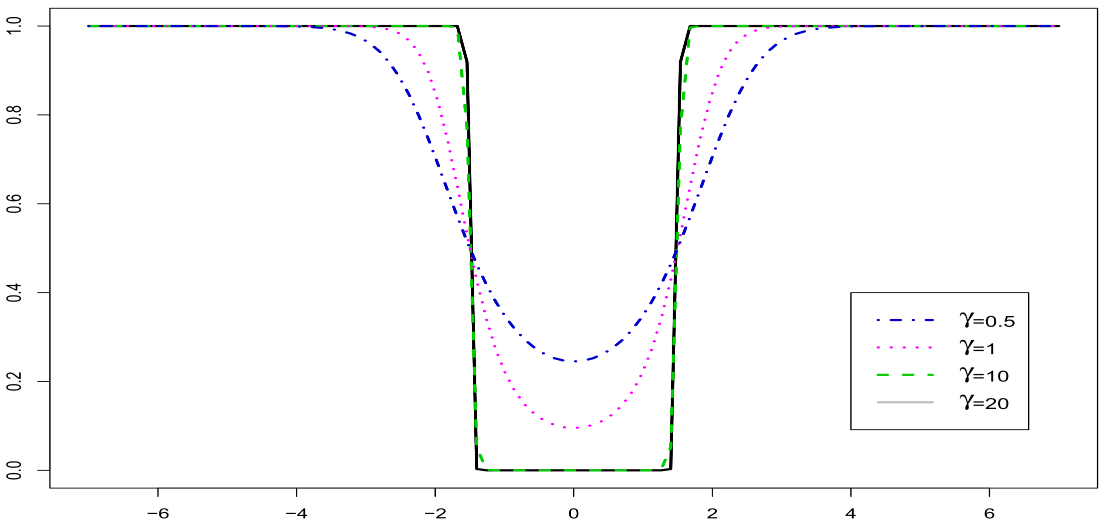


*Figure 1. Plots of the second-order ST function for $c_1$=-1.5 and $c_2$=1.5.*

It is useful to note that the parameter space still can accommodate some explosive lower regime (Gerlach and Chen 2008) with restrictions as follows:

$$(7)\quad \alpha_0^{(1)} < b_1, \beta_i^{(1)} < b_2,\ \sum_{i=1}^{g}\alpha_i^{(1)} + \sum_{j=1}^{q}\beta_j^{(1)} < b_3,$$

where $b_1$, $b_2$, and $b_3$ are user-specified. In this study, we let $b_2$, $b_3 \geq 1$ to allow explosive behavior.

## 3. BAYESIAN INFERENCE

In this section we use the Bayesian approach to carry out our parameter estimations. As observations in the extremes may have different effects, we further extend this class of models in (1) to incorporate different speeds of smooth transition functions for the mean and variance, i.e. $F_i(z_{t-d};\gamma_i,\boldsymbol{c})$ in (3), $i = 1, 2$ where $\gamma_1$ and $\gamma_2$ are the smooth parameters in the mean and variance, respectively. Based on the empirical evidence, the empirical density function has a higher peak and longer tails than the normal density. This phenomenon is common for daily stock returns. We consider $D(0, 1) = t^*_\nu$ to be a standardized Student's t distribution with $\nu$ degrees of freedom, which captures the conditional leptokurtosis observed in financial return data. Let $\boldsymbol{\theta} = (\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{c}, \boldsymbol{\gamma}, \nu, d)'$, where $\boldsymbol{\phi}_j = (\phi_0^{(j)}, \ldots, \phi_p^{(j)})'$, $\boldsymbol{\alpha}_j = (\alpha_0^{(j)}, \ldots, \alpha_g^{(j)}, \beta_1^{(j)}, \ldots, \beta_q^{(j)})'$. Note that $\boldsymbol{c} = (c_1, c_2)'$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)'$ if the ST function belongs to the second-order ST-GARCH models, otherwise, $\boldsymbol{c} = c$ and $\gamma_1 = \gamma_2$. We allow for a higher degree of flexibility in model parameters for this model. The notation $\boldsymbol{y}^{1,n}$ denotes $(y_1, \ldots, y_n)$.

The conditional likelihood function for the double ST model is thus:

$$p(\boldsymbol{y}^{s+1,n} \mid \boldsymbol{\theta}) = \prod_{t=s+1}^{n}\left\{\frac{1}{\sqrt{h_t}}p_\epsilon\left(\frac{y_t-\mu_t}{\sqrt{h_t}}\right)\right\},$$

where $p_\epsilon$ is the density function for $\epsilon_t$, the term $1/\sqrt{h_t}$ is the Jacobin of the transformation, $n$ is the sample size, $s = \max\{p, g, q, d_0\}$ with maximum delay $d_0$, $h_t = \mathrm{Var}(y_t|\mathcal{F}_{t-1})$, and $\mu_t = E(y_t|\mathcal{F}_{t-1})$, with $\mathcal{F}_{t-1}$ being the information set. The proposed model contains the lagged AR(1) (i.e. $p = 1$) effect in each regime, which allows one to recognize whether the return series exhibits asymmetry mean reversion or market efficiency. Generally, a GARCH model with $g = q = 1$ is sufficient to capture volatility clustering in most financial applications (cf. Bollerslev, Chou, and Kroner 1992). In summary, we assume $p = g = q = 1$. One can use the deviance information criterion (DIC) of Spiegelhalter et al. (2002) to determine the best lag for the proposed model.

The conditional likelihood function becomes:

$$p(\boldsymbol{y}^{s+1,n} \mid \boldsymbol{\theta}) = \prod_{t=s+1}^{n}\left\{\zeta\frac{1}{\sqrt{h_t}}\left[1+\frac{(y_t-\mu_t)^2}{(\nu-2)h_t}\right]^{-\frac{\nu+1}{2}}\right\},$$

$$\text{where } \zeta = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{(\nu-2)\pi}}.$$

Aside from fat tails, empirical distributions of asset returns may also be skewed. To handle this additional characteristic of asset returns, the Student's t distribution has been modified to become a skew Student's t distribution. There are many versions of skew Student's t distribution, but we adopt the approach of Hansen (1994), which has zero mean and unit variance. The probability density function of skew Student's t defined by Hansen (1994) is as follows:

$$p_\epsilon(\epsilon_t|\nu,\eta) = \begin{cases} bc\left[1+\frac{1}{\nu-2}\left(\frac{b\epsilon_t+a}{1-\eta}\right)^2\right]^{-\frac{\nu+1}{2}} & \text{if } \epsilon_t < -\frac{a}{b} \\ bc\left[1+\frac{1}{\nu-2}\left(\frac{b\epsilon_t+a}{1+\eta}\right)^2\right]^{-\frac{\nu+1}{2}} & \text{if } \epsilon_t \geq -\frac{a}{b} \end{cases},$$

where degrees of freedom $\nu$ and skewness parameter $\eta$ satisfy $2 < \nu < \infty$ and $-1 < \eta < 1$, respectively. We define the constants $a$, $b$, and $c$ as:

$$a = 4\eta c\left(\frac{\nu-2}{\nu-1}\right), \quad b^2 = 1 + 3\eta^2 - a^2, \quad c = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi(\nu-2)}}.$$

This distribution already has zero mean and unit variance. We use the notation $\mathrm{St}(\nu, \eta)$. The standardized Student's t distribution is a special case of this skew Student's t distribution, when $\eta = 0$. Our parameter space thus becomes $\boldsymbol{\theta}^* = (\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{c}, \boldsymbol{\gamma}, \nu, \eta, d)'$.

## 3.1 Prior specification

Bayesian inference requires specifying *a prior* distribution for the unknown parameters, combined with the likelihood function. We assume the parameters $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{c}, \boldsymbol{\gamma}, \nu, d)$, are a priori independent. An estimation of the smoothing parameter and its identification as it tends to zero have proven a challenge for both classical and Bayesian approaches, because the likelihood function is not integrable for this parameter in an ST-GARCH model. To alleviate this identifiability problem as the speed of the transition parameter tends to zero, we adopt a specific prior formulation for the mean in Equation (1), based on George and McCulloch (1993) and extended by Gerlach and Chen (2008). Note that Gerlach and Chen (2008) only handle the ST-GARCH model with the first-order logistic function. We define the latent variable $\delta_j^{(i)}$, which determines the prior distribution of $\phi_j^{(i)}$, via a mixture of two normals:

$$\phi_j^{(i)}|\delta_j^{(i)} \sim (1-\delta_j^{(i)})N(0, k^2{\tau_j^{(i)}}^2) + \delta_j^{(i)}N(0, {\tau_j^{(i)}}^2), j = 1, \ldots, p$$

$$(8) \qquad \delta_j^{(i)}|\gamma = \begin{cases} 1, & \text{if } i = 1 \text{ or } \gamma > \xi \\ 0, & \text{if } i = 2 \text{ and } \gamma \leq \xi, \end{cases}$$

where $i = 1, 2$ denotes the regime, and $j = 1, \ldots, p$ denotes the lag order of the AR mean terms in $\boldsymbol{\phi_j}$. Here, $\xi$ is a specified threshold, and $\gamma \leq \xi$ indicates that $\mathrm{F}(z_{t-d}; \gamma, c) \to 0.5$; that is, an AR-GARCH model. As suggested in Gerlach and Chen (2008), we choose $k$ to be a small positive value, so that if $\gamma \leq \xi$ and $\delta_j^{(2)} = 0$, then the posterior value for the parameters $\phi_j^{(2)}$ will be weighted by the prior value towards 0.

A constrained uniform prior is taken for $p(\boldsymbol{\alpha})$, the constraint defined by the indicator $I(S)$, where $S$ defines the constraints in Equations (5), (6), and (7). For $\gamma$, we choose the log-normal distribution, $\gamma \sim LN(\mu_\gamma, \sigma_\gamma^2)$. The prior for the delay lag, $d$, is a discrete, uniform variable:

$$\Pr(d) = \frac{1}{d_0},$$

where $d = 1, \ldots, d_0$. For $\nu$ degrees of freedom, we define $\rho = \nu^{-1}$ and set it to $I(\rho \in [0, 0.25])$ (see Chen, Chiang, and So, 2003, for more details). For the skew parameter, we set a flat prior over $\eta \in (-1, 1)$. Finally, we choose the flat priors for the threshold parameters in three ST functions, described as follows.

**The first-order and the exponential ST function:** When we consider a double ST model with one threshold value, a flat prior on the threshold limit $c$ is $\mathrm{Unif}(lb, ub)$, where $(lb, ub)$ are chosen as suitable percentiles of $z$ to allow a reasonable sample size in each regime for inference.

**The second-order ST function:** Two threshold values in the second-order ST function are much more complicated and need to be constrained in two ways: the first ensures that $c_1 < c_2$ as required, while the second ensures that a sufficient sample size exists in each regime for estimation. For this second constraint, a set of ranges can be set, as relevant percentiles of the sample size $n$, to ensure that at least $100h$ $(0 < h < 1)$ percent of the observations are contained in each regime, as suggested by Chen, Gerlach, and Lin (2010) and Chen, Gerlach, and Liu (2011). The general priors for $c_1$ and $c_2$ are:

$$c_1 \sim \mathrm{Unif}(lb_1, ub_1);$$
$$c_2|c_1 \sim \mathrm{Unif}(lb_2, ub_2),$$

where $lb_1$ and $ub_1$ are the $\wp_{h_1}$ and $\wp_{1-h_1-h_2}$ percentiles of $z_t$, respectively. For example, if $h_1 = h_2 = 0.1$, then $c_1 \in (\wp_{0.1}, \wp_{0.8})$. Furthermore, we set $ub_2 = \wp_{(1-h_2)}$ and $lb_2 = c_1 + c^*$, where $c^*$ is a selected number that ensures $c_1 + c^* \leq c_2$ and at least $100h_2\%$ of observations are in the range $(c_1, c_2)$.

## 3.2 Posteriors

The posteriors are proportional to the product of the likelihood function and the priors, or in other words:

$$p(\boldsymbol{\theta}_l \mid \boldsymbol{y}^{s+1,n}, \boldsymbol{\theta}_{\neq l}) \propto p(\boldsymbol{y}^{s+1,n} \mid \boldsymbol{\theta}) \cdot p(\boldsymbol{\theta}_l \mid \boldsymbol{\theta}_{\neq l}),$$

where $\boldsymbol{\theta}_l$ is a parameter group, $p(\boldsymbol{\theta}_l)$ is its prior density, and $\boldsymbol{\theta}_{\neq l}$ is the vector of all model parameters, except for $\boldsymbol{\theta}_l$. We obtain the delay parameter $d$ by sampling from the conditional multinomial distribution with posterior probabilities as follows:

$$(9) \quad \Pr(d = j \mid \boldsymbol{y}^{s+1,n}, \boldsymbol{\theta}_{\neq d}) = \frac{p(\boldsymbol{y}^{s+1,n} \mid d = j, \boldsymbol{\theta}_{\neq d})}{\sum_{j=1}^{d_0} p(\boldsymbol{y}^{s+1,n} \mid d = j, \boldsymbol{\theta}_{\neq d})}, \; j = 1, \ldots, d_0.$$

Since the posterior distributions for parameters $(\boldsymbol{\phi}_j, \boldsymbol{\alpha}_j, \boldsymbol{c}, \gamma, \nu, \eta)$, with $j = 1, 2$, are not standard forms, we turn to the MCMC method. For our parameters $(\boldsymbol{\phi}_j, \boldsymbol{c}, \gamma, \nu, \eta)$, and $j = 1, 2$, we estimate the parameters by exercising the Metropolis-Hasting (MH) algorithm. For the GARCH parameter $\boldsymbol{\alpha}_j$, we apply a random walk MH algorithm before burn-in period and use the independent kernel MH (IK-MH) algorithm after the burn-in period, since IK-MH would speed up the convergence (see Gerlach and Chen, 2008, for more details). For details on the MCMC sampling scheme, random walk MH, and the IK-MH algorithm, please refer to Chen and So (2006).

## 4. FORECASTING OF VAR AND VOLATILITY

Understanding volatility is vital for financial time series analysis and predicting it is crucial for many functions in financial markets, such as VaR estimation, options pricing, asset allocation, and many other applications.

Assuming the long position, the VaR forecast satisfies:

$$\Pr(y_{n+1} \leq -\text{VaR}_{n+1} \mid \mathcal{F}_n) = \alpha,$$

where $\mathcal{F}_n$ is the information available up to $n$. A one-step-ahead VaR is the $\alpha\%$ quantile level of the conditional distribution $y_{n+1}|\mathcal{F}_n \sim D(\mu_{n+1}, h_{n+1})$, where $h_{n+1}$ is given by one of the parametric models, and $D$ is the relevant error distribution. We estimate this predictive distribution via the MCMC simulation. The quantile VaR is then given by:

$$\text{VaR}^{[j]}_{n+1} = -\left[\mu^{[j]}_{n+1} + D^{-1}_{\alpha}(\boldsymbol{\theta}^{[j]})\sqrt{h^{[j]}_{n+1}}\right], \tag{10}$$

where $D^{-1}$ is the inverse CDF for the distribution $D$. For standardized Student's t errors;

$$D^{-1}_{\alpha} = \frac{t_{\alpha}(\nu^{[j]})}{\sqrt{\nu^{[j]}/(\nu^{[j]}-2)}},$$

where $t_{\alpha}(\nu^{[j]})$ is the $\alpha th$ quantile of a Student's t distribution with $\nu^{[j]}$ degrees of freedom, and $\nu^{[j]}$ is the $jth$ iteration of $\nu$. Hence, $\sqrt{\nu^{[j]}/(\nu^{[j]}-2)}$ is an adjustment term for a standardized Student's t with $\nu^{[j]}$ degrees of freedom.

The final forecasted one-step-ahead VaR is the Monte Carlo posterior mean estimate:

$$\text{VaR}_{n+1} = \frac{1}{N-M}\sum_{j=M+1}^{N} \text{VaR}^{[j]}_{n+1}, \tag{11}$$

where $N$ is the number of MCMC iterations, and $M$ is the size of the burn-in sample. Alternatively, one can compute $\text{VaR}_{n+1}$ as the $\alpha$-percentile of the MCMC sample of $y_{n+1}$ (see Takahashi, Watanabe, Omori 2016). Based on our experience, the performances of these two methods show no difference.

### 4.1 VaR forecasting evaluation

In this section we provide the criteria for comparing and testing the VaR forecast models. The Basel Committee on Banking Supervision (established in 1996) proposed evaluating the worst expected loss over 250 trading days at the 1% level as a form of backtesting, so that at least one year of actual returns is compared with VaR forecasts.

The common guides for comparing the performance are the number of violations ($I(y_t < -\text{VaR}_t)$) and the violation rate (VRate):

$$\text{VRate} = \frac{1}{m}\sum_{t=n+1}^{n+m} I\left(y_t < -\text{VaR}_t\right),$$

where $n$ is the in-sample period size, and $m$ is the forecast size. Naturally, a VRate close to nominal $\alpha$ is desirable. Furthermore, under the Basel Accord, models that over-estimate risk (VRate $< \alpha$) are preferable to those that under-estimate risk levels.

We are greatly interested in the magnitude of the VaR exceedance rather than simply whether or not an exceedance occurred. A backtest can be based on a function of the observed profit or loss and the corresponding model VaR. This would result in the construction of a general loss function, $L(\text{VaR}_t, y_t)$, which could be evaluated using past data on profits and losses and the reported VaR series. Lopez (1999 a,b) suggests this approach to backtesting as an alternative to the approach that focuses exclusively on the hit series. We consider two loss functions below that measure the difference between the observed loss and the VaR in cases where the loss exceeds the reported VaR measure.

(12)

$$\Psi_1(\text{VaR}_t, y_t) = \begin{cases} 1 + \left(y_t - (-\text{VaR}_t)\right)^2 & \text{if } y_t < -\text{VaR}_t \\ 0 & \text{if } y_t \geq -\text{VaR}_t \end{cases},$$

(13)

$$\Psi_2(\text{VaR}_t, y_t) = \begin{cases} 1 + \left| y_t - (-\text{VaR}_t) \right| & \text{if } y_t < -\text{VaR}_t \\ 0 & \text{if } y_t \geq -\text{VaR}_t \end{cases}.$$

When an exception takes place, the risk model is penalized. Hence, we prefer to have a lower average loss value (between two models), defined as the average of these penalty scores:

$$\boldsymbol{\Psi}_i = \frac{1}{m}\sum_{t=n+1}^{n+m} \Psi_i(\text{VaR}_t, y_t),\ i = 1, 2.$$

### 4.2 Backtesting methods

We further consider three backtesting methods for evaluating and testing the accuracy of VaR models: the unconditional coverage (UC) test of Kupiec (1995) - a likelihood ratio test that the true violation rate equals $\alpha$; the conditional coverage (CC) test of Christoffersen (1998) - a joint test, combining a likelihood ratio test for the independence of violations and the UC test; and the dynamic quantile (DQ) test of Engle and Manganelli (2004). We present the details of the processes below.

**The UC test of Kupiec (1995):** As stated in Christoffersen (1998), the UC test looks at the unconditional probability of a violation that must be equal to the coverage rate

$\alpha$, with the LRT being:

$$LR_{uc} = 2 \cdot \log \left[ \frac{\hat{\alpha}^X (1-\hat{\alpha})^{m-X}}{\alpha^X (1-\alpha)^{m-X}} \right] \sim \chi_1^2,$$

where $X$ = number of violations, $m$ = forecast period size, and $\hat{\alpha} = X/m$.

**The CC test of Christoffersen (1998):** The CC test is a joint test that combines a likelihood ratio test for independence of violations and the UC test, where the independence hypothesis stands for VaR violations observed at two different dates being independently distributed.

$$LR_{ind} = 2 \cdot \log \left( \frac{L_1}{L_0} \right) \ ; \ LR_{ind} \sim \chi_1^2.$$

We define $T_{ij}$ as the number of days when condition $j$ occurred under present status, and assuming that condition $i$ occurred on the previous day, we get:

$$\begin{aligned} i, j &= \begin{cases} 1, & \text{if violation occurs} \\ 0, & \text{if no violation occurs,} \end{cases} \\ L_1 &= \prod_{i=0}^{1} (1-\pi_{i1})^{T_{i0}} \pi_{i1}^{T_{i1}}, \\ L_0 &= (1-\pi)^{\sum_{i=0}^{1} T_{i0}} \pi^{\sum_{i=0}^{1} T_{i1}}, \\ \pi_{i1} &= \frac{T_{i1}}{(T_{i0}+T_{i1})}, \text{and } \pi = \frac{(T_{01}+T_{11})}{m}, \end{aligned}$$

with $m$ being the forecast period size. Thus, the joint CC test is a chi-square test, in which $LR_{cc} = LR_{uc} + LR_{ind}$, when $LR_{cc} \sim \chi_2^2$.

**The DQ test of Engle and Manganelli (2004):** The DQ test is based on a linear regression model of the hits variable on a set of explanatory variables, including a constant, the lagged values of the hit variable, and any function of the past information set suspected of being informative. $\text{H}_0$: $H_t = I(y_t < -\text{VaR}_t) - \alpha$ is independent of $\boldsymbol{W}$. The test statistic is:

$$\text{DQ}(q) = \frac{\boldsymbol{H}'\boldsymbol{W} \left( \boldsymbol{W}'\boldsymbol{W} \right)^{-1} \boldsymbol{W}'\boldsymbol{H}}{\alpha(1-\alpha)},$$

where $\boldsymbol{W}$ is lagged 'hits', lagged VaR forecasts, or other relevant regressors over time which is discussed in detail by Engle and Manganelli(2004). Under the $\text{H}_0$, $\text{DQ}(q) \sim \chi_q^2$. The DQ test is recognized to be more powerful than the CC test.

### 4.3 Volatility proxies

Though volatility is unobservable, we consider the following two proxy variables. They are range-based proxies, using mean square error (MSE) and mean absolute deviation (MAD) as loss functions. The first range-based proxy is like that of Parkinson (1980), and the second is based on Alizadeh, Brandt, and Diebold (2002) and employed by Lin, Chen, and Gerlach (2012). The formulae for the two volatility proxies are the following.

1. $\hat{\sigma}_{1,t} = R_t / \sqrt{4 \ln(2)}$; (Parkinson, 1980).
2. $\hat{\sigma}_{2,t} = \exp[\ln(R_t) - 0.43 + 0.29^2/2]$; (Alizadeh, Brandt, and Diebold, 2002),

where $R_t = (\max P_t - \min P_t) \times 100$, $P_t$ is the log price index at time $t$. Note that $\hat{\sigma}_{i,t}$, $i = 1, 2$ are known as the unbiased estimators of $\sqrt{h_t}$. We consider the two loss functions:

$$\begin{aligned} \text{MSE} &= \frac{1}{m} \sum_{t=n+1}^{n+m} (e_{i,t})^2, \\ \text{MAD} &= \frac{1}{m} \sum_{t=n+1}^{n+m} |e_{i,t}|, \\ \text{where } e_{i,t} &= \hat{\sigma}_{i,t} - \sqrt{h_t}, \qquad i = 1, 2. \end{aligned}$$

When comparing the models, it is favorable to have smaller error values under these two criteria.

## 5. SIMULATION STUDY

We perform simulation studies for the Bayesian estimation to examine the effectiveness of the MCMC sampling scheme. Considering finite sample properties and consistencies of the MCMC estimators, 500 replications are generated, with sample size $n = 2000$. We take the second-order ST-GARCH model with a skew Student's t distribution as follows:

(14)

$$\begin{aligned} & y_t = (0.1 + 0.4 y_{t-1}) + F_1(z_{t-1})(0.1 - 0.25 y_{t-1}) + a_t, \\ & a_t = \sqrt{h_t} \epsilon_t, \qquad \epsilon_t \overset{\text{i.i.d.}}{\sim} SK(7, -0.4) \\ & h_t = (0.15 + 0.2 a_{t-1}^2 + 0.7 h_{t-1}) + \\ & F_2(z_{t-1})(-0.1 - 0.1 a_{t-1}^2 - 0.2 h_{t-1}), \\ & F_i(z_{t-1}) = \frac{1}{1 + \exp\left\{ \frac{-\gamma_i (z_{t-1} - (-0.35))(z_{t-1} - 0.3)}{s_z} \right\}}, \end{aligned}$$

where $(\gamma_1, \gamma_2) = (4, 10)$ and $(10, 20)$, $z_t$ is the daily returns of the S&P500 index, and $s_z$ is its sample standard deviation. Since under the proposed model, it would be difficult to generate a series of $z_t$, we use S&P500 returns instead. Orders $p$, $g$, and $q$ are all set to 1. We choose two combinations of the smoothness parameters in (14). The choice $(\gamma_1, \gamma_2) = (10, 20)$ reflects more extreme $\gamma_i$, because $\gamma_i \geq 10$ is effectively a sharp transition. Actually, the second-order ST functions' shapes of $\gamma$, which are equal to 10 or 20 and illustrated in Figure 1, are indistinguishable from one other. We choose the maximum delay, $d_0$, to be 3. The initial values for each parameter are $\boldsymbol{\phi}_1 = (0, 0)$, $\boldsymbol{\phi}_2 = (0, 0)$, $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2 = (0.01, 0.1, 0.1)$, $\nu = 100$, $\gamma = 30$, and $(c_1, c_2) = (0, 0.1)$.

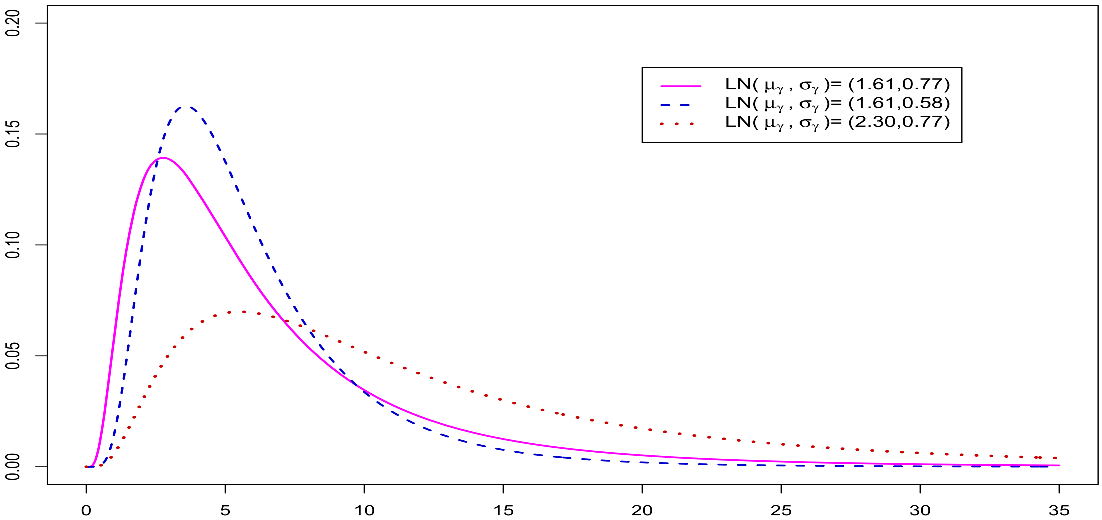


*Figure 2. Three prior densities for $\gamma_i$.*

Based on Figure 1, $F(.)$ turns into a sharp or abrupt transition when $\gamma > 10$. Therefore, we consider only choices of $(\mu_\gamma, \sigma_\gamma)$ that ensure the prior density becomes small for $\gamma > 20$. We establish two set-ups of prior information for $\gamma_i, i = 1, 2$, $LN(\mu_\gamma, \sigma_\gamma) = (1.61, 0.77)$ and $(1.61, 0.58)$ in which the densities are in Figure 2. We set the hyper-parameters to $(\xi, k) = (0.5, 0.001)$ in the mixture specification (8), with $\tau_i = 0.35$, $c_1 \sim$Unif$(\wp_{0.2}, \wp_{0.7})$, and $c_2|c_1 \sim$Unif$(c_1+c^*, \wp_{0.8})$. Hence, $c^*$ is chosen, which leads to at least 10% of observations in-between. We also set $b_1 = s_y^2$, $b_2 = 1$, and $b_3 = 1.1$ in (7), allowing for possible explosiveness in the variance equation. Note that these settings are suggested by Gerlach and Chen (2008).

We extensively examine trace plots and the autocorrelation function (ACF) plots to confirm convergence and to infer adequate coverage. We observe the trace plots for parameters that converge immediately. The ACF plots cut off fairly quickly, which means that the MCMC mixing is fast and the autocorrelation is low. Those plots are not shown to save space.

We use a burn-in sample of $M$ = 10,000 and a total sample of $N$ = 30,000 iterations, but utilize only every 2nd iteration in the sample period for inference. It takes less than 3 minutes of CPU time for one simulated dataset using author-written FORTRAN code to complete 30,000 iterations with a sample size of 2000.

Table 1 displays the summary statistics from 500 replications for each parameter with two prior choices. The columns are the true values, the averages of 500 posterior mean, median estimates, and standard deviations. Table 2 lists the averages of posterior mean, median, standard deviation, and 95% credible interval for 500 replications for the second set of parameters based on the third choice of $(\mu_\gamma, \sigma_\gamma) = (2.30, 0.77)$. The last row in both tables are the average, median, and standard deviation among 500 posterior modes of the parameter $d$, which is the delay lag. They 100% correctly indicate $d$=1 for each 500 replications.

We assume the same prior density for $\gamma_1$ and $\gamma_2$ - that is, we do not have any restriction about the magnitudes of $\gamma_1$ and $\gamma_2$. The estimates of $(\gamma_1,\gamma_2) = (4, 10)$ are $(5.04, 10.03)$ and $(4.95, 8.68)$ based on priors 1 and 2, which are sound. The average standard deviations for prior 2 are slightly smaller versus those of prior 1. Table 2 lists the posterior means of the second combination $(\gamma_1,\gamma_2)$ to be $(9.76, 17.34)$. Again, the shape of the second-order ST function of $\gamma_2 = 17.34$ is not distinguishable from that of the true value of $\gamma_2 = 20$. These simulation results indicate that the posterior estimates obtained by the proposed sampling scheme are reliable. Non-Bayesian methods are unable to accomplish this desired purpose for estimation.

## 6. EMPIRICAL STUDY

To examine the performance of the models under highly varied market conditions, this study examines two distinct forecasting periods. The first complete dataset is divided into two: an in-sample period from January 1, 2004 to October 31, 2011, and 500 out-of-sample forecast days, from November 1, 2011 to late October or mid-November 2013. Small differences in end-dates across markets did occur due to different market trading days. This time frame covers the period after the effects of the global financial crisis hit the markets. The study includes 10 European stock markets, 7 Asian stock markets, 1 North American market, and 1 South American market, making 19 stock markets in all. We utilize three regions for the daily closing prices of stock markets, including (i) Americas: the S&P500 (U.S.) and the Bovespa Index (Brazil); (ii) Asia: KOSPI (South Korea), HANG SENG Index (Hong Kong), Nikkei 225 (Japan), CNX 500

Table 1. Simulation results for the second-order ST-GARCH model in (14) based on n = 2000 and obtained from 500 replications

| | True | Mean | Med | Std | Mean | Med | Std |
|---|---|---|---|---|---|---|---|
| | | Prior 1 | | | Prior 2 | | |
| $\phi_0^{(1)}$ | -0.10 | -0.0992 | -0.0991 | 0.0345 | -0.1012 | -0.1009 | 0.0347 |
| $\phi_1^{(1)}$ | 0.30 | 0.3072 | 0.3048 | 0.0833 | 0.3019 | 0.2997 | 0.0834 |
| $\phi_0^{(2)}$ | 0.10 | 0.0980 | 0.0966 | 0.0412 | 0.1026 | 0.1021 | 0.0412 |
| $\phi_1^{(2)}$ | -0.25 | -0.2605 | -0.2579 | 0.0993 | -0.2467 | -0.2472 | 0.0991 |
| $\alpha_0^{(1)}$ | 0.15 | 0.1629 | 0.1672 | 0.0295 | 0.1632 | 0.1651 | 0.0289 |
| $\alpha_1^{(1)}$ | 0.20 | 0.2080 | 0.2034 | 0.0762 | 0.2170 | 0.2153 | 0.0768 |
| $\beta_1^{(1)}$ | 0.70 | 0.6503 | 0.6524 | 0.1141 | 0.6651 | 0.6678 | 0.1116 |
| $\alpha_0^{(2)}$ | -0.10 | -0.1084 | -0.1110 | 0.0330 | -0.1089 | -0.1106 | 0.0320 |
| $\alpha_1^{(2)}$ | -0.10 | -0.1083 | -0.1001 | 0.0845 | -0.1140 | -0.1109 | 0.0858 |
| $\beta_1^{(2)}$ | -0.20 | -0.1819 | -0.1859 | 0.1394 | -0.1905 | -0.1933 | 0.1349 |
| $\nu$ | 7.00 | 7.1252 | 6.9782 | 1.0762 | 7.0909 | 6.9636 | 1.0733 |
| $\eta$ | -0.40 | -0.4016 | -0.4018 | 0.0292 | -0.3963 | -0.3966 | 0.0291 |
| $\gamma_1$ | 4.00 | 5.0494 | 4.3081 | 3.1122 | 4.9478 | 4.4164 | 2.3318 |
| $\gamma_2$ | 10.00 | 10.0281 | 9.0847 | 4.5053 | 8.6667 | 8.1056 | 3.1059 |
| $c_1$ | -0.35 | -0.3356 | -0.3326 | 0.1077 | -0.3333 | -0.3303 | 0.1076 |
| $c_2$ | 0.30 | 0.2886 | 0.2866 | 0.1075 | 0.2849 | 0.2815 | 0.1068 |
| $d^*$ | 1.00 | 1 | 1 | 0 | 1 | 1 | 0 |

$^*$: Average of posterior modes and median of posterior modes for $d$.

Table 2. Simulation results for the second-order ST-GARCH model in (14) based on n = 2000 and obtained from 500 replications

| Parameters | True | Mean | Med | Std | 2.5% | 97.5% |
|---|---|---|---|---|---|---|
| $\phi_0^{(1)}$ | 0.1 | -0.1048 | -0.1057 | 0.0342 | -0.1751 | -0.0399 |
| $\phi_1^{(1)}$ | 0.4 | 0.3143 | 0.3120 | 0.0894 | 0.1372 | 0.4878 |
| $\phi_0^{(2)}$ | 0.1 | 0.1049 | 0.1081 | 0.0390 | 0.0304 | 0.1884 |
| $\phi_1^{(2)}$ | -0.25 | -0.2679 | -0.2634 | 0.1006 | -0.4619 | -0.0670 |
| $\alpha_0^{(1)}$ | 0.15 | 0.1712 | 0.1726 | 0.0257 | 0.1152 | 0.2174 |
| $\alpha_1^{(1)}$ | 0.2 | 0.2239 | 0.2221 | 0.0903 | 0.0768 | 0.3959 |
| $\beta_1^{(1)}$ | 0.7 | 0.6663 | 0.6784 | 0.1255 | 0.4075 | 0.8714 |
| $\alpha_0^{(2)}$ | -0.1 | -0.1183 | -0.1197 | 0.0287 | -0.1709 | -0.0612 |
| $\alpha_1^{(2)}$ | -0.1 | -0.1182 | -0.1112 | 0.0961 | -0.3004 | 0.0507 |
| $\beta_1^{(2)}$ | -0.2 | -0.1843 | -0.2054 | 0.1510 | -0.4440 | 0.1482 |
| $\nu$ | 7 | 7.1687 | 7.0050 | 1.1809 | 5.4498 | 10.1743 |
| $\eta$ | -0.4 | -0.3979 | -0.3991 | 0.0299 | -0.4526 | -0.3412 |
| $\gamma_1$ | 10 | 9.7613 | 9.1084 | 3.8261 | 4.5725 | 18.0951 |
| $\gamma_2$ | 20 | 17.3461 | 16.9047 | 4.9817 | 9.1169 | 29.4396 |
| $c_1$ | -0.35 | -0.3217 | -0.3193 | 0.0629 | -0.4563 | -0.2098 |
| $c_2$ | 0.3 | 0.2681 | 0.2662 | 0.0602 | 0.1581 | 0.3885 |
| $d^*$ | 1 | 1 | 1 | 0 | 1 | 1 |

$^*$: Average of posterior modes and median of posterior modes for $d$.
Prior 3 ($LN(2.3, 0.77)$) is adopted for the ($\gamma_1$,$\gamma_2$).

(India), SHANGHAI SE A SHARE (China), TAIEX (Taiwan), and SET Index (Thailand); and (iii) Europe: FTSE 100 (U.K.), DAX 30 (Germany), CAC 40 (France), AEX Index (Netherlands), PSI 20 (Portugal), MIB Index (Italy), ISEQ (Ireland), Athex Composite Index (Greece), RTS Index (Russia), and IBEX 35 (Spain).

To examine how the models perform during the recent financial crisis period (2007–2009) and to evaluate how the crisis affects risk management, we take on a second time span: a learning period from January 4, 2000 to December 31, 2006 and a second validation or out-of-sample forecast evaluation window: January 3, 2007 to December 30, 2009.

*Table 3. Summary statistics of market returns:[a] in-sample period from January 1, 2004 to October 31, 2011*

| | Mean | Min | Max | Std | Skewness | Kurtosis | JB test[b] | Q(5)[c] | $Q^2(5)$[c] |
|---|---|---|---|---|---|---|---|---|---|
| Americas | | | | | | | | | |
| Brazil | 0.050 | -12.100 | 13.680 | 1.9463 | -0.081 | 8.263 | < 0.001 | 0.024 | < 0.001 |
| U.S. | 0.006 | -9.470 | 10.960 | 1.3848 | -0.312 | 13.161 | < 0.001 | < 0.001 | < 0.001 |
| Europe | | | | | | | | | |
| France | -0.005 | -9.472 | 10.590 | 1.4814 | 0.065 | 10.559 | < 0.001 | < 0.001 | < 0.001 |
| Germany | 0.022 | -7.433 | 10.800 | 1.4383 | 0.052 | 10.354 | < 0.001 | 0.025 | < 0.001 |
| Greece | -0.053 | -10.210 | 13.430 | 1.7603 | 0.047 | 7.956 | < 0.001 | 0.036 | < 0.001 |
| Ireland | -0.030 | -13.960 | 9.733 | 1.6616 | -0.575 | 9.966 | < 0.001 | 0.536 | < 0.001 |
| Italy | -0.026 | -8.598 | 10.880 | 1.5155 | -0.043 | 10.172 | < 0.001 | 0.001 | < 0.001 |
| Netherlands | -0.005 | -9.590 | 10.030 | 1.4455 | -0.174 | 11.957 | < 0.001 | < 0.001 | < 0.001 |
| Portugal | -0.007 | -10.380 | 10.200 | 1.2099 | -0.114 | 13.874 | < 0.001 | 0.134 | < 0.001 |
| Russia | 0.052 | -21.200 | 20.200 | 2.3665 | -0.454 | 14.085 | < 0.001 | < 0.001 | < 0.001 |
| Spain | 0.007 | -9.586 | 13.480 | 1.5068 | 0.150 | 11.685 | < 0.001 | 0.001 | < 0.001 |
| U.K. | 0.011 | -9.266 | 9.384 | 1.2913 | -0.150 | 11.307 | < 0.001 | < 0.001 | < 0.001 |
| Asia | | | | | | | | | |
| China | 0.026 | -9.261 | 9.033 | 1.8159 | -0.294 | 5.920 | < 0.001 | 0.078 | < 0.001 |
| Hong Kong | 0.024 | -13.580 | 13.410 | 1.7202 | 0.044 | 11.652 | < 0.001 | 0.184 | < 0.001 |
| India | 0.052 | -12.880 | 15.030 | 1.7253 | -0.488 | 11.131 | < 0.001 | < 0.001 | < 0.001 |
| Japan | -0.009 | -12.110 | 13.230 | 1.6112 | -0.569 | 12.236 | < 0.001 | 0.230 | < 0.001 |
| South Korea | 0.044 | -11.170 | 11.280 | 1.5320 | -0.578 | 9.212 | < 0.001 | 0.233 | < 0.001 |
| Taiwan | 0.013 | -6.912 | 6.525 | 1.397 | -0.432 | 5.996 | < 0.001 | 0.0216 | < 0.001 |
| Thailand | 0.012 | -16.060 | 10.580 | 1.4913 | -0.928 | 15.513 | < 0.001 | 0.221 | < 0.001 |

[a] The summary statistics results exclude the out-of-sample forecasting period.
[b] Jarque-Bera normality test
[c] $Q(5)$ and $Q^2(10)$ are the p-values of Ljung-Box test for autocorrelation in the level of returns and the squared returns up to the 5th lag.

We mainly focus on S&P500 and Nikkei 225 for this financial turmoil period.

All data are obtained from Datastream International. The returns are the difference of the logarithm of the daily price index:

$$y_t = [\ln(P_t) - \ln(P_{t-1})] \times 100,$$

where $P_t$ is the closing index on day $t$. Table 3 shows summary statistics for the 19 markets during the in-sample period from January 1, 2004 to October 31, 2011. The statistics include stock-index return means, extreme values, standard deviations, skewness, kurtosis, the Jarque-Bera normality (JB) test, and the Ljung-Box Q(5) values for both returns and squared returns. As per the characteristics of financial data, the daily return has heavy tails and is negatively skewed (exceptions are for Hong Kong, Germany, France, Greece, and Spain). The normality test exhibits a clear rejection for each market by the Jarque-Bera normality test under a 1% significant level. Furthermore, high volatility is clearly evident during the global financial crisis period (around late-2008 to 2009).

The DT-GARCH model proposed by Chen and So (2006) is a special case of the first-order ST-GARCH model when the smoothness parameter $\gamma$ goes to infinity. We state the DT-GARCH(1,1) model as follows:

$$\begin{aligned} y_t &= \begin{cases} \phi_0^{(1)} + \phi_1^{(1)} y_{t-1} + a_t & y_{t-d} < c \\ \phi_0^{(2)} + \phi_1^{(2)} y_{t-1} + a_t & y_{t-d} \ge c \end{cases}, \\ a_t &= \sqrt{h_t}\epsilon_t, \qquad \epsilon_t \overset{\text{i.i.d.}}{\sim} t_\nu^*, \\ h_t &= \begin{cases} \alpha_0^{(1)} + \alpha_1^{(1)} a_{t-1}^2 + \beta_1^{(1)} h_{t-1} & y_{t-d} < c \\ \alpha_0^{(2)} + \alpha_1^{(2)} a_{t-1}^2 + \beta_1^{(2)} h_{t-1} & y_{t-d} \ge c \end{cases}, \end{aligned} \tag{15}$$

where $\epsilon_t$ has a student's t distribution with degrees of freedom $v$, standardized to have unit variance; $d$ is the delay lag and $c$ is the threshold value. We estimate the parameters in the DT-GARCH model, $\phi_0^{(j)}$, $\phi_1^{(j)}$, $\alpha_0^{(j)}$, $\alpha_1^{(j)}$, $\beta_1^{(j)}$, (where $j = 1, 2,$) $c$, $d$, and $\nu$, by the Bayesian method. When $h_t = \alpha_0 + \alpha_1 a_{t-1}^2 + \gamma I(a_{t-1} < 0) a_{t-1}^2 + \beta_1 h_{t-1}$ in Equation (15), then this model becomes a special case: GJR-GARCH model. We will employ both models for VaR forecasting later.

We report Bayesian estimates for the U.S. market during the in-sample period based on the five nonlinear GARCH models: first-order logistic function with Student's t error (1ST-GARCH), second-order logistic function with Student's t and skew Student's t errors (2ST-GARCH and 2STsk-GARCH), the exponential function with Student's t error (EST-GARCH), and the DT-GARCH model with Student's t errors. The priors' settings for ST-GARCH are

*Table 4. Bayesian estimation of parameters for 1ST, 2ST, 2STsk, EST-GARCH specifications, and DT-GARCH for the S&P500 index*

| | 1ST-GARCH | | 2ST-GARCH | | 2STsk-GARCH | | EST-GARCH | | DT-GARCH | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Med | Std | Med | Std | Med | Std | Med | Std | Med | Std |
| $\phi_0^{(1)}$ | 0.533 | 0.302 | 0.029 | 0.055 | -0.036 | 0.079 | 0.027 | 0.028 | 0.033 | 0.035 |
| $\phi_1^{(1)}$ | -0.177 | 0.162 | -0.008 | 0.069 | -0.150 | 0.103 | -0.008 | 0.042 | -0.084 | 0.040 |
| $\phi_0^{(2)}$ | -0.908 | 0.605 | 0.088 | 0.091 | 0.104 | 0.109 | 0.102 | 0.156 | 0.034 | 0.074 |
| $\phi_1^{(2)}$ | 0.274 | 0.313 | -0.087 | 0.102 | 0.099 | 0.132 | -0.098 | 0.075 | -0.033 | 0.064 |
| $\alpha_0^{(1)}$ | 0.206 | 0.006 | 0.185 | 0.010 | 0.030 | 0.025 | 0.006 | 0.005 | 0.007 | 0.008 |
| $\alpha_1^{(1)}$ | 0.345 | 0.013 | 0.273 | 0.008 | 0.128 | 0.039 | 0.006 | 0.006 | 0.133 | 0.018 |
| $\beta_1^{(1)}$ | 0.354 | 0.007 | 0.331 | 0.012 | 0.681 | 0.057 | 0.885 | 0.018 | 0.947 | 0.019 |
| $\alpha_0^{(2)}$ | 0.188 | 0.023 | 0.196 | 0.016 | 0.004 | 0.039 | 0.312 | 0.092 | 0.008 | 0.007 |
| $\alpha_1^{(2)}$ | 0.167 | 0.017 | 0.133 | 0.016 | -0.059 | 0.048 | 0.105 | 0.022 | 0.005 | 0.006 |
| $\beta_1^{(2)}$ | 0.286 | 0.011 | 0.273 | 0.007 | 0.266 | 0.065 | 0.014 | 0.037 | 0.883 | 0.018 |
| $\nu$ | 4.081 | 0.112 | 4.174 | 0.231 | 6.955 | 1.298 | 6.784 | 1.194 | 7.220 | 1.309 |
| $\eta$ | | | | | -0.149 | 0.029 | | | | |
| $\gamma$ | 0.247 | 0.103 | 6.714 | 1.657 | | | 0.238 | 0.041 | - | - |
| $\gamma_1$ | | | | | 5.562 | 3.424 | | | | |
| $\gamma_2$ | | | | | 4.545 | 2.867 | | | | |
| $c$ | 0.054 | 0.377 | - | - | | | 0.672 | 0.052 | 0.325 | 0.269 |
| $c_1$ | - | - | -0.326 | 0.186 | 0.183 | 0.131 | - | - | - | - |
| $c_2$ | - | - | 0.568 | 0.123 | 0.518 | 0.094 | - | - | - | - |
| $d^*$ | 1 | | 1 | | 2 | | 1 | | 1 | |

* denotes the posterior mode for $d$.

the same as in the simulation study, and the estimation is based on a total of 30,000 MCMC iterations, discarding the first 10,000 iterations as a burn-in period.

Table 4 presents the estimated posterior median and standard deviation of parameters. To save space, we do not report 95% credible intervals here. The majority of coefficients in mean equations are insignificant, which are indicated by the 95% credible intervals. Allowing AR(1) in the conditional mean helps account for possible asymmetric autocorrelations in the returns. The delay lag $d$ is not always fixed and swings from 1 to 3, based on learning periods. In order to show further justification about the same effect or the same smooth transition function in the mean and variance, we allow Model "2STsk-GARCH" to incorporate different ST functions in the mean and variance with skew Student's t errors. The estimated skew parameter and degrees of freedom are respectively $-0.149$ and 6.955, indicating the skew Student's t assumption is appropriate. However, the estimates of $\gamma_1$ and $\gamma_2$ are 5.562 and 4.545, respectively. It seems that the effects in the mean and variance are indistinguishable during the in-sample period. In order to further examine about whether we should include different effects in mean and variance, we plan to use these five models for VaR forecasting.

In the out-of-sample period, we employ a rolling window approach to produce a one-step-ahead forecasting of $h_{n+1}$, 1% VaR, and 5% VaR under the following 2 HS methods and 12 risk models. The HS methods encompass the short-term HS with an observation window of 25 days (HS-ST) and the long-term HS with an observation window of 100 days (HS-LT). The risk models include the five asymmetric models employed for Table 4. In addition, there are RiskMetrics (RM), AR-GARCH, and AR-GJR-GARCH with three error distributions. For the global financial crisis period, we utilize RV models with three error distributions for the U.S. and Japan stock markets.

### Post-global financial crisis period (2011–2013)

Tables 5 and 6 report the empirical results for the 19 markets: The ratio of VRate to $\alpha$ under the 1% and 5% confidence levels based on a forecast period of 500 trading days. The ratio VRate/$\alpha = 1$ indicates a good VaR method/model. We summarize our results as follows.

1) The Greek government-debt crisis started in 2009 and was still ongoing as of the out-of-sample period from 2011 to 2013. Our results reveal that most of Europe's markets had a hard time trying to avoid any of the negative effects from the Greek government-debt crisis. All models and methods massively underestimate this risk, especially for Greece, Italy, and Portugal markets, at the 1% level. The range of VRate for the Greece stock market varies from [1.2, 5.2] at the 1% level, which is far away from the 1%. The worst performance for most of the risk models occurs in the Greece market.
2) We use HS methods due to their popularity and ease of implementation. As expected, the performance is poor since most of the markets are unstable. Nevertheless,

*Table 5. Post-global financial crisis period: VaR prediction performance using 2 HS methods and 12 risk models and 500 forecasted stock returns under $\alpha = 1\%$. VRate/$\alpha$ are given*

| Markets | HS-ST | HS-LT | RM | G-n | G-t | G-sk | GJRn | GJRt | GJRsk | 1ST GARCH | 2ST GARCH | 2STsk GARCH | EST GARCH | DT GARCH |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Americas | (5.1) | (1.5) | (1.7) | (1.5) | (1.0) | (0.8) | (1.3) | (0.7) | (0.4) | (0.6) | (0.8) | (0.7) | (0.9) | (0.8) |
| Brazil | 5.0 | 1.6 | 1.4 | 0.8 | 0.8 | 0.6 | 0.8 | 0.2 | 0.2 | 0.4 | 0.8 | 0.6 | 0.4 | 0.4 |
| U.S. | 5.2 | 1.4 | 2.0 | 2.2 | 1.2 | 1.0 | 1.8 | 1.2 | 0.6 | 0.8 | 0.8 | 0.8 | 1.4 | 1.2 |
| Europe | (4.20) | (1.68) | (1.88) | (1.81) | (1.49) | (1.28) | (1.92) | (1.66) | (1.44) | (1.56) | (1.44) | (1.42) | (1.66) | (1.62) |
| France | 3.2 | 1.4 | 1.6 | 1.6 | 1.2 | 1.2 | 1.6 | 1.6 | 1.4 | 1.4 | 1.4 | 1.6 | 1.2 | 1.4 |
| Germany | 4.0 | 1.6 | 2.0 | 1.6 | 1.2 | 1.4 | 2.0 | 1.6 | 1.4 | 1.2 | 1.4 | 1.6 | 1.6 | 1.8 |
| Greece | 5.2 | 1.2 | 1.8 | 2.2 | 1.8 | 1.6 | 2.4 | 2.2 | 2.0 | 2.0 | 1.8 | 1.8 | 2.0 | 2.0 |
| Ireland | 4.8 | 1.6 | 1.4 | 1.6 | 1.4 | 1.2 | 1.6 | 1.6 | 1.2 | 1.2 | 1.2 | 1.4 | 1.4 | 1.6 |
| Italy | 3.6 | 1.6 | 2.2 | 2.5 | 2.3 | 1.4 | 2.4 | 2.0 | 2.0 | 2.2 | 2.0 | 1.8 | 2.4 | 2.0 |
| Netherlands | 4.6 | 1.8 | 2.2 | 2.0 | 1.6 | 1.4 | 1.8 | 1.6 | 1.6 | 1.8 | 1.2 | 1.2 | 1.8 | 1.6 |
| Portugal | 4.2 | 2.0 | 2.0 | 2.2 | 2.0 | 1.8 | 2.2 | 1.6 | 1.4 | 1.8 | 2.0 | 1.8 | 1.8 | 1.8 |
| Russia | 4.0 | 1.8 | 2.2 | 1.6 | 1.2 | 0.8 | 1.0 | 0.8 | 0.6 | 1.0 | 1.2 | 1.0 | 1.2 | 0.6 |
| Spain | 4.0 | 2.0 | 1.8 | 1.6 | 1.2 | 1.2 | 2.4 | 2.0 | 1.8 | 1.6 | 1.2 | 1.2 | 1.8 | 2.0 |
| UK | 4.4 | 1.8 | 1.6 | 1.2 | 1.0 | 0.8 | 1.8 | 1.6 | 1.0 | 1.4 | 1.0 | 0.8 | 1.4 | 1.4 |
| Asia | (4.89) | (2.00) | (1.74) | (1.36) | (1.01) | (0.60) | (1.03) | (0.86) | (0.66) | (0.86) | (0.97) | (0.66) | (1.06) | (0.86) |
| China | 4.8 | 1.4 | 2.4 | 2.3 | 1.5 | 0.4 | 1.4 | 1.0 | 0.4 | 0.6 | 0.8 | 0.4 | 1.0 | 1.0 |
| Hong Kong | 4.0 | 1.8 | 2.0 | 1.8 | 1.8 | 1.4 | 1.4 | 1.4 | 1.4 | 1.4 | 1.6 | 1.6 | 1.6 | 1.4 |
| India | 4.4 | 3.0 | 1.6 | 1.0 | 0.8 | 0.2 | 1.2 | 0.8 | 0.4 | 0.6 | 1.0 | 0.4 | 1.0 | 0.6 |
| Japan | 4.6 | 2.6 | 1.4 | 1.4 | 1.0 | 0.8 | 1.4 | 1.2 | 1.0 | 1.0 | 1.2 | 1.0 | 1.6 | 1.2 |
| South Korea | 5.6 | 1.6 | 2.2 | 0.8 | 0.6 | 0.4 | 0.2 | 0.2 | 0.0 | 0.4 | 0.6 | 0.2 | 0.6 | 0.2 |
| Taiwan | 4.6 | 1.2 | 1.8 | 1.2 | 0.6 | 0.4 | 0.4 | 0.4 | 0.4 | 0.6 | 0.6 | 0.2 | 0.6 | 0.6 |
| Thailand | 6.2 | 2.4 | 0.8 | 1.0 | 0.8 | 0.6 | 1.2 | 1.0 | 1.0 | 1.4 | 1.0 | 0.8 | 1.0 | 1.0 |

Note that the values in (.) are the average VRate/$\alpha$ for each method/model in each region.

*Table 6. Post-global financial crisis period: VaR prediction performance using 2 HS methods and 12 risk models and 500 forecasted stock returns under $\alpha = 5\%$. VRate/$\alpha$ are given*

| Markets | HS-ST | HS-LT | RM | G-n | G-t | G-sk | GJRn | GJRt | GJRsk | 1ST GARCH | 2ST GARCH | 2STsk GARCH | EST GARCH | DT GARCH |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Americas | (1.74) | (1.08) | (1.06) | (0.98) | (1.09) | (0.96) | (0.96) | (1.00) | (0.88) | (0.94) | (1.00) | (0.84) | (1.04) | (0.90) |
| Brazil | 1.80 | 1.16 | 1.12 | 1.00 | 1.06 | 0.96 | 0.88 | 0.88 | 0.84 | 0.88 | 0.92 | 0.88 | 0.96 | 0.80 |
| U.S. | 1.68 | 1.00 | 1.00 | 0.96 | 1.12 | 0.96 | 1.04 | 1.12 | 0.92 | 1.00 | 1.08 | 0.80 | 1.12 | 1.00 |
| Europe | (1.59) | (0.95) | (1.07) | (1.07) | (1.13) | (1.05) | (1.05) | (1.10) | (1.00) | (1.16) | (1.14) | (1.05) | (1.26) | (1.10) |
| France | 1.60 | 0.88 | 1.00 | 1.12 | 1.20 | 1.00 | 1.08 | 1.08 | 1.00 | 1.20 | 1.24 | 1.16 | 1.20 | 1.20 |
| Germany | 1.68 | 0.92 | 0.92 | 1.08 | 1.08 | 1.00 | 1.04 | 1.04 | 1.00 | 1.20 | 1.16 | 1.08 | 1.20 | 1.24 |
| Greece | 1.68 | 1.12 | 1.16 | 1.08 | 1.16 | 1.28 | 1.24 | 1.28 | 1.28 | 1.36 | 1.12 | 1.24 | 1.64 | 1.12 |
| Ireland | 1.60 | 1.12 | 0.92 | 0.88 | 0.96 | 0.96 | 0.92 | 0.96 | 0.92 | 1.00 | 1.00 | 1.00 | 1.08 | 1.04 |
| Italy | 1.52 | 0.80 | 1.04 | 1.20 | 1.26 | 1.04 | 1.12 | 1.16 | 1.04 | 1.20 | 1.24 | 1.08 | 1.56 | 1.12 |
| Netherlands | 1.60 | 0.80 | 1.12 | 0.84 | 1.12 | 0.96 | 1.00 | 1.00 | 0.96 | 1.16 | 1.08 | 0.96 | 1.24 | 1.12 |
| Portugal | 1.64 | 1.08 | 1.12 | 1.24 | 1.16 | 1.24 | 1.16 | 1.28 | 1.16 | 1.24 | 1.24 | 1.08 | 1.24 | 1.08 |
| Russia | 1.40 | 1.00 | 1.16 | 0.96 | 1.00 | 0.80 | 0.76 | 0.84 | 0.76 | 0.80 | 1.00 | 0.80 | 0.84 | 0.84 |
| Spain | 1.72 | 1.00 | 1.04 | 1.12 | 1.12 | 1.12 | 1.12 | 1.24 | 1.04 | 1.12 | 1.12 | 0.96 | 1.28 | 1.12 |
| UK | 1.44 | 0.80 | 1.20 | 1.20 | 1.24 | 1.12 | 1.04 | 1.12 | 0.88 | 1.28 | 1.20 | 1.16 | 1.28 | 1.16 |
| Asia | (1.66) | (1.10) | (1.06) | (0.95) | (1.11) | (0.88) | (0.83) | (0.95) | (0.82) | (0.93) | (1.01) | (0.87) | (0.97) | (0.97) |
| China | 1.64 | 0.96 | 1.16 | 0.96 | 1.26 | 0.72 | 0.76 | 0.96 | 0.72 | 0.76 | 0.84 | 0.64 | 0.76 | 0.76 |
| Hong Kong | 1.56 | 1.04 | 1.04 | 0.96 | 1.04 | 0.92 | 0.88 | 0.92 | 0.88 | 0.96 | 1.12 | 0.96 | 0.96 | 0.84 |
| India | 1.48 | 1.16 | 1.08 | 1.08 | 1.24 | 0.92 | 0.84 | 0.84 | 0.76 | 0.84 | 1.04 | 0.80 | 0.84 | 0.96 |
| Japan | 1.52 | 1.24 | 0.92 | 0.92 | 0.92 | 0.88 | 1.04 | 1.04 | 0.92 | 0.92 | 0.96 | 0.88 | 1.08 | 1.12 |
| South Korea | 1.80 | 0.84 | 1.16 | 1.12 | 1.20 | 1.00 | 0.92 | 1.04 | 0.92 | 1.08 | 1.16 | 1.04 | 1.16 | 1.08 |
| Taiwan | 1.72 | 0.92 | 1.04 | 0.92 | 1.12 | 0.76 | 0.76 | 0.88 | 0.72 | 0.92 | 0.96 | 0.76 | 1.04 | 1.04 |
| Thailand | 1.92 | 1.52 | 1.04 | 0.68 | 1.00 | 0.96 | 0.64 | 1.00 | 0.84 | 1.00 | 0.96 | 1.00 | 0.96 | 0.96 |

Note that the values in (.) are the average VRate/$\alpha$ for each method/model in each region.

*Table 7. Post-global financial crisis period: Evaluating VaR prediction performance using the 19 markets with 500 out-of-sample forecasting*

| | Mean | Med | Ave loss* | Min | Max |
|---|---|---|---|---|---|
| 1% | | | | | |
| HS-ST | 4.55 | 4.60 | 1.131 | 3.20 | 6.20 |
| HS-LT | 1.78 | 1.60 | 1.008 | 1.20 | 3.00 |
| RM | 1.81 | 1.80 | 0.955 | 0.80 | 2.40 |
| G-n | 1.61 | 1.60 | 0.796 | 0.75 | 2.50 |
| G-t | 1.26 | 1.20 | 0.634 | 0.60 | 2.25 |
| G-sk | **0.98** | **1.00** | **0.475** | 0.20 | 1.80 |
| GJRn | 1.53 | 1.60 | 0.796 | 0.20 | 2.40 |
| GJRt | 1.26 | 1.40 | 0.635 | 0.20 | 2.20 |
| GJRsk | 1.04 | 1.00 | 0.476 | 0.00 | 2.00 |
| 1ST-GARCH | 1.20 | 1.20 | 0.581 | 0.40 | 2.20 |
| 2ST-GARCH | 1.20 | 1.20 | 0.581 | 0.60 | 2.00 |
| 2STsk-GARCH | **1.06** | **1.00** | **0.475** | 0.20 | 1.80 |
| EST-GARCH | 1.36 | 1.40 | 0.688 | 0.40 | 2.40 |
| DT-GARCH | 1.25 | 1.40 | 0.635 | 0.20 | 2.00 |
| 5% | | | | | |
| HS-ST | 8.16 | 8.20 | 5.104 | 7.00 | 9.60 |
| HS-LT | 5.09 | 5.00 | 2.111 | 4.00 | 7.60 |
| RM | 5.33 | 5.20 | 3.687 | 4.60 | 6.00 |
| G-n | 5.08 | 5.00 | 2.371 | 3.40 | 6.20 |
| G-t | 5.59 | 5.60 | 3.953 | 4.60 | 6.25 |
| G-sk | 4.89 | **4.80** | **1.318** | 3.60 | 6.40 |
| GJRn | 5.08 | 5.00 | 2.370 | 3.20 | 6.20 |
| GJRt | 5.18 | 5.20 | 2.635 | 4.20 | 6.40 |
| GJRsk | **4.62** | **4.60** | **1.054** | 3.60 | 6.40 |
| 1ST-GARCH | 5.22 | 5.00 | 2.374 | 3.80 | 6.80 |
| 2ST-GARCH | 5.38 | 5.40 | 3.162 | 4.20 | 6.20 |
| 2STsk-GARCH | **4.81** | **4.80** | 1.844 | 3.20 | 6.20 |
| EST-GARCH | 5.64 | 5.60 | 3.436 | 3.80 | 8.20 |
| DT-GARCH | 5.16 | 5.40 | 3.161 | 3.80 | 6.20 |

The values in boldface indicate the best two favored models.
Ave loss:

$$\Psi(\text{VRte}) = \begin{cases} \alpha + (\text{VRate} - \alpha)^2 & \text{if VRate} > \alpha \\ 0 & \text{if VRate} \leq \alpha \end{cases},$$

the HS-LT method is relatively better than other models for the Greece market at the 1% level.

3) The 2ST-GARCH model with Student's t error outperforms among all other models for the Ireland and Spain markets. We note that the banking crisis in Ireland in November 2010 further dented confidence in an already uncertain global financial market. It is estimated that Ireland owed over $130 billion each to banks in Germany and the UK. The wide exposure of the Euro crisis to the rest of the European market would likely weaken market confidence in that region in the near future. Under these circumstances, the 2ST-GARCH model is a comparatively better choice for these markets during the post-global financial crisis period at the 1% level.
4) Except for the Hong Kong market, three models - GARCH, GJR-GARCH, and 2ST-GARCH with skew Student's t errors - stand out as performing the best across Russia, UK, Asia, and Americas markets at the 1% level. These three models also work well across the markets of Europe, Asia, and Americas regions at the 5% level.

Figure 3 displays boxplots for VaR prediction performance over the 19 markets and 2 HS methods and 12 risk models at 1% and 5% levels. The figure illustrates that models with Gaussian errors and HS methods underestimate the 1% risk level in all or most markets. Among these models, the top three are G-sk, GJRsk, and 2STsk-GARCH models, in which the means of VRate are closest to nominal at the 1% level. These three best models have skew Student's t errors; clearly, fat tails with additional skew characteristic are required in this forecasting time period. The results are different from $\alpha = 1\%$ to 5% level. There are several models whose violation rates are equal to or less than one at the 5% level.

Table 7 confirms these results. The values in boldface

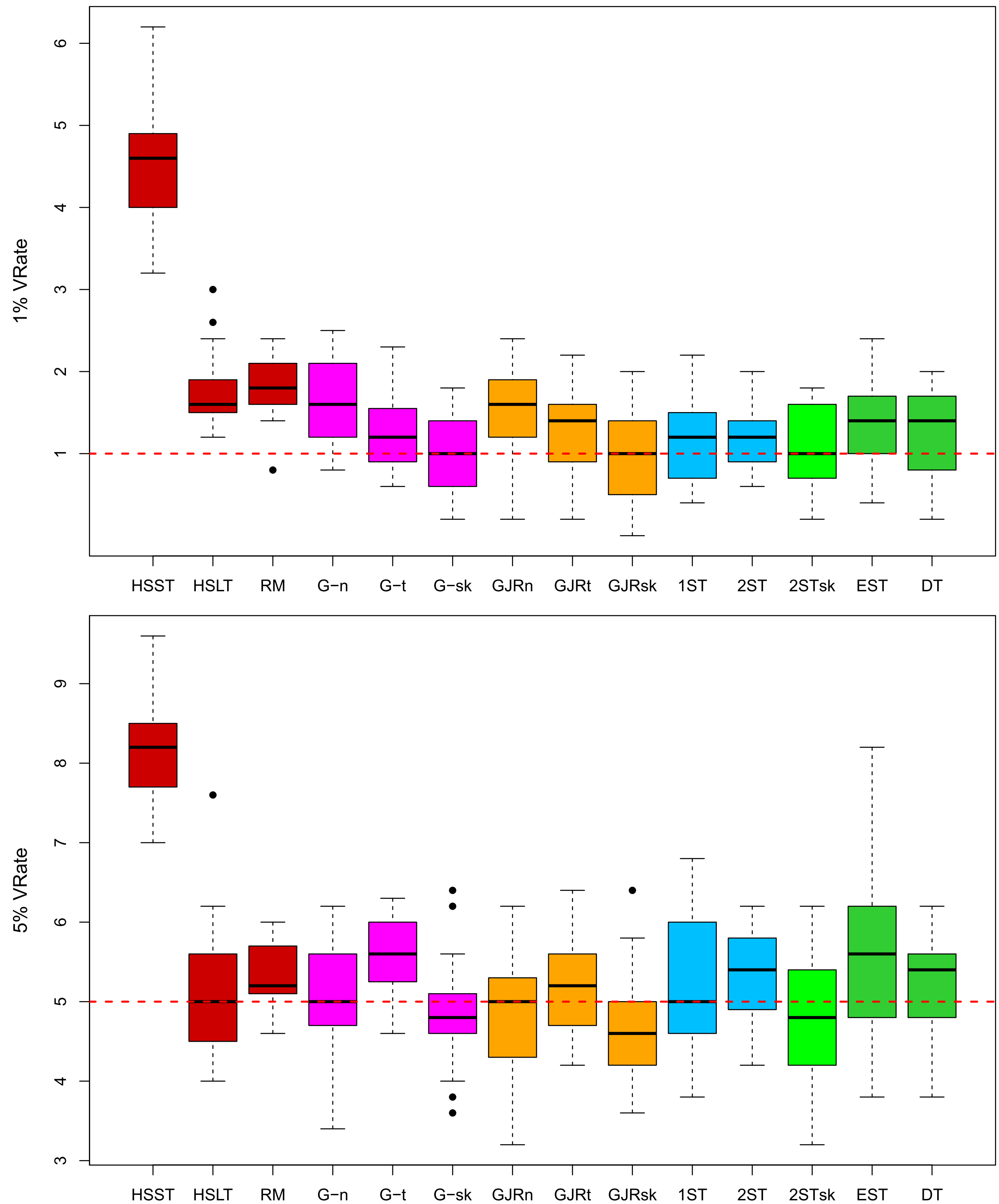


*Figure 3. The boxplots for VaR prediction performance over the 19 markets and 2 HS methods and 12 risk models at the 1% and 5% levels.*

are two favored models. Based on the idea of Lopez (1999 a, b), we construct the following loss function in Table 7 to evaluate the performance of risk models (methods) that penalize those violation rates exceeding the $\alpha$ level by using the squared difference.

$$(16) \quad \Psi(\text{VRate}) = \begin{cases} \alpha + (\text{VRate} - \alpha)^2 & \text{if VRate} > \alpha \\ 0 & \text{if VRate} \leq \alpha \end{cases}.$$

Three models, GARCH, GJR-GARCH, and 2ST-GARCH models with skew Student's t, perform the best at both 1%

*Table 8. Post-global financial crisis period: Evaluating VaR estimates based on Lopez's loss functions using the 19 markets with 500 out-of-sample forecasting*

| | Mean | Med | Std | Min | Max(1) | Mean | Med | Std | Min | Max(2) |
|---|---|---|---|---|---|---|---|---|---|---|
| | Quadratic Loss | | | | | Absolute Loss | | | | |
| 1% | | | | | | | | | | |
| HS-ST | 8.63 | 7.94 | 3.03 | 5.75 | 18.25 | 7.49 | 7.48 | 1.13 | 6.06 | 11.17 |
| HS-LT | 3.05 | 2.57 | 1.73 | 1.55 | 9.25 | 3.24 | 2.56 | 2.08 | 1.78 | 11.04 |
| RM | 3.04 | 2.89 | 1.11 | 1.70 | 5.60 | 2.84 | 2.97 | 0.67 | 1.56 | 3.98 |
| G-n | 2.87 | 2.42 | 1.69 | 1.10 | 8.11 | 2.56 | 2.69 | 0.94 | 1.25 | 4.73 |
| G-t | 2.15 | 1.83 | 1.44 | 0.71 | 6.57 | 1.93 | 1.78 | 0.80 | 0.81 | 3.77 |
| G-sk | **1.73** | 1.39 | 1.41 | 0.33 | 6.21 | **1.53** | 1.60 | 0.82 | 0.36 | 3.44 |
| GJRn | 2.88 | 2.12 | 2.22 | 0.25 | 9.39 | 2.50 | 2.33 | 1.26 | 0.30 | 5.49 |
| GJRt | 3.38 | 1.84 | 6.17 | 0.21 | 28.13 | 2.11 | 2.03 | 1.43 | 0.24 | 6.66 |
| GJRsk | 1.90 | **1.36** | 1.87 | 0.00 | 7.25 | **1.61** | **1.52** | 1.07 | 0.00 | 4.25 |
| 1ST-GARCH | 2.34 | 1.72 | 2.00 | 0.43 | 8.02 | 1.99 | 2.00 | 1.18 | 0.48 | 4.68 |
| 2ST-GARCH | 2.21 | 1.52 | 1.64 | 0.68 | 7.20 | 1.94 | 1.67 | 0.90 | 0.79 | 3.93 |
| 2STsk-GARCH | **1.83** | **1.36** | 1.55 | 0.23 | 6.75 | 1.63 | **1.56** | 0.94 | 0.27 | 3.71 |
| EST-GARCH | 2.65 | 2.00 | 2.30 | 0.46 | 9.64 | 2.25 | 1.98 | 1.20 | 0.53 | 5.08 |
| DT-GARCH | 2.32 | 1.72 | 1.92 | 0.42 | 7.67 | 1.98 | 1.92 | 1.04 | 0.49 | 4.47 |
| | Mean | Med | Std | Min | Max(1) | Mean | Med | Std | Min | Max(3) |
| | Quadratic Loss | | | | | Absolute Loss | | | | |
| 5% | | | | | | | | | | |
| HS-ST | 16.54 | 14.95 | 6.06 | 10.56 | 36.91 | 15.47 | 14.11 | 6.76 | 11.11 | 41.91 |
| HS-LT | 9.97 | 8.78 | 4.04 | 5.64 | 20.07 | 8.65 | 8.32 | 1.79 | 6.06 | 12.44 |
| RM | 10.08 | 8.93 | 3.39 | 6.69 | 21.08 | 8.92 | 8.53 | 1.40 | 6.88 | 13.07 |
| G-n | 10.00 | 8.12 | 4.16 | 6.24 | 23.25 | 8.63 | 8.35 | 1.83 | 5.81 | 12.79 |
| G-t | 10.93 | 9.11 | 4.23 | 7.27 | 24.68 | 9.42 | 8.89 | 1.68 | 7.35 | 13.53 |
| G-sk | 9.30 | 7.77 | 4.31 | 4.95 | 24.18 | 8.15 | 7.80 | 1.94 | 5.49 | 13.97 |
| GJRn | **9.23** | **6.95** | 4.87 | 4.62 | 25.24 | **7.98** | **7.39** | 2.19 | 5.20 | 14.04 |
| GJRt | 10.00 | 8.78 | 4.95 | 5.15 | 26.31 | 8.65 | 8.04 | 2.13 | 5.79 | 14.57 |
| GJRsk | **8.86** | 7.74 | 4.86 | 4.40 | 25.60 | **7.65** | **6.96** | 2.18 | 4.90 | 14.27 |
| 1ST-GARCH | 10.70 | 8.46 | 5.77 | 5.84 | 29.92 | 8.94 | 8.58 | 2.59 | 5.95 | 16.06 |
| 2ST-GARCH | 10.79 | 9.03 | 4.40 | 6.76 | 24.57 | 9.25 | 8.78 | 1.81 | 7.31 | 13.33 |
| 2STsk-GARCH | 9.28 | **7.46** | 4.42 | 4.92 | 24.28 | 8.06 | 8.10 | 2.01 | 5.47 | 13.74 |
| EST-GARCH | 11.69 | 9.09 | 6.49 | 6.43 | 32.97 | 9.73 | 9.20 | 3.06 | 6.24 | 18.29 |
| DT-GARCH | 10.37 | 8.53 | 4.86 | 5.81 | 26.04 | 8.74 | 8.04 | 2.00 | 6.02 | 13.50 |

NOTE: The values in boldface indicate the best two models.
(1): All extreme quadratic losses occurred in the Greece market, except for "HS-LT" method.
(2): All extreme absolute losses occurred in the Greece market, except for "HS-LT" and RiskMetrics.
(3): All extreme absolute losses occurred in the Greece market, except for two "HS" methods.

and 5% levels, yielding the three lowest average loss values based on (16).

To evaluate the efficiency of risk measurement, Table 8 presents the results of Lopez's loss functions. The best model concerning smallest mean and median loss is highlighted in bold. Two models with skew Student's t errors, GJRsk, and 2STsk-GARCH models, are the best two over quadratic loss and absolute loss at the 1% level. It is clear that the choice of error distribution is highly important during this period. The best two models turn out to be 2STsk-GARCH, GJR-n, and GJR-sk at the 5% level. The family of GJR models is a good choice when we only consider risk at the 5% level.

Table 9 briefly describes the number of rejections for each model, over the 19 markets, at the 5% significance level for each of the three tests considered: the UC, CC, and the DQ tests. Four lags are used, as stated in Engle and Manganelli (2004) for the DQ test. The DQ statistic is evidently the most powerful test and rejects the most models in the most markets. The "Total" states are the number of markets rejected by any backtests under each model. Using the HS-ST method, all markets fail under the three backtests. At $\alpha = 1\%$ level, the HS-ST method, the HS-LT method, and the RM model are rejected in most of the markets, mainly by the DQ test. The 1ST-GARCH model and the DT-GARCH model have fewer rejections among tests in all the markets. At $\alpha = 5\%$ level, there are 10 models with only 0 or 1 rejections across markets.

*Table 9. Post-global financial crisis period: Counts of model rejections for three backtests across the 19 markets at the 5% significant level*

| | 1% | | | | 5% | | | |
|---|---|---|---|---|---|---|---|---|
| Model | UC | CC | $DQ_4$ | Total | UC | CC | $DQ_4$ | Total |
| HS-ST | 19 | 19 | 19 | 19 | 19 | 19 | 19 | 19 |
| HS-LT | 5 | 3 | 12 | 13 | 1 | 2 | 7 | 7 |
| RM | 9 | 2 | 14 | 15 | 0 | 2 | 4 | 4 |
| G-n | 5 | 0 | 7 | 9 | 0 | 0 | 0 | 0 |
| G-t | 3 | 1 | 4 | 6 | 1 | 1 | 0 | 1 |
| G-sk | 2 | 0 | 4 | 6 | 0 | 0 | 0 | 0 |
| GJRn | 4 | 2 | 3 | 5 | 0 | 0 | 0 | 0 |
| GJRt | 6 | 3 | 5 | 7 | 1 | 0 | 0 | 1 |
| GJRsk | 5 | 0 | 4 | 6 | 0 | 0 | 0 | 0 |
| 1ST-GARCH | 2 | 0 | 3 | 4 | 0 | 0 | 0 | 0 |
| 2ST-GARCH | 2 | 0 | 6 | 7 | 0 | 0 | 1 | 1 |
| 2STsk-GARCH | 2 | 0 | 4 | 6 | 0 | 0 | 1 | 1 |
| EST-GARCH | 3 | 2 | 4 | 5 | 2 | 2 | 3 | 3 |
| DT-GARCH | 4 | 0 | 2 | 4 | 0 | 0 | 0 | 0 |

Global financial crisis period (2007–2009)

We next consider the recent financial crisis period (2007–2009) as an out-of-sample forecast evaluation window. We also investigate the realized GARCH models proposed by Hansen, Huang and Shek (2012), where the daily returns and realized measure of volatility calculated using the intraday returns are jointly modelled. The realized measure of volatility calculated using the intraday returns may be subject to the bias caused by microstructure noise and non-trading hours. The realized GARCH model can adjust the bias in the realized measure.

As a realized measure of volatility, we use the realized kernel calculated by taking account of the bias caused by microstructure noise (Barndorff-Nielsen et al., 2008). The realized kernel of the S&P500 index is downloaded from the Oxford-Man Institute Realized Library (Heber et al., 2009) and that of the Nikkei 225 index is calculated using one-minute returns of the Nikkei 225 index obtained from the Nikkei NEEDS-tick data (Ubukata and Watanabe, 2014). We describe the realized GARCH model with three error distributions, given by the following three equations:

$$(17) \qquad y_t = \sigma_t \zeta_t, \quad \zeta_t \overset{iid}{\sim} D(0,1)$$

$$(18) \qquad \ln \sigma_t^2 = \omega + \beta \ln \sigma_{t-1}^2 + \gamma \ln x_{t-1}$$

$$(19) \qquad \ln x_t = \xi + \varphi \ln \sigma_{t-1}^2 + \tau(\zeta_t) + u_t,$$

where $y_t$ is the return, and $x_t$ is the realized kernel. In Equation (19), we utilize $\tau(\zeta_t) = \tau_1 \zeta_t + \tau_2(\zeta_t^2 - 1)$ to generate an asymmetric response in volatility to return shocks. Here, $\zeta_t \overset{iid}{\sim} D(0,1)$, $D(0,1)$ indicates a distribution that has mean 0 and variance 1, $u_t \overset{iid}{\sim} N(0, \sigma_u^2)$, and $\sigma_t^2 = \text{var}(y_t|\mathcal{F}_{t-1})$ with $\mathcal{F}_t = \sigma(y_t, x_t, y_{t-1}, x_{t-1}, \ldots)$.

Equation (19) is called a measurement equation, which relates the realized measure of volatility to the true volatility. If the realized measure were an unbiased estimator of the true volatility, then $\xi$ and $\varphi$ would be 0 and 1, respectively. Realized volatility, however, has a bias caused by microstructure noise and non-trading hours. Since we use the realized kernel, the bias caused by microstructure noise may be negligible. New York Stock Exchange and Tokyo Stock Exchange are open only for 6.5 hours and 5 hours, respectively, within a normal trading day, and our realized kernels are calculated using the intraday returns only when the market is open. Thus, we should expect $\xi < 0$ or $\varphi < 1$.

We use three error distributions for the i.i.d. disturbances in each RV-type model in Equation (17). The choice D(0,1) is a standard Gaussian and labelled as RV-n. The Student's t (RV-t) and skew Student's t (RV-sk) distributions need to be standardized to have unit variance. We take the classical estimator, employing the "rugarch" package in R software, for modelling and forecasting RV models in (17)–(19) (see Ghalanos 2014). To save space, we do not provide the parameter estimation for the RV models here, which are available from the authors upon request. However, we do observe that the estimated $\xi$ is significantly negative and $\varphi$ is significantly less than one for both U.S. and Japan stock markets. The estimate of $\tau_1$ is significantly negative for both stock markets, indicating a negative correlation between today's return and tomorrow's volatility.

We construct the results of VRate/$\alpha$ and three backtests in Table 10. In the global financial crisis forecast period, however, all models significantly underestimate risk levels at the 1% and 5% quantiles and no model could be recommended as being accurate. All VRate/$\alpha$ values are greater than 1. For the Japan market, the violation rates of G-n, GJR-sk, 2ST-GARCH, and 2STsk-GARCH reach a minimum value (equal to 1.5) at the 1% level. For the U.S. market, RV-sk has a minimum VRate at the 1% level. Clearly, during the financial turmoil period, a skew error distribution

*Table 10. Evaluating VaR prediction performance over the time period from January 2007 to December 2009 at the 1% and 5% levels*

| Method/Model | Japan | | | | U.S. | | | |
|---|---|---|---|---|---|---|---|---|
| | 1% VRate/$\alpha$ | UC | CC | DQ | 1% VRate/$\alpha$ | UC | CC | DQ |
| 1% | | | | | | | | |
| HS-ST | 4.17 | | | | 5.33 | | | |
| HS-LT | 2.50 | | | | 3.17 | | | |
| MR | 2.18 | | | | 2.80 | | | |
| G-n | 1.77 | √ | √ | | 3.34 | | | |
| G-t | 1.64 | √ | √ | | 1.87 | | √ | |
| G-sk | 1.50 | √ | √ | | 1.60 | √ | √ | |
| GJRn | 1.91 | | √ | | 3.20 | | | |
| GJRt | 1.64 | √ | √ | √ | 1.87 | | √ | |
| GJRsk | 1.50 | √ | √ | √ | 1.47 | √ | √ | √ |
| 1ST-GARCH | 2.18 | | √ | | 2.80 | | | |
| 2ST-GARCH | 1.50 | √ | √ | | 2.67 | | | |
| 2STsk-GARCH | 1.50 | √ | √ | | 2.00 | | | |
| EST-GARCH | 3.68 | √ | √ | | 3.74 | √ | | |
| DT | 1.77 | √ | √ | √ | 2.54 | | | |
| RV-n | 2.86 | | | | 2.94 | | | |
| RV-t | 2.73 | | | | 2.40 | | | |
| RV-sk | 2.46 | | | | 1.34 | √ | √ | √ |
| | 5% VRate/$\alpha$ | UC | CC | DQ | 5% VRate/$\alpha$ | UC | CC | DQ |
| 5% | | | | | | | | |
| HS-ST | 1.73 | | | | 1.90 | | √ | |
| HS-LT | 1.17 | √ | | | 1.40 | | √ | |
| MR | 1.58 | | | | 1.44 | | | |
| G-n | 1.64 | | | | 1.55 | | | |
| G-t | 1.64 | | | | 1.60 | | | |
| G-sk | 1.50 | | | | 1.47 | | | |
| GJRn | 1.45 | | | | 1.52 | | | |
| GJRt | 1.50 | | | √ | 1.50 | | | |
| GJRsk | 1.39 | | | | 1.42 | | | |
| 1ST-GARCH | 1.58 | | | | 1.44 | | √ | |
| 2ST-GARCH | 1.66 | | | | 1.60 | | | |
| 2STsk-GARCH | 1.56 | | | | 1.50 | | | |
| EST-GARCH | 2.02 | | | | 2.00 | | | |
| DT | 1.53 | | | | 1.60 | | | |
| RV-n | 2.18 | | | | 1.63 | | | |
| RV-t | 2.21 | | | | 1.60 | | | |
| RV-sk | 1.99 | | | | 1.39 | | | |

"√" indicates that we fail to reject $H_0$ at the 5% significance level.

with fat tails is very important to capture risk dynamics and level, at the 1% level, under a 1-day horizon. Most backtests for both markets are rejected at the 5% level. The measurements in Table 11 are based on Lopez's loss functions. It turns out that GJR-sk is the best model for the Japan market, while RV-sk and GJR-sk have the best performance for the U.S. market during financial turmoil periods.

Table 12 provides summary statistics of standard deviation ($\sqrt{h_t}$) forecasts based on two proxies and two loss functions across markets. As the loss functions are judged under MSE and MAD, we prefer the model with the smallest value. The performances of standard deviation forecasts and VaR are in contrast to one another. Apparently, the RiskMetrics model is suitable in regards to the lowest MSE and MAE values (GJRsk ranks in the top 2 in some cases), but the VaR forecasts for RiskMetrics is not the best among the risk models. VaR estimates depend much more on the choice of distribution than volatility estimates do. However, when comparing the performance of the 2ST-GARCH model and the rest of the models, the differences do not seem too large. We conclude that the 2ST-GARCH model is not excellent, but is still acceptable in volatility forecasting.

## 7. CONCLUSION

This study evaluates the performances of VaR forecasts across a range of competing parametric heteroskedastic models and non-parametric methods. We employ three vari-

*Table 11. Evaluating VaR estimates based on Lopez's loss functions over the time period from January 2007 to December 2009 at the 1% and 5% levels*

| | Quadratic Loss | | Absolute Loss | |
|---|---|---|---|---|
| | NK225 | SP500 | NK225 | SP500 |
| 1% | | | | |
| HS-ST | 14.19 | 12.41 | 8.41 | 9.79 |
| HS-LT | 16.82 | 7.69 | 6.75 | 5.75 |
| MR | 8.53 | 4.85 | 4.94 | 4.42 |
| G-n | 6.54 | 5.60 | 3.99 | 5.13 |
| G-t | 5.94 | 3.44 | 3.68 | 3.08 |
| G-sk | 5.17 | 2.84 | 3.33 | 2.56 |
| GJRn | 4.84 | 5.04 | 3.65 | 4.70 |
| GJRt | 4.04 | 3.20 | 3.09 | 2.92 |
| GJRsk | 3.47 | 2.48 | 2.73 | 2.26 |
| 1ST-GARCH | 4.31 | 4.56 | 3.49 | 4.14 |
| 2ST-GARCH | 5.75 | 4.21 | 3.57 | 3.98 |
| 2STsk-GARCH | 5.14 | 3.16 | 3.33 | 3.03 |
| EST-GARCH | 5.88 | 6.71 | 3.17 | 6.30 |
| DT | 6.35 | 4.28 | 5.45 | 4.05 |
| RV-n | 7.34 | 5.94 | 5.68 | 4.72 |
| RV-t | 8.47 | 4.55 | 7.26 | 3.73 |
| RV-sk | 7.39 | 2.81 | 6.54 | 2.26 |
| | Quadratic Loss | | Absolute Loss | |
| | NK225 | SP500 | NK225 | SP500 |
| 5% | | | | |
| HS-ST | 28.93 | 23.82 | 16.82 | 18.14 |
| HS-LT | 55.01 | 24.82 | 16.73 | 14.06 |
| MR | 26.32 | 17.11 | 15.13 | 13.61 |
| G-n | 24.12 | 19.25 | 15.66 | 14.89 |
| G-t | 25.72 | 19.77 | 15.98 | 15.34 |
| G-sk | 23.39 | 17.60 | 14.52 | 14.01 |
| GJRn | 20.15 | 17.17 | 13.86 | 13.90 |
| GJRt | 21.19 | 17.00 | 14.31 | 13.82 |
| GJRsk | 19.24 | 14.90 | 13.05 | 12.60 |
| 1ST-GARCH | 22.27 | 18.63 | 15.03 | 13.81 |
| 2ST-GARCH | 26.43 | 19.69 | 16.47 | 15.50 |
| 2STsk-GARCH | 24.20 | 17.53 | 15.14 | 14.28 |
| EST-GARCH | 24.82 | 25.71 | 15.40 | 19.51 |
| DT | 24.49 | 19.30 | 17.12 | 15.13 |
| RV-n | 31.80 | 18.46 | 20.88 | 14.29 |
| RV-t | 34.45 | 18.08 | 23.46 | 14.05 |
| RV-sk | 30.67 | 15.43 | 21.27 | 11.99 |

ant ST functions in order to capture the asymmetry in nonlinear, double threshold GARCH models. For a comparison, we also consider two popular asymmetric families: GJR-GARCH and DT-GARCHs models. We apply Bayesian MCMC methods on all heteroskedastic models (except the RV model) for estimation, inference, and forecasts. A simulation study shows that model parameters are well estimated for the 2ST-GARCH model with a different effect (smooth transition function) for the mean and variance and skew Student's t errors.

We evaluate two out-of-sample periods in light of the recent global financial crisis. For the post-global financial crisis period, our results suggest that Eurozone countries found it hard to avoid contagion from the Greek debt crisis. Except for Russia and UK markets, all models underestimate the extreme risk. In general, GARCH, GJR, and 2ST-GARCH with skew Student's t errors perform best at the 1% level based on Lopez's loss functions. The results show that the 2ST-GARCH model with skew Student's t errors is very apt for Russia, UK, Asia, and Americas markets at the 1% level during the post-global financial crisis period, when compared to a range of existing alternatives. Both DT-GARCH and EST-GARCH models have a favorable out-of-sample volatility forecasting performance. Based on the empirical application, we conclude that higher moments (skewness and kurtosis) need to be explicitly modeled in order to obtain better VaR predictions.

For the global financial crisis period, all risk models un-

*Table 12. Evaluation of volatility forecasting based on two proxies and two loss functions*

| | MSE | | | MAD | | |
|---|---|---|---|---|---|---|
| Proxy 1 | Mean | Med | Std | Mean | Med | Std |
| RM | 0.397 | 0.257 | 0.394 | 0.491 | 0.417 | 0.235 |
| G-n | 0.416 | 0.284 | 0.428 | 0.525 | 0.474 | 0.241 |
| G-t | 0.413 | 0.283 | 0.430 | 0.522 | 0.467 | 0.242 |
| G-sk | 0.405 | 0.277 | 0.427 | 0.514 | 0.453 | 0.242 |
| GJRn | 0.403 | 0.268 | 0.446 | 0.507 | 0.449 | 0.249 |
| GJRt | 0.404 | 0.279 | 0.448 | 0.507 | 0.463 | 0.249 |
| GJRsk | 0.400 | 0.272 | 0.448 | 0.504 | 0.459 | 0.249 |
| 1ST-GARCH | 0.427 | 0.317 | 0.480 | 0.525 | 0.482 | 0.256 |
| 2ST-GARCH | 0.413 | 0.282 | 0.425 | 0.521 | 0.457 | 0.239 |
| 2STsk-GARCH | 0.418 | 0.288 | 0.420 | 0.524 | 0.456 | 0.237 |
| EST-GARCH | 0.400 | 0.285 | 0.427 | 0.510 | 0.457 | 0.240 |
| DT-GARCH | 0.413 | 0.297 | 0.465 | 0.516 | 0.471 | 0.254 |
| Proxy 2 | Mean | Med | Std | Mean | Med | Std |
| RM | 0.375 | 0.228 | 0.393 | 0.460 | 0.388 | 0.238 |
| G-n | 0.386 | 0.247 | 0.433 | 0.486 | 0.425 | 0.247 |
| G-t | 0.383 | 0.240 | 0.436 | 0.483 | 0.418 | 0.248 |
| G-sk | 0.377 | 0.236 | 0.432 | 0.476 | 0.400 | 0.248 |
| GJRn | 0.374 | 0.230 | 0.451 | 0.469 | 0.415 | 0.255 |
| GJRt | 0.375 | 0.234 | 0.454 | 0.469 | 0.416 | 0.255 |
| GJRsk | 0.372 | 0.232 | 0.453 | 0.466 | 0.411 | 0.256 |
| 1ST-GARCH | 0.402 | 0.268 | 0.486 | 0.488 | 0.437 | 0.263 |
| 2ST-GARCH | 0.384 | 0.244 | 0.432 | 0.483 | 0.416 | 0.245 |
| 2STsk-GARCH | 0.389 | 0.256 | 0.426 | 0.486 | 0.417 | 0.243 |
| EST-GARCH | 0.384 | 0.239 | 0.436 | 0.479 | 0.431 | 0.247 |
| DT-GARCH | 0.382 | 0.248 | 0.471 | 0.475 | 0.423 | 0.261 |

derestimate the risk levels. We find that volatility asymmetry is most important for capturing risk, with skew errors also prominent, especially at the 1% level during the global financial period. In further works, we shall also focus on expected shortfalls - that is, the expected number on the worst side, under a given percentage, that is more sensitive than VaR.

The use of our proposed Bayesian forecasting of nonlinear ST models to deal with some complex derivatives and to calculate their corresponding VaR formulae is of practical importance and theoretical interests. The Bayesian approach provides risk traders with the flexibility of adjusting their VaR models according to their subjective opinions. The findings of this research contribute to a better understanding of the performance of Bayesian forecasting of VaR based on various nonlinear ST models and hence could help securities traders or commercial banks concerning valuating their risky portfolios.

## ACKNOWLEDGEMENTS

The authors thank the AE and referees for their precious time and valuable comments to improve the quality of this paper. Cathy W.S. Chen's research is funded by the Ministry of Science and Technology, Taiwan (MOST 103-2118-M-035-002-MY2 and MOST 104-2410-H-035-004). Toshiaki Watanabe's financial supports are from the Ministry of Education, Culture, Sports, Science and Technology of the Japanese Government through Grant-in-Aid for Scientific Research (No.25245034; 25245037; 26245028), Center for Financial Engineering Education at Hitotsubashi University, and the Hitotsubashi Institute for Advanced Study.

*Received 10 January 2016*

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