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G and related distributions with applications in reliability growth analysis

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Motivated by four unsolved issues on the *mean time between failures* (MTBFs) in nonhomogeneous Poisson processes (NHPP) with power law intensity function for complete/incomplete observations, in this article, we first study some important properties on three new distributions (i.e., the G, inverse G, and RG distributions). Next, we develop three methods (i.e., the Lagrange multiplier, quantile-based and sampling-based methods) to establish the shortest confidence intervals for the MTBF in a single repairable system and for the MTBF ratio in two independent repairable systems; and also develop two methods (i.e., the density-based and sampling-based methods) within the framework of the critical region and p-value approaches to test hypotheses on the MTBF and the MTBF ratio. Simulation studies are performed to compare the proposed methods. Two real data sets are used to illustrate the proposed statistical methods.

Keywords and phrases: G distribution, Hypothesis testing, Inverse G distribution, Nonhomogeneous Poisson process, RG distribution, Shortest confidence interval.

1. INTRODUCTION

Suppose that a reliability growth test on a repairable system is performed and the number of failures, denoted by $N(t)$, in the time interval $(0, t]$ is observed. Furthermore, we assume that $\{N(t), t > 0\}$ follows a *non-homogeneous Poisson process* (NHPP) with power law intensity function (Crow, [2]; [3])

$$\lambda(t) = \alpha\beta t^{\beta-1}, \quad \alpha,\ \beta > 0. \tag{1.1}$$

This process is also known as the Weibull process, the power–law process, or the Army Materiel Systems Analysis Activity (AMSAA) model in the literature (Crow, [2]; [3]; Crow & Basu, [4]). For the failure-truncated case, let $\{x_i\}_{i=1}^n$ be the successive failure times with n being predetermined. Further, we assume that $x_1, \ldots, x_{r-1} (1 \leqslant r < n)$ are missing (Crow & Basu, [4]; Yu *et al.*, [17], [18]) and the observed data are then denoted by $Y_{\rm obs}^{\rm ft} = \{x_i\}_{i=r}^n$, where "ft" stands for "failure-truncated" and $r = 1$ indicates no missing data. At the time of the n-th failure, x_n, the intensity of failure is $\lambda(x_n) = \alpha\beta x_n^{\beta-1}$. If improvements are not made to the system after time x_n, the failures would occur at the constant rate $\lambda(x_n)$ with further operation. Then, the subsequent times between failures of the system independently follow an exponential distribution with the common failure rate $\lambda(x_n)$. The *mean time between failures* (MTBF) of the system with further operation after x_n is defined by $M(x_n) = 1/\lambda(x_n)$ (Crow, [3]). Usually, $M(x_n)$ is called the achieved MTBF of the system and is estimated by $\hat{M}(x_n) = x_n^{1-\hat{\beta}}/(\hat{\alpha}\hat{\beta}) = x_n/(n\hat{\beta})$, where

$$\hat{\alpha} = n/x_n^{\hat{\beta}} \quad \text{and} \quad \hat{\beta} = \frac{n-r+1}{\sum_{i=r+1}^{n-1}\log(x_n/x_i) + r\log(x_n/x_r)} \tag{1.2}$$

are the *maximum likelihood estimates* (MLEs) of α and β, respectively (Yu *et al.*, [18]). For a given $t_0 > 0$, the current system reliability is defined by $R(t_0) = \exp\{-t_0/M(x_n)\}$, which is the reliability of the exponential distribution with mean $M(x_n)$. It is clear that $R(t_0)$ is a monotonic increasing function of $M(x_n)$. Thus, statistical inferences on $R(t_0)$ will follow if an optimal test or a confidence interval for $M(x_n)$ is available. In fact, after the data $\{x_i\}_{i=r}^n$ are collected, x_n is fixed and $M(x_n)$ is a function of the parameters α and β. Thus, we can discuss testing hypotheses and confidence intervals on $M(x_n)$ with a known x_n. From this viewpoint, the statistical inferences on $M(x_n)$ are of importance.

Let $W = 2(n-r+1)\beta/\hat{\beta}$ and $S = 2\alpha x_n^{\beta}$. Yu *et al.*[18] proved that $W \sim \chi^2(2n-2r)$, $S \sim \chi^2(2n)$,

$$W \times S = \frac{4n(n-r+1)\hat{M}(x_n)}{M(x_n)}, \tag{1.3}$$

and W is independent of S (denoted by $W \perp\!\!\!\perp S$). It is of great interest to consider the following two statistical issues related to $M(x_n)$:

Issue 1. To construct the shortest *confidence interval* (CI) of $M(x_n)$ for the given x_n;

Issue 2. For a given constant M_0, to consider the following hypothesis testing problem

$$H_0\colon M(x_n) = M_0 \quad \text{against} \quad H_1\colon M(x_n) \neq M_0. \tag{1.4}$$

To our best knowledge, even for the complete-observation case (corresponding to $r = 1$), Issues 1 and 2 are not addressed in the literature. For example, for the Weibull process with complete observations, Crow [3] only computed

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equal-tailed CIs for $M(x_n)$ with confidence coefficients 0.80, 0.90, 0.95 and 0.98 (see Table 1 of Crow [3]). For the Weibull process with incomplete observations, Yu, Tian and Tang [18] derived the equal-tailed CI for $M(x_n)$ for a given confidence coefficient. In general, the density function of $W \times S$ is very skew, hence, for a given confidence coefficient the equal-tailed CI of $M(x_n)$ is not the shortest. Next, for Issue 2, the traditional critical-region approach employs the *equal-tail* method (instead of the equal-height method), yielding an incorrect critical region; while the traditional p-value approach *approximately* calculates the p-value as

$$2 \times \min\{\Pr(Z_0 \leqslant Z_{0,\,\text{obs}}|H_0),\ \Pr(Z_0 \geqslant Z_{0,\,\text{obs}}|H_0)\}, \tag{1.5}$$

where $Z_{0,\,\text{obs}}$ denotes the observed value of the test statistic Z_0 defined by (5.10).

Now, we consider two independent repairable systems and assume their failures follow two different NHPPs with intensity functions $\lambda_1(t) = \alpha_1\beta_1 t^{\beta_1-1}$ and $\lambda_2(t) = \alpha_2\beta_2 t^{\beta_2-1}$, respectively. Furthermore, we assume that $x_1, \dots, x_{r_1-1} (1 \leqslant r_1 < n)$ and $y_1, \dots, y_{r_2-1} (1 \leqslant r_2 < m)$ are missing for Systems 1 and 2, respectively. For the failure-truncated case, their observed data are respectively denoted by $X_{\text{obs}}^{\text{ft}} = \{x_i\}_{i=r_1}^{n}$ and $Y_{\text{obs}}^{\text{ft}} = \{y_j\}_{j=r_2}^{m}$. In particular, when $r_1 = r_2 = 1$, these observations become complete observations. Let $M_1(x_n)$ and $M_2(y_m)$ denote the achieved MTBFs for Systems 1 and 2, respectively. For the purpose of comparison, we are interested in the following two issues:

Issue 3. To construct the shortest CI for the MTBF ratio $M_2(y_m)/M_1(x_n)$ for the given (x_n, y_m);

Issue 4. For a given constant ρ_0, to consider the following hypothesis testing problem

$$H_0'\colon \frac{M_2(y_m)}{M_1(x_n)} = \rho_0 \quad \text{against} \quad H_1'\colon \frac{M_2(y_m)}{M_1(x_n)} \neq \rho_0. \tag{1.6}$$

To our knowledge, even for the complete-observation case, solutions to Issues 3 and 4 are not yet available.

To address Issues 1–4 above, in this paper we will develop three methods (i.e., the Lagrange multiplier, quantile-based and sampling-based methods) to establish the shortest CIs for the MTBF in a single repairable system and for the MTBF ratio in two independent repairable systems; and also develop two methods (i.e., the density-based and sampling-based methods) within the framework of the critical region and p-value approaches to test hypotheses on the MTBF and the MTBF ratio. However, the implementation of the five methods involve three new distributions: (i) the distribution of the product of two independent chi-square random variables, which is called G distribution in this paper first time; (ii) the distribution of the reciprocal of a G random variable, which is called the inverse G distribution; and (iii) the distribution of the ratio of two independent G random variables, which is named as RG distribution. Therefore, the second objective of this paper is to study some important properties on the three distributions by paying special attention to the G distribution.

The distributions of the product $Z = XY$ have been investigated by several authors, especially when X and Y are independent random variables and come from the same distribution family. For example, with the operational method of Mellin transforms, Wells, Anderson and Cell [15] only derived the density function of the product of two independent random variables distributed according to non-central chi-squared distributions. Although a chi-squared distribution is a special case of a non-central chi-squared distribution, their derivation is rather complicated. Joarder [7] and Joarder and Omar [8] obtained the joint density of a bivariate chi-squared distribution with two correlated components, which reduces to the density of the product of two independent chi-square variables with the same degrees of freedom if the correlation coefficient is set to be zero. For the gamma distribution family, Stuart [12] proved that $Z \sim \text{Gamma}(q, 1)$ if $X \sim \text{Gamma}(p, 1)$ is independent of $Y \sim \text{Beta}(q, p-q)$. Withers and Nadarajah [16] studied the distribution of the product of two independent gamma variables. However, we note the following facts: (i) the motivation for these papers is mainly from the theoretical study rather than from the practical demand; (ii) several important properties (e.g., cumulative distribution function in accessible forms, the computation of the quantile, the calculation of the mode of the density) for the G distribution are not yet available. Therefore, there is a need to thoroughly investigate the G and related distributions.

The rest of the paper is organized as follows. Some basic notation and preliminary results are introduced in Section 2. In Sections 3 and 4, we study some important properties for the G, inverse G, and RG distributions, respectively. Statistical inferences on MTBFs for Issues 1–4 are developed in Section 5. Two real data sets are used to illustrate the proposed statistical methods in Section 6. In Section 7, we conduct several simulation studies to compare the proposed statistical methods. Finally, a discussion is presented in Section 8.

2. NOTATION AND PRELIMINARIES

The calculation in this paper involves several special functions, including the gamma function $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1}\mathrm{e}^{-t}\,\mathrm{d}t$ and the incomplete gamma function $\gamma(\alpha, x) = \int_0^{x} t^{\alpha-1}\mathrm{e}^{-t}\,\mathrm{d}t$. We use Laplace$(\mu, b)$ to denote the Laplace distribution with density $0.5b\exp(-b|x-\mu|)$, where $x \in \mathbb{R}$, $-\infty < \mu < \infty$ and $b > 0$. Gamma(α, β) denotes the gamma distribution with density $\beta^{\alpha}x^{\alpha-1}\mathrm{e}^{-\beta x}/\Gamma(\alpha)$, where $x \in \mathbb{R}_+$, $\alpha > 0$ and $\beta > 0$. In particular, Gamma$(1, \beta)$ = Exponential(β) and Gamma$(\nu/2, 1/2) = \chi^2(\nu)$. IBeta$(\alpha, \beta)$ denotes the inverted beta distribution with density $x^{\alpha-1}/[B(\alpha, \beta)(1+x)^{\alpha+\beta}]$, where $x \in \mathbb{R}_+$, $\alpha > 0$ and $\beta > 0$. If two random variables X and Y have the same distribution, we denote this by $X \stackrel{d}{=} Y$. The notation $X \perp\!\!\!\perp Y$

denotes that X is independent of Y. We also need the following important results.

Lemma 1. *(Waston, [14]). For any $v \in \mathbb{R}$, we have*

$$\int_0^{+\infty} t^{-(v+1)} \exp\left(-t - \frac{z^2}{4t}\right) \mathrm{d}t = 2\left(\frac{z}{2}\right)^{-v} K_v(z), \tag{2.1}$$

where $K_v(z)$ is the modified Bessel function of the second kind of order v (Gradshteyn et al., [6]), which can be calculated by the built-in R *function* `besselK`(z, ν).

Lemma 2. *The incomplete gamma function $\gamma(\alpha, x)$ can be expanded as the power series*

$$\gamma(\alpha, x) = x^{\alpha}\Gamma(\alpha)\, e^{-x} \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(\alpha + k + 1)}, \tag{2.2}$$

which can be calculated by the built-in R *function* `pgamma`$(x, \alpha) \times$ `gamma`(α).

Lemma 3. *(Waston, [14]). For any integer $k \geqslant 0$, we have*

$$K_{k+\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \sum_{r=0}^{k} \frac{(k+r)!}{r!(k-r)!(2z)^r}. \tag{2.3}$$

In particular, $K_{1/2}(z) = [\pi/(2z)]^{1/2} e^{-z}$.

Lemma 4. *(Equation (2.10.1.12), Prudnikov et al., [11], Vol. 2). For $r, p, c, v > 0$ and $\alpha < 0$, we have*

$$\begin{aligned} &\int_0^{+\infty} t^{\alpha-1} \exp(-pt^{-r})\gamma(v, ct)\, \mathrm{d}t \\ = \;& \frac{c^v p^{\frac{\alpha+v}{r}}}{r} \sum_{r=0}^{+\infty} \frac{\Gamma(-\frac{\alpha+v+k}{r})(-cp^{\frac{1}{r}})^k}{k!(k+v)} \\ &+ c^{-\alpha} \sum_{r=0}^{+\infty} \frac{\Gamma(\alpha+v-rk)(-c^r p)^k}{k!(rk-\alpha)}. \end{aligned} \tag{2.4}$$

3. THE G DISTRIBUTION

Motivated by (1.3) and Issues 1–2 in Section 1, in this section we study some important properties on the distribution of the product of two independent chi-squared random variables. Since the letter "G" comes after the letter "F", we call this distribution the G *distribution* by mimicking the F distribution (named after R. A. Fisher), which is the distribution of the ratio (up to a constant) of two independent chi-squared random variables. We also introduce the *inverse G distribution*, which plays a crucial role in the construction of the shortest CI for MTBF.

3.1 Definition

Definition 1. (G and inverse G distributions). Let $X \sim \chi^2(n)$, $Y \sim \chi^2(m)$, and $X \perp\!\!\!\perp Y$. (i) The distribution of $Z = XY$ is called G distribution with n and m degrees of freedom, denoted by $Z \sim G(n, m)$. (ii) The distribution of $T = 1/(XY)$ is called inverse G distribution with n and m degrees of freedom, denoted by $T \sim \mathrm{IG}(n, m)$. ‖

It is clear that $G(n, m) = G(m, n)$. Examples 1 and 2 below show the close relationship between the G distribution and Laplace/uniform distributions.

Example 1. (Connection with Laplace distribution). Let $X, Y \overset{\text{iid}}{\sim} \text{Laplace}(\mu, b)$, then

$$4b^2|X-\mu| \times |Y-\mu| \sim G(2,2).$$

In fact, since $|X-\mu|$, $|Y-\mu| \overset{\text{iid}}{\sim} \text{Gamma}(1, b)$, we have

$$2b|X-\mu|,\ 2b|Y-\mu| \overset{\text{iid}}{\sim} \text{Gamma}\left(1, \frac{1}{2}\right) = \chi^2(2).$$

By the definition of the G distribution, we obtain $4b^2|X-\mu| \times |Y-\mu| \sim G(2,2)$. ‖

Example 2. (Connection with uniform distribution). Let $X, Y \overset{\text{iid}}{\sim} U(0,1)$, then

$$4\log(X) \times \log(Y) \sim G(2,2).$$

In fact, since $-\log(X)$, $-\log(Y) \overset{\text{iid}}{\sim} \text{Gamma}(1,1)$, we have $-2\log(X)$, $-2\log(Y) \overset{\text{iid}}{\sim} \chi^2(2)$. By the definition of the G distribution, we have $4\log(X) \times \log(Y) \sim G(2,2)$. ‖

3.2 Density and cumulative distribution function

Lemma 5 below gives an explicit expression for the *probability density function* (pdf) of the G distribution $G(n, m)$ in terms of the modified Bessel function of the second kind.

Lemma 5. *The pdf of $Z \sim G(n, m)$ is given by*

$$f_Z(z; n, m) = \frac{z^{\frac{n+m}{4}-1} K_{\frac{n-m}{2}}(\sqrt{z})}{2^{\frac{n+m}{2}-1}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}, \quad z \geqslant 0, \tag{3.1}$$

where $K_v(z)$ defined by (2.1) is the modified Bessel function of the second kind of order v. ‖

Proof. Let $X \sim \chi^2(n)$, $Y \sim \chi^2(m)$, and $X \perp\!\!\!\perp Y$. Furthermore, let $f_X(x; n)$ and $f_Y(y; m)$ be the pdfs of X and Y, respectively. Then, the pdf of $Z = XY$ is given by

$$\begin{aligned} &f_Z(z; n, m) \\ = \;& \int_0^{+\infty} \frac{1}{y} f_X\!\left(\frac{z}{y}; n\right) f_Y(y; m)\, \mathrm{d}y \\ = \;& \frac{1}{2^{\frac{n+m}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})} \int_0^{+\infty} \frac{1}{y}\left(\frac{z}{y}\right)^{\frac{n}{2}-1} \exp\left(-\frac{z}{2y}\right) \end{aligned}$$

$$
\begin{aligned}
&\quad \cdot\, y^{\frac{m}{2}-1}\exp\left(-\frac{y}{2}\right)\mathrm{d}y\\
&= \frac{z^{\frac{n}{2}-1}}{2^{\frac{n+m}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}\int_0^{+\infty} y^{\frac{m}{2}-\frac{n}{2}-1}\\
&\quad \cdot\, \exp\left(-\frac{y}{2}-\frac{z}{2y}\right)\mathrm{d}y \qquad [\text{Let } y/2=t]\\
&= \frac{z^{\frac{n}{2}-1}}{2^{\frac{n+m}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}\int_0^{+\infty} (2t)^{\frac{m}{2}-\frac{n}{2}-1}\\
&\quad \cdot\, \exp\left(-t-\frac{z}{4t}\right)2\,\mathrm{d}t\\
&= \frac{z^{\frac{n}{2}-1}}{2^{n}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}\int_0^{+\infty} t^{-(\frac{n}{2}-\frac{m}{2}+1)}\\
&\quad \cdot\, \exp\left[-t-\frac{(\sqrt{z})^2}{4t}\right]\mathrm{d}t\\
&\overset{(2.1)}{=} \frac{z^{\frac{n}{2}-1}}{2^{n}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}2\left(\frac{\sqrt{z}}{2}\right)^{-(\frac{n}{2}-\frac{m}{2})}K_{\frac{n-m}{2}}(\sqrt{z})\\
&= \frac{z^{\frac{n+m}{4}-1}K_{\frac{n-m}{2}}(\sqrt{z})}{2^{\frac{n+m}{2}-1}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})},
\end{aligned}
$$

which implies (3.1). □

Remark 1. (i) Armed with the Mellin transforms, Wells, Anderson and Cell [15] derived the pdf of the product of two independent random variables distributed according to the non-central chi-squared distribution with non-central parameters Δ_1 and Δ_2 and degrees of freedom n and m, respectively. By simply setting $\Delta_1 = \Delta_2 = 0$, they obtained the pdf of the product of two independent chi-squared random variables, which is identical to (3.1). However, our proof to (3.1) is more straightforward. (ii) Note that the modified Bessel function of the second kind of order v satisfies $K_v(z) = K_{-v}(z)$ for any $v \in \mathbb{R}$, then the G distribution possesses the symmetry property; that is, $G(n,m)$ is identical to $G(m,n)$. ‖

Figure 1 shows four plots of the density of the G distribution with four different combinations of n and m. When $n = 1, 2$ or $m = 1, 2$ or $n+m < 8$, the density of $Z \sim G(n,m)$ is monotone decreasing; when $n, m \geqslant 3$ and $n + m \geqslant 8$, the density of $Z \sim G(n,m)$ is unimodal.

Next, we will derive the *cumulative distribution function* (cdf) of the G distribution for different cases. Theorem 1 below gives the cdf of the $G(n,m)$ distribution in terms of the modified Bessel function of the second kind for the general case. When $|n-m|$ is odd, two alternative expressions for the cdf of the $G(n,m)$ are provided by Theorem 2. Figure 2 shows plots of the cdf of the G distribution with four different combinations of n and m.

Theorem 1. *The cdf of $Z \sim G(n,m)$ is given by*

$$
(3.2)\qquad F_Z(z;n,m) = \frac{z^{\frac{n+m}{4}}}{2^{\frac{n+m}{2}-1}\Gamma(\frac{n}{2})}\sum_{k=0}^{+\infty}\frac{z^{\frac{k}{2}}K_{\frac{n-m}{2}+k}(\sqrt{z})}{2^k\Gamma(\frac{m}{2}+k+1)},
$$

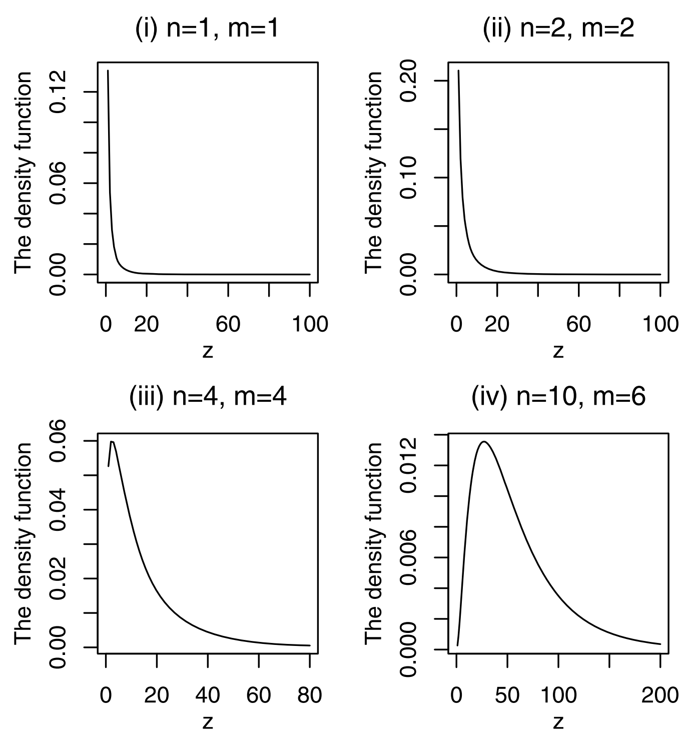

Figure 1. Plots of the pdf $f_Z(z;n,m)$ defined by (3.1) for the G distribution with different combinations of the degrees of freedom. (i) $n = m = 1$; (ii) $n = m = 2$; (iii) $n = m = 4$; (iv) $n = 10$, $m = 6$.

where $K_v(z)$ defined by (2.1) is the modified Bessel function of the second kind of order v. ‖

Proof. Let $X \sim \chi^2(n)$, $Y \sim \chi^2(m)$, and $X \perp\!\!\!\perp Y$. Furthermore, let $f_X(x;n)$ be the pdf of X and $F_Y(y;m) = \gamma(m/2, y/2)/\Gamma(m/2)$ be the cdf of Y. Then the cdf of $Z = XY$ is given by

$$
\begin{aligned}
&F_Z(z;n,m)\\
&= \int_0^{+\infty} f_X(x;n)\,F_Y\Big(\frac{z}{x};m\Big)\,\mathrm{d}x,\\
(3.3)&= \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}\int_0^{+\infty} x^{\frac{n}{2}-1}\mathrm{e}^{-\frac{x}{2}}\gamma\left(\frac{m}{2},\frac{z}{2x}\right)\mathrm{d}x\\
&\overset{(2.2)}{=} \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}\int_0^{+\infty} x^{\frac{n}{2}-1}\mathrm{e}^{-\frac{x}{2}}\left(\frac{z}{2x}\right)^{\frac{m}{2}}\Gamma\left(\frac{m}{2}\right)\\
&\quad \cdot\, \mathrm{e}^{-\frac{z}{2x}}\sum_{k=0}^{+\infty}\frac{\left(\frac{z}{2x}\right)^k}{\Gamma(\frac{m}{2}+k+1)}\mathrm{d}x\\
&= \frac{z^{\frac{m}{2}}}{2^{\frac{n+m}{2}}\Gamma(\frac{n}{2})}\sum_{k=0}^{+\infty}\frac{\left(\frac{z}{2}\right)^k}{\Gamma(\frac{m}{2}+k+1)}\\
&\quad \cdot \int_0^{+\infty} x^{\frac{n}{2}-\frac{m}{2}-k-1}\mathrm{e}^{-\frac{x}{2}-\frac{z}{2x}}\,\mathrm{d}x. \qquad [\text{Let } x/2=t]
\end{aligned}
$$

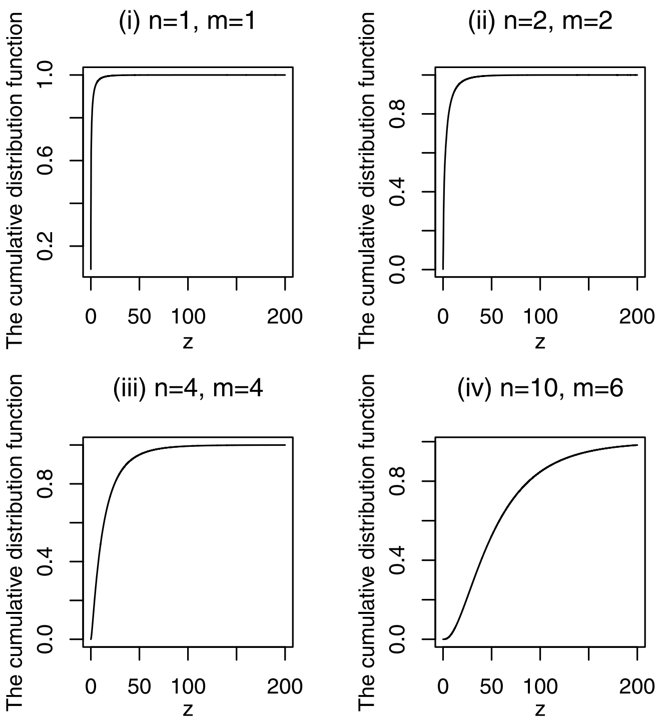

Figure 2. Plots of the cdf $F_Z(z;n,m)$ defined by (3.2) for the G distribution with different combinations of the degrees of freedom. (i) $n=m=1$; (ii) $n=m=2$; (iii) $n=m=4$; (iv) $n=10$, $m=6$.

$$
\begin{aligned}
&= \frac{z^{\frac{m}{2}}}{2^{\frac{n+m}{2}}\Gamma(\frac{n}{2})}\sum_{k=0}^{+\infty}\frac{\left(\frac{z}{2}\right)^k}{\Gamma(\frac{m}{2}+k+1)} \\
&\quad\cdot\int_0^{+\infty}(2t)^{\frac{n}{2}-\frac{m}{2}-k-1}\mathrm{e}^{-t-\frac{z}{4t}}2\,\mathrm{d}t \\
&= \frac{z^{\frac{m}{2}}}{2^m\Gamma(\frac{n}{2})}\sum_{k=0}^{+\infty}\frac{\left(\frac{z}{4}\right)^k}{\Gamma(\frac{m}{2}+k+1)} \\
&\quad\cdot\int_0^{+\infty}t^{-(\frac{m}{2}-\frac{n}{2}+k+1)}\mathrm{e}^{-t-\frac{(\sqrt{z})^2}{4t}}\,\mathrm{d}t \\
&\overset{(2.1)}{=} \frac{z^{\frac{m}{2}}}{2^m\Gamma(\frac{n}{2})}\sum_{k=0}^{+\infty}\frac{\left(\frac{z}{4}\right)^k}{\Gamma(\frac{m}{2}+k+1)}2\left(\frac{\sqrt{z}}{2}\right)^{\frac{m}{2}-\frac{n}{2}+k} \\
&\quad\cdot K_{\frac{m}{2}-\frac{n}{2}+k}(\sqrt{z}\,) \\
&= \frac{z^{\frac{n+m}{4}}}{2^{\frac{n+m}{2}-1}\Gamma(\frac{n}{2})}\sum_{k=0}^{+\infty}\frac{z^{\frac{k}{2}}K_{\frac{n-m}{2}+k}(\sqrt{z}\,)}{2^k\Gamma(\frac{m}{2}+k+1)},
\end{aligned}
$$

which implies (3.2). □

Theorem 2. *If $|n-m|$ is odd, i.e., there is a non-negative integer k such that $\frac{|n-m|}{2}=k+\frac{1}{2}$, then the cdf of $Z\sim G(n,m)$ can be expressed as a sum of finite terms*

$$
\begin{aligned}
(3.4)\qquad & F_Z(z;n,m) \\
&= \frac{\sqrt{\pi}}{2^{\frac{n+m-3}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})} \\
&\quad\times\sum_{r=0}^{k}\frac{(k+r)!}{r!(k-r)!2^r}\gamma\left(\frac{n+m-1}{2}-r,\sqrt{z}\right).
\end{aligned}
$$

Alternatively, the cdf can be expressed as a sum of infinite terms

$$
\begin{aligned}
(3.5)\qquad & F_Z(z;n,m) \\
&= \frac{z^{\frac{m}{2}}}{2^m\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}\sum_{k=0}^{+\infty}\frac{(-\frac{z}{4})^k\Gamma(\frac{n-m}{2}-k)}{k!(k+\frac{m}{2})} \\
&\quad+\frac{z^{\frac{n}{2}}}{2^n\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}\sum_{k=0}^{+\infty}\frac{(-\frac{z}{4})^k\Gamma(\frac{m-n}{2}-k)}{k!(k+\frac{n}{2})}.
\end{aligned}
$$

‖

Proof. From (3.1), we have

$$
\begin{aligned}
& F_Z(z;n,m) \\
&= \frac{1}{2^{\frac{n+m}{2}-1}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}\int_0^z t^{\frac{n+m}{4}-1} \\
&\quad\cdot K_{\frac{n-m}{2}}(\sqrt{t}\,)\,\mathrm{d}t \qquad [\text{Let } w=\sqrt{t}\,] \\
&= \frac{1}{2^{\frac{n+m}{2}-2}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}\int_0^{\sqrt{z}} w^{\frac{n+m}{2}-1} \\
&\quad\cdot K_{\frac{n-m}{2}}(w)\,\mathrm{d}w \qquad \left[\because\ \frac{|n-m|}{2}=k+\frac{1}{2}\right] \\
&\overset{(2.3)}{=} \frac{1}{2^{\frac{n+m}{2}-2}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}\int_0^{\sqrt{z}} w^{\frac{n+m}{2}-1}\left(\frac{\pi}{2w}\right)^{\frac{1}{2}}\mathrm{e}^{-w} \\
&\quad\cdot\sum_{r=0}^{k}\frac{(k+r)!}{r!(k-r)!(2w)^r}\,\mathrm{d}w \\
&= \frac{\sqrt{\pi}}{2^{\frac{n+m-3}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}\sum_{r=0}^{k}\frac{(k+r)!}{r!(k-r)!2^r} \\
&\quad\cdot\int_0^{\sqrt{z}} w^{\frac{n+m}{2}-r-\frac{3}{2}}\mathrm{e}^{-w}\,\mathrm{d}w \\
&= \frac{\sqrt{\pi}}{2^{\frac{n+m-3}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})} \\
&\quad\cdot\sum_{r=0}^{k}\frac{(k+r)!}{r!(k-r)!2^r}\gamma\left(\frac{n+m-1}{2}-r,\sqrt{z}\right),
\end{aligned}
$$

which implies (3.4).

In (3.3), after making the transformation $t=1/x$, we obtain

$$
(3.6)\qquad F_Z(z;n,m)=\frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}\int_0^{+\infty}t^{-\frac{n}{2}-1}\mathrm{e}^{-\frac{1}{2t}}\gamma\left(\frac{m}{2},\frac{z}{2}t\right)\mathrm{d}t.
$$

In (2.4) let $\alpha=-n/2$, $p=1/2$, $r=1$, $v=m/2$, $c=z/2$, then (3.6) becomes

$$\begin{aligned} & F_Z(z;n,m) \\ = & \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}\left[\left(\frac{z}{2}\right)^{\frac{m}{2}}\left(\frac{1}{2}\right)^{-\frac{n}{2}+\frac{m}{2}}\right. \\ & \cdot\sum_{k=0}^{+\infty}\frac{\Gamma(-(-\frac{n}{2}+\frac{m}{2}+k))(-\frac{z}{4})^k}{k!(k+\frac{m}{2})} \\ & \left.+\left(\frac{z}{2}\right)^{\frac{n}{2}}\sum_{k=0}^{+\infty}\frac{\Gamma(-\frac{n}{2}+\frac{m}{2}-k)(-\frac{z}{4})^k}{k!(k+\frac{m}{2})}\right] \\ = & \frac{z^{\frac{m}{2}}}{2^m\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}\sum_{k=0}^{+\infty}\frac{(-\frac{z}{4})^k\Gamma(\frac{n-m}{2}-k)}{k!(k+\frac{m}{2})} \\ & +\frac{z^{\frac{n}{2}}}{2^n\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}\sum_{k=0}^{+\infty}\frac{(-\frac{z}{4})^k\Gamma(\frac{m-n}{2}-k)}{k!(k+\frac{n}{2})}, \end{aligned}$$

which indicates (3.5). □

In practice, the cdf of the $G(n,m)$ distribution can be calculated by

$$F_Z(z;n,m)=\int_0^z f_Z(t;n,m)\,\mathrm{d}t,$$

where $f_Z(\cdot;n,m)$ is the pdf of the $G(n,m)$ distribution and is defined by (3.1). The built-in `R` function `integrate`$(f_Z(t;n,m),0,z)$ can facilitate the computation of the integral.

Remark 2. The α-th quantile of the $G(n,m)$ distribution, denoted by $G(\alpha;n,m)$, is the solution to the equation

$$h(z)\,\hat{=}\int_0^z f_Z(t;n,m)\,\mathrm{d}t-\alpha=0.$$

The built-in `R` function `uniroot`$(h(z),c(a,b))$ can be used to find the solution, where (a,b) is the isolation interval such that $h(a)\times h(b)<0$. A suggestion on choosing a and b is that we could set a small a and a large b. For example, let $a=0$, then $h(a)=h(0)=-\alpha<0$, then we only need to choose a large $b=1{,}000{,}000$, say, such that $h(b)>0$. ‖

3.3 Moments, mode and moment generating function

Other important distributional properties such as the k-th moment, the mode of the density, the skewness, kurtosis and *moment generating function* (mgf) of the G distribution are given in the following theorem.

Theorem 3. *Let $Z\sim G(n,m)$, we have the following results.*

(1) The k-th moment of Z is given by

$$E(Z^k)=\prod_{i=1}^{k}(n+2k-2i)(m+2k-2i),\quad k\geqslant 1. \tag{3.7}$$

In particular,

$$\begin{aligned} E(Z) &= nm,\quad E(Z^2)=(n+2)n(m+2)m,\quad and \\ Var(Z) &= 2nm(n+m+2). \end{aligned}$$

(2) When $n,m\geqslant 3$ and $n+m\geqslant 8$, the mode ω of the density of Z is determined by the equation

$$\begin{aligned} H(\omega) \ \hat{=}\ & (m-2)\omega^{\frac{n+m}{4}-2}K_{\frac{n-m}{2}}(\sqrt{\omega}\,) \\ & -\omega^{\frac{n+m}{4}-\frac{3}{2}}K_{\frac{n-m}{2}-1}(\sqrt{\omega}\,)=0. \end{aligned} \tag{3.8}$$

(3) The skewness of Z is

$$\gamma_1=\frac{4[(n+2)^2+(m+2)^2+3nm+2(n+m)]}{\sqrt{2nm}(n+m+2)^{\frac{3}{2}}}. \tag{3.9}$$

(4) The kurtosis of Z is

$$\begin{aligned} & \gamma_2 \\ = & \frac{(n+6)(n+4)(n+2)n(m+6)(m+4)(m+2)m}{4nm(n+m+2)^2} \\ & -3. \end{aligned} \tag{3.10}$$

(5) The mgf of Z is

$$\begin{aligned} & M_Z(t;n,m) \\ = & \frac{1}{\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{n}{2}+k)\Gamma(\frac{m}{2}+k)(4t)^k}{k!}. \end{aligned} \tag{3.11}$$

‖

Proof. (1) Using the moments of the chi-squared distribution with ν degrees of freedom, we immediately obtain the k-th moment of $Z\sim G(n,m)$, given by (3.7).

(2) When $n,m\geqslant 3$ and $n+m\geqslant 8$, the density of Z is unimodal. Let $f_Z(z;n,m)$ be the pdf of Z defined by (3.1). The mode ω of $f_Z(z;n,m)$ is determined by

$$\frac{\mathrm{d}f_Z(\omega;n,m)}{\mathrm{d}\omega}=0.$$

Using the following differential property of $K_v(z)$, i.e.,

$$\frac{\mathrm{d}K_v(\omega)}{\mathrm{d}\omega}=-K_{v-1}(\omega)-\frac{v}{\omega}K_v(\omega),$$

we immediately obtain (3.8).

(3) Let $\mu=E(Z)$ and $\sigma^2=\mathrm{Var}(Z)$, then

$$\begin{aligned} \gamma_1 &= E\left(\frac{Z-\mu}{\sigma}\right)^3=\frac{EZ^3-3\mu\sigma^2-\mu^3}{\sigma^3} \\ &= \frac{4[(n+2)^2+(m+2)^2+3nm+2(n+m)]}{\sqrt{2nm}(n+m+2)^{\frac{3}{2}}}. \end{aligned}$$

(4) Let $\mu_4=E(Z^4)$, then

$$\gamma_2 = \frac{\mu_4}{\sigma^4}-3$$

$$= \frac{(n+6)(n+4)(n+2)(m+6)(m+4)(m+2)}{4nm(n+m+2)^2} - 3.$$

(5) The mgf of Z is given by

$$\begin{aligned} & M_Z(t;n,m) \\ = \ & E(\mathrm{e}^{tZ}) = E\sum_{k=0}^{\infty}\frac{(tZ)^k}{k!} = \sum_{k=0}^{\infty}\frac{t^k E(Z)^k}{k!} \\ \overset{(3.7)}{=} \ & \frac{1}{\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{n}{2}+k)\Gamma(\frac{m}{2}+k)(4t)^k}{k!}. \end{aligned}$$

□

Remark 3. The built-in R function uniroot$(H(\omega), c(a,b))$ can be used to find the solution to the equation (3.8), where (a,b) is the isolation interval such that $H(a)\times H(b) < 0$. By this way, we can obtain the mode of the density of $Z\sim G(n,m)$ provided that it exists. ‖

4. THE DISTRIBUTION OF THE RATIO OF TWO INDEPENDENT G RANDOM VARIABLES

Motivated by Issues 3–4 in Section 1, in this section we study the distribution of the ratio of two independent G random variables.

Definition 2. (RG distribution). Let $Z_1\sim G(n_1,m_1)$, $Z_2\sim G(n_2,m_2)$, and $Z_1 \perp\!\!\!\perp Z_2$. The distribution of the ratio $V = Z_1/Z_2$ is called RG distribution with (n_1,m_1) and (n_2,m_2) degrees of freedom, denoted by $V\sim \mathrm{RG}(n_1,m_1;n_2,m_2)$. ‖

Let $V\sim\mathrm{RG}(n_1,m_1;n_2,m_2)$, then the RG distribution has the following properties:

(1) $\mathrm{RG}(m_1,n_1;m_2,n_2) = \mathrm{RG}(n_1,m_1;n_2,m_2)$;
(2) $V^{-1}\sim\mathrm{RG}(n_2,m_2;n_1,m_1)$; and
(3) $V \overset{d}{=} \left(\frac{n_1m_1}{n_2m_2}\right) W_1\times W_2$, where $W_1\sim F(n_1,n_2)$, $W_2\sim F(m_1,m_2)$, and $W_1\perp\!\!\!\perp W_2$.
(4) $V\overset{d}{=} U_1\times U_2$, where $U_1\sim \mathrm{IBeta}(n_1/2,n_2/2)$, $U_2\sim \mathrm{IBeta}(m_1/2,m_2/2)$, and $U_1\perp\!\!\!\perp U_2$.

Pham-Gia and Turkkan [10] investigated the distribution of the product of two independent generalized-F random variables. Since the inverted beta distribution is a special case of the generalized-F distribution, from the formula (9) in Pham-Gia and Turkkan [10], the pdf of V is given by

$$\begin{aligned} (4.1)\quad & f_V(v) \\ = \ & \frac{B(\frac{m_1+n_2}{2},\frac{n_1+m_2}{2})}{B(\frac{n_1}{2},\frac{n_2}{2})B(\frac{m_1}{2},\frac{m_2}{2})} v^{-(\frac{n_2}{2}+1)} \\ & \times F_D^{(2)}\left(\frac{m_1+n_2}{2};\frac{n_1+n_2}{2},\frac{m_1+m_2}{2}; \frac{n_1+n_2+m_1+m_2}{2};1-\frac{1}{v},0\right),\ v>0, \end{aligned}$$

where $F_D^{(2)}$ is Appell's first hypergeometric function, which can be calculated by the R package appell.

5. STATISTICAL INFERENCES ON MTBFS

In this section, we address Issues 1–4 arisen in Section 1.

5.1 Issue 1: shortest confidence interval for the MTBF $M(x_n)$

From (1.3) and the definition of the G distribution, we obtain

$$(5.1)\quad Z \hat{=} W\times S = \frac{4n(n-r+1)\hat{M}(x_n)}{M(x_n)} \sim G(2n-2r,2n).$$

Let $[L_1,\,L_2]$ be a $(1-\alpha)100\%$ CI of $M(x_n)$, then we have

$$\begin{aligned} (5.2)\quad 1-\alpha &= \Pr\{L_1\leqslant M(x_n)\leqslant L_2\} \\ &= \Pr\left\{\frac{4n(n-r+1)\hat{M}(x_n)}{L_2}\leqslant Z \leqslant \frac{4n(n-r+1)\hat{M}(x_n)}{L_1}\right\}. \end{aligned}$$

To derive the shortest CI of $M(x_n)$, we can in general use one of the three methods: The Lagrange multiplier method, the quantile-based method, and the sampling-based method.

5.1.1 The Lagrange multiplier method

In (5.2), let $a = 4n(n-r+1)\hat{M}(x_n)/L_2$ and $b = 4n(n-r+1)\hat{M}(x_n)/L_1$, then

$$[L_1,\,L_2] = \left[\frac{4n(n-r+1)\hat{M}(x_n)}{b},\ \frac{4n(n-r+1)\hat{M}(x_n)}{a}\right]$$

with width $L(a,b) = 4n(n-r+1)\hat{M}(x_n)(1/a-1/b)$. We want to minimize $L(a,b)$ subject to (5.2), i.e.,

$$\int_a^b f_Z(z;2n-2r,2n)\,\mathrm{d}z = 1-\alpha,$$

where $f_Z(z;2n-2r,2n)$ denotes the pdf of Z defined in (5.1). To use the Lagrange multiplier method, we define

$$l(a,b) = L(a,b) + \lambda\left\{\int_a^b f_Z(z;2n-2r,2n)\,\mathrm{d}z - 1 + \alpha\right\}.$$

Let $0 = \partial l(a,b)/\partial a = \partial l(a,b)/\partial b$, we obtain

$$(5.3)\quad a^2 f_Z(a;2n-2r,2n) = b^2 f_Z(b;2n-2r,2n).$$

On the other hand, $a = G(\alpha_1;2n-2r,2n)$ and $b = G(1-\alpha+\alpha_1;2n-2r,2n)$ for any $\alpha_1\in[0,\alpha]$, where $G(\alpha;2n-2r;2n)$ denotes the α-th quantile of the $G(2n-2r,2n)$ distribution. In Remark 2, we discussed the computation of $G(\alpha;2n-$

$2r; 2n)$. It is now clear that (5.3) is a one-dimensional nonlinear equation with argument $\alpha_1 \in [0, \alpha]$. Therefore, the solution α_1^* to (5.3) can be obtained by, e.g., the bisection method. Hence, the shortest CI of $M(x_n)$ with confidence level $1-\alpha$ is given by

(5.4)
$$[L_1,\, L_2] = \left[\frac{4n(n-r+1)\hat{M}(x_n)}{G(1-\alpha+\alpha_1^*; 2n-2r; 2n)},\ \frac{4n(n-r+1)\hat{M}(x_n)}{G(\alpha_1^*; 2n-2r; 2n)}\right].$$

The Lagrange multiplier method involves the computation of both the quantile and pdf of the $G(2n-2r, 2n)$ distribution. To avoid the computation of the pdf, we could consider the quantile-based method in the next subsection.

5.1.2 The quantile-based method

Note that (5.2) is equivalent to

$$\begin{aligned} \alpha_1 &= \Pr\{Z \leqslant G(\alpha_1; 2n-2r; 2n)\} \quad \text{and} \\ 1-\alpha+\alpha_1 &= \Pr\{Z \leqslant G(1-\alpha+\alpha_1; 2n-2r; 2n)\} \end{aligned}$$

for any $\alpha_1 \in [0, \alpha]$. Therefore, we obtain

(5.5)
$$[L_1,\, L_2] = \left[\frac{4n(n-r+1)\hat{M}(x_n)}{G(1-\alpha+\alpha_1; 2n-2r; 2n)},\ \frac{4n(n-r+1)\hat{M}(x_n)}{G(\alpha_1; 2n-2r; 2n)}\right].$$

In particular, when $\alpha_1 = \alpha/2$, (5.5) reduces to the $(1-\alpha)100\%$ equal-tail CI of $M(x_n)$. By using the grid method, we can numerically find

(5.6)
$$\alpha_1^* = \arg \min_{0 \leqslant \alpha_1 \leqslant \alpha} L(\alpha_1),$$

where

(5.7)
$$L(\alpha_1) = 4n(n-r+1)\hat{M}(x_n)\left\{\frac{1}{G(\alpha_1; 2n-2r, 2n)} - \frac{1}{G(1-\alpha+\alpha_1; 2n-2r, 2n)}\right\}$$

denotes the width of the CI (5.5). Hence, the shortest CI of $M(x_n)$ with confidence level $1-\alpha$ is given by (5.5) with $\alpha_1 = \alpha_1^*$.

We note that when the grid method is applied to (5.6), the optimization is rather time-consuming because the explicit expression for the quantile of the $G(2n-2r, 2n)$ distribution is not available. To avoid computing the quantile of the G distribution, we could consider the sampling-based method in the next subsection.

5.1.3 The sampling-based method

The random samples from a G distribution or an inverse G distribution can be easily obtained by sampling from two independent chi-squared distributions. From (5.1) and the definition of an inverse G distribution, we obtain

(5.8)
$$T \hat{=} \frac{1}{W \times S} = \frac{M(x_n)}{4n(n-r+1)\hat{M}(x_n)} \sim \mathrm{IG}(2n-2r, 2n).$$

We rewrite (5.2) as

$$\begin{aligned} 1-\alpha &= \Pr\{L_1 \leqslant M(x_n) \leqslant L_2\} \\ &= \Pr\left\{\frac{L_1}{4n(n-r+1)\hat{M}(x_n)} \leqslant T \leqslant \frac{L_2}{4n(n-r+1)\hat{M}(x_n)}\right\}. \end{aligned}$$

The sampling-based method for constructing the shortest CI of $M(x_n)$ is as follows:

Step 1. To generate $\{W^{(j)}\}_{j=1}^J \overset{\text{iid}}{\sim} \chi^2(2n-2r)$ and independently generate $\{S^{(j)}\}_{j=1}^J \overset{\text{iid}}{\sim} \chi^2(2n)$. Let $T^{(j)} = 1/[W^{(j)}S^{(j)}]$ for $j = 1, \ldots, J$, then $\{T^{(j)}\}_{j=1}^J \overset{\text{iid}}{\sim} \mathrm{IG}(2n-2r, 2n)$.

Step 2. Based on $\{T^{(j)}\}_{j=1}^J$, to construct the $(1-\alpha)100\%$ shortest CI $[L_1', L_2']$ for the quantity $T = M(x_n)/[4n(n-r+1)\hat{M}(x_n)]$.

Step 3. The $(1-\alpha)100\%$ shortest CI of $M(x_n)$ is then given by

$$\begin{aligned} [L_1,\, L_2] = \Big[&4n(n-r+1)\hat{M}(x_n)L_1', \\ &4n(n-r+1)\hat{M}(x_n)L_2'\Big]. \end{aligned}$$

It is noted that Step 2 is a crucial step for the sampling-based method. We wrote an `R` code to implement Step 2, which is illustrated as follows. For the purpose of demonstration, let $J = 1{,}000$ and $\alpha = 0.05$. We first sort the i.i.d. samples $\{T^{(j)}\}_{j=1}^J$ to obtain the order statistics $\{T_{(j)}\}_{j=1}^J$. Next, we have 50 CIs for the quantity T with 95% confidence level, i.e.,

$$[T_{(j)},\, T_{(950+j)}], \quad j = 1, \ldots, 50.$$

The 95% shortest CI for the T is $[T_{(j^*)}, T_{(950+j^*)}]$, where

(5.9)
$$T_{(950+j^*)} - T_{(j^*)} = \min_{1 \leqslant j \leqslant 50}\{T_{(950+j)} - T_{(j)}\}.$$

In fact, the 95% equal-tail CI for the quantity T is given by $[T_{(25)}, T_{(975)}]$.

Remark 4. (i) When $J \to \infty$, the $(1-\alpha)100\%$ shortest CI $[L_1', L_2']$ for the quantity T tends to the $(1-\alpha)100\%$ equal-height CI for the T. (ii) In particular, when the central area $1-\alpha \to 0$ (i.e., $\alpha \to 1$), the corresponding shortest CI $[L_1', L_2']$ approaches to a point, i.e., the mode of the density of the $\mathrm{IG}(2n-2r, 2n)$ distribution. ‖

5.2 Issue 2: testing hypotheses on $M(x_n)$

Consider to test the null hypothesis H_0 against the alternative hypothesis H_1 specified by (1.4). From (5.1), we know that Z is a pivotal quantity. The test statistic is defined by

$$Z_0 \hat{=} \frac{4n(n-r+1)\hat{M}(x_n)}{M_0}. \tag{5.10}$$

When H_0 is true, we have $Z_0 = Z \sim G(2n-2r, 2n)$. The critical region approach and the p-value approach will be employed.

5.2.1 The critical region approach

Since $\alpha = \alpha_1 + \alpha_2 = \Pr(Z_0 \leqslant k_1|H_0) + \Pr(Z_0 \geqslant k_2|H_0)$, the critical region is given by

$$\mathbb{C} = \{(x_r, \ldots, x_n)\colon Z_{0,\,\mathrm{obs}} \leqslant k_1 \ \text{ or } \ Z_{0,\,\mathrm{obs}} \geqslant k_2\}, \tag{5.11}$$

where $Z_{0,\,\mathrm{obs}}$ denotes the observed value of the test statistic Z_0 specified by (5.10),

$$k_1 = G(\alpha_1; 2n-2r, 2n) \quad \text{and} \quad k_2 = G(1-\alpha+\alpha_1; 2n-2r, 2n). \tag{5.12}$$

It is clear that finding k_1 and k_2 is equivalent to finding α_1. In what follows, two methods are used to determine the α_1 or the k_1 and k_2.

(a) The density-based method. We first consider the density-based method. Again, let $f_Z(z; 2n-2r, 2n)$ denote the pdf of the $G(2n-2r, 2n)$ distribution. The equal-height approach means that k_1 and k_2 should satisfy $f_Z(k_1; 2n-2r, 2n) = f_Z(k_2; 2n-2r, 2n)$, so α_1 should satisfy

$$\begin{aligned} & f_Z(G(\alpha_1; 2n-2r, 2n); 2n-2r, 2n) \\ = \ & f_Z(G(1-\alpha+\alpha_1; 2n-2r, 2n); 2n-2r, 2n). \end{aligned} \tag{5.13}$$

Using the grid method, we can find the solution α_1 to the equation (5.13) as demonstrated in Section 5.1.2.

(b) The sampling-based method. Alternatively, the sampling-based method for finding k_1 and k_2 can be described as follows:

Step 1. To generate $\{W^{(j)}\}_{j=1}^J \overset{\text{iid}}{\sim} \chi^2(2n-2r)$ and independently generate $\{S^{(j)}\}_{j=1}^J \overset{\text{iid}}{\sim} \chi^2(2n)$. Let $Z^{(j)} = W^{(j)}S^{(j)}$ for $j = 1, \ldots, J$, then $\{Z^{(j)}\}_{j=1}^J \overset{\text{iid}}{\sim} G(2n-2r, 2n)$.

Step 2. Based on $\{Z^{(j)}\}_{j=1}^J$, to compute the $(1-\alpha)100\%$ shortest (or equal-height) CI $[k_1, k_2]$ for the quantity $Z = 4n(n-r+1)\hat{M}(x_n)/M(x_n)$.

5.2.2 The p-value approach

As shown in Figure 7, the p-value for testing H_0 versus H_1 in (1.4) can be calculated by

$$p = p_1 + p_2 = \Pr(Z_0 \leqslant b_1|H_0) + \Pr(Z_0 \geqslant b_2|H_0), \tag{5.14}$$

where $Z_0|H_0 \sim G(2n-2r, 2n)$, b_1 or b_2 is the observed value $Z_{0,\,\mathrm{obs}}$ of the test statistic Z_0 specified by (5.10), and b_1 and b_2 satisfy

$$f_Z(b_1; 2n-2r, 2n) = f_Z(b_2; 2n-2r, 2n). \tag{5.15}$$

In what follows, two methods are used to determine b_1 and b_2.

(a) The density-based method. We first consider the density-based method. Let ω denote the mode of the pdf of $G(2n-2r, 2n)$. If $Z_{0,\,\mathrm{obs}} < \omega$, let $b_1 = Z_{0,\,\mathrm{obs}}$ (see Figure 7). Then, we can determine b_2 by solving (5.15) subject to $b_2 > \omega$. If $Z_{0,\,\mathrm{obs}} > \omega$, let $b_2 = Z_{0,\,\mathrm{obs}}$. Then, we can determine b_1 by solving (5.15) subject to $0 < b_1 < \omega$.

(b) The sampling-based method. Next, we consider the sampling-based method. Based on the i.i.d. samples $\{Z^{(j)}\}_{j=1}^J$ from $G(2n-2r, 2n)$, we can find an estimated density of the true density $f_Z(z; 2n-2r, 2n)$ by a kernel density smoother, denoted by $\tilde{f}_Z(z; 2n-2r, 2n)$. If $Z_{0,\,\mathrm{obs}} < \omega$, let $b_1 = Z_{0,\,\mathrm{obs}}$. Then, we can determine b_2 by solving

$$\tilde{f}_Z(b_1; 2n-2r, 2n) = \tilde{f}_Z(b_2; 2n-2r, 2n) \tag{5.16}$$

subject to $b_2 > \omega$. The p-value in (5.14) can be approximated by

$$p = p_1 + p_2 \approx \frac{1}{J}\sum_{j=1}^J I(Z^{(j)} \leqslant b_1) + \frac{1}{J}\sum_{j=1}^J I(Z^{(j)} \geqslant b_2), \tag{5.17}$$

where $I(\cdot)$ is the indicator function. Similarly, we can deal with the case of $Z_{0,\,\mathrm{obs}} > \omega$.

It is noted that the key for the sampling-based method is how to find the mode ω based on the i.i.d. samples $\{Z^{(j)}\}_{j=1}^J$ and find the b_2 from the equation (5.16). We wrote an `R` code to find the ω, b_2 and the p-value in (5.17), which is illustrated as follows.

- First, using the built-in `R` function `density`$(\{Z^{(j)}\}_{j=1}^J, K)$, we can obtain K pairs of x- and y-coordinate of the estimated density $\tilde{f}_Z(\cdot; 2n-2r, 2n)$, denoted by $\{ds\$x_k, ds\$y_k\}_{k=1}^K$, where K is the number of equally spaced points at which the density is to be estimated. Usually, K is a power of two and the default value is 512.
- Second, we can find the index k^* such that $ds\$y_{k^*} = \max_{1 \leqslant k \leqslant K} ds\$y_k$. Thus, the mode $\omega \approx ds\$x_{k^*}$.
- Third, we can find the index k_1 $(k_1 < k^*)$ such that $ds\$x_{k_1}$ is the nearest x-coordinate to b_1 and the index k_2 $(k_2 > k^*)$ such that $ds\$y_{k_2}$ is the nearest y-coordinate to $ds\$y_{k_1}$. Thus, $b_2 \approx ds\$y_{k_2}$.

5.3 Issue 3: shortest confidence interval for $M_2(y_m)/M_1(x_n)$

From (1.3), we have

$$\frac{4n(n-r_1+1)\hat{M}_1(x_n)}{M_1(x_n)} \sim G(2n-2r_1, 2n),$$

$$\frac{4m(m-r_2+1)\hat{M}_2(y_m)}{M_2(y_m)} \quad \sim \quad G(2m-2r_2, 2m),$$

and they are independent. According to the definition of the RG distribution, we obtain

$$(5.18)\quad V \hat{=} \frac{c\,M_2(y_m)}{M_1(x_n)} \sim \mathrm{RG}(2n-2r_1, 2n;\ 2m-2r_2, 2m),$$

where

$$(5.19)\qquad c = \frac{n(n-r_1+1)\hat{M}_1(x_n)}{m(m-r_2+1)\hat{M}_2(y_m)}.$$

The pdf of V has the form of (4.1). Since the pdf (4.1) involves Appell's first hypergeometric function, it is rather difficult in practice to compute its quantiles. As we saw in Section 5.1, both the Lagrange multiplier and the quantile-based methods involve the computation of quantiles, therefore, only the sampling-based method is considered to construct the shortest CI for $M_2(y_m)/M_1(x_n)$.

Let $[L_3,\ L_4]$ be a $(1-\alpha)100\%$ CI of $M_2(y_m)/M_1(x_n)$, then we have

$$1-\alpha = \Pr\left\{L_3 \leqslant \frac{M_2(y_m)}{M_1(x_n)} \leqslant L_4\right\} = \Pr(cL_3 \leqslant V \leqslant cL_4).$$

The sampling-based method for constructing the shortest CI of $M_2(y_m)/M_1(x_n)$ is as follows:

Step 1. To generate $\{Z_1^{(j)}\}_{j=1}^J \overset{\text{iid}}{\sim} G(2n-2r_1, 2n)$ and independently generate $\{Z_2^{(j)}\}_{j=1}^J \overset{\text{iid}}{\sim} G(2m-2r_2, 2m)$. Let $V^{(j)} = Z_1^{(j)}/Z_2^{(j)}$ for $j = 1, \ldots, J$, then $\{V^{(j)}\}_{j=1}^J \overset{\text{iid}}{\sim} \mathrm{RG}(2n-2r_1, 2n;\ 2m-2r_2, 2m)$.

Step 2. Based on $\{V^{(j)}\}_{j=1}^J$, to construct the $(1-\alpha)100\%$ shortest CI $[L_3', L_4']$ for the quantity $V = cM_2(y_m)/M_1(x_n)$.

Step 3. The $(1-\alpha)100\%$ shortest CI of $M_2(y_m)/M_1(x_n)$ is $[L_3,\ L_4] = [L_1'/c,\ L_4'/c]$.

5.4 Issue 4: testing hypotheses on $M_2(y_m)/M_1(x_n)$

Consider to test the null hypothesis H_0' against the alternative hypothesis H_1' specified by (1.6). From (5.18), we know that V is a pivotal quantity. The test statistic is defined by

$$(5.20)\qquad V_0 \hat{=} c\rho_0,$$

where c is defined by (5.19). When H_0' is true, we have $V_0 = V \sim \mathrm{RG}(2n-2r_1, 2n; 2m-2r_2, 2m)$. The critical region approach and the p-value approach will be employed.

5.4.1 The critical region approach

Since $\alpha = \alpha_1 + \alpha_2 = \Pr(V_0 \leqslant k_1|H_0') + \Pr(V_0 \geqslant k_2|H_0')$, the critical region is given by

$$(5.21)\qquad \mathbb{C} = \{(x_{r_1}, \ldots, x_n; y_{r_2}, \ldots, y_m)\colon V_{0,\,\text{obs}} \leqslant k_1 \ \text{ or }\ V_{0,\,\text{obs}} \geqslant k_2\},$$

where $V_{0,\,\text{obs}}$ denotes the observed value of the test statistic V_0 specified by (5.20). The sampling-based method for finding k_1 and k_2 can be described as follows:

Step 1. To generate $\{V^{(j)}\}_{j=1}^J \overset{\text{iid}}{\sim} \mathrm{RG}(2n-2r_1, 2n;\ 2m-2r_2, 2m)$.

Step 2. Based on $\{V^{(j)}\}_{j=1}^J$, to compute the $(1-\alpha)100\%$ shortest (or equal-height) CI $[k_1, k_2]$ for the quantity $V = cM_2(y_m)/M_1(x_n)$.

5.4.2 The p-value approach

As shown in Figure 12, the p-value for testing H_0' versus H_1' in (1.6) can be calculated by

$$(5.22)\qquad p = p_1 + p_2 = \Pr(V_0 \leqslant b_1|H_0') + \Pr(V_0 \geqslant b_2|H_0'),$$

where $V_0|H_0' \sim \mathrm{RG}(2n-2r_1, 2n;\ 2m-2r_2, 2m)$. Based on the samples

$$\{V^{(j)}\}_{j=1}^J \overset{\text{iid}}{\sim} \mathrm{RG}(2n-2r_1, 2n;\ 2m-2r_2, 2m),$$

we can compute the mode ω of the density and obtain an approximate density of the distribution $\mathrm{RG}(2n-2r_1, 2n; 2m-2r_2, 2m)$ similar to the process as shown in Section 5.2.2. We denoted the approximate density by $\tilde{f}_V(v; 2n-2r_1, 2n;\ 2m-2r_2, 2m)$.

If $V_{0,\,\text{obs}} < \omega$, let $b_1 = V_{0,\,\text{obs}}$. Then, we can determine b_2 by solving

$$(5.23)\qquad \begin{aligned} &\tilde{f}_V(b_1; 2n-2r_1, 2n;\ 2m-2r_2, 2m) \\ =\ &\tilde{f}_V(b_2; 2n-2r_1, 2n;\ 2m-2r_2, 2m) \end{aligned}$$

subject to $b_2 > \omega$. The p-value in (5.22) can be approximated by

$$(5.24)\qquad p \approx \frac{1}{J}\sum_{j=1}^J I(V^{(j)} \leqslant b_1) + \frac{1}{J}\sum_{j=1}^J I(V^{(j)} \geqslant b_2).$$

If $V_{0,\,\text{obs}} > w$, let $b_2 = V_{0,\,\text{obs}}$. Then, we can determine b_1 by solving (5.23) subject to $0 < b_1 < \omega$. The corresponding p-value is still given by (5.24).

6. ILLUSTRATIONS

In this section, we illustrate the proposed statistical methods for Issues 1–4 based on the developed properties for G and related distributions by two real data sets.

Table 1. The equal-tailed and shortest CIs for $M(x_n) = M(x_{40}) = M(8063)$

Method	Type of CI	α_1	Lower bound	Upper bound	Width
Quantile-based	Equal-tailed CI	0.02500	202.4210	501.0324	298.6187
	Shortest CI	0.01084	188.4262	476.7423	288.3161
Sampling-based	Equal-tailed CI	0.02500	202.5701	501.4049	298.8348
	Shortest CI	0.01090	188.6817	477.2435	288.5618

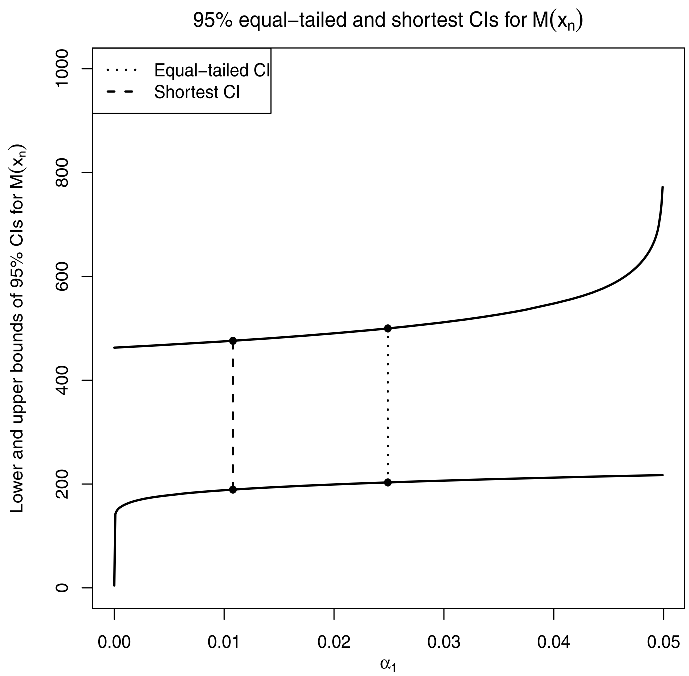

Figure 3. The lower and upper bounds of 95% CIs for $M(x_n)$ obtained by the quantile-based method versus α_1. The equal-tailed and shortest CIs are denoted by the dotted and dashed lines, respectively.

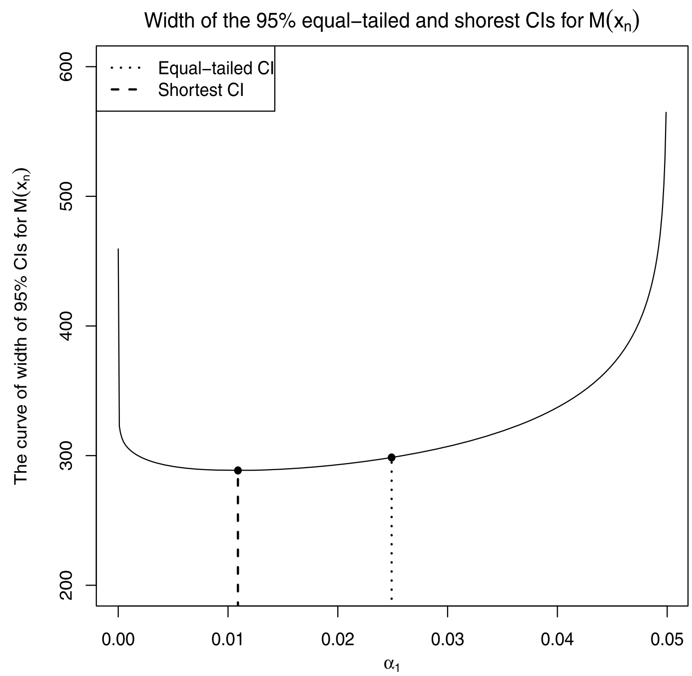

Figure 4. The width of 95% CIs for $M(x_n)$ obtained by the quantile-based method versus α_1.

6.1 Engine failure data

Yu, Tian and Tang [18] proposed likelihood-based inference and prediction methods for a NHPP with incomplete observations. The engine failure data set described by Zhou and Weng ([19], p. 51–52) is used to illustrate their methodologies. A total of 40 failure times in the time interval (0, 8063 h] for some engine undergoing development testing were reported as follows: $*$, $*$, $*$, 171, 234, 274, 377, 530, 533, 941, 1074, 1188, 1248, 2298, 2347, 2347, 2381, 2456, 2456, 2500, 2913, 3022, 3038, 3728, 3873,4724, 5147, 5179, 5587, 5626, 6824, 6983, 7106, 7106, 7568, 7568, 7593, 7642, 7928, 8063 h, where the symbol $*$ represents that the exact failure time of the failure is unknown. With a goodness-of-fit testing method, Yu, Tian and Tang [18] showed that the data are coming from a NHPP with power law intensity function (1.1) at the 0.95 confidence level. Now, $n = 40$ and $r = 4$. The MLE of the achieved MTBF at $x_{40} = 8,063$ is $\hat{M}(8063) = 298.15$.

6.1.1 Shortest confidence interval for MTBF $M(x_n)$

We first use the quantile-based method introduced in Section 5.1.2. The 95% equal-tailed and shortest CIs for $M(x_n) = M(x_{40}) = M(8063)$ are reported in the second and third rows of Table 1. Figure 3 shows lower and upper bounds of 95% CIs for $M(x_n)$ versus α_1. Figure 4 shows the width of 95% CIs for $M(x_n)$ versus α_1. When $\alpha_1 = 0.01084$, the width of the corresponding 95% CI for $M(x_n)$ is the shortest.

Next, we use the sampling-based method by drawing i.i.d. samples $\{T^{(j)}\}_{j=1}^{J}$ from $\mathrm{IG}(72, 80)$ as shown in Step 1 of Section 5.1.3, where the sample size is $J =$ 10,000,000. The 95% equal-tailed CI and shortest CIs for $M(8063)$ are shown in the fourth and fifth rows of Table 1. From Table 1, we found that the results derived by the quantile- and sampling-based methods are almost identical. Figure 5 shows lower and upper bounds of 95% CIs for $M(x_n)$ versus α_1. Figure 6 shows the estimated density function of the $\mathrm{IG}(72, 80)$ distribution and the 95% equal-tailed and shortest CIs for $M(x_n)$ obtained by the sampling-based method.

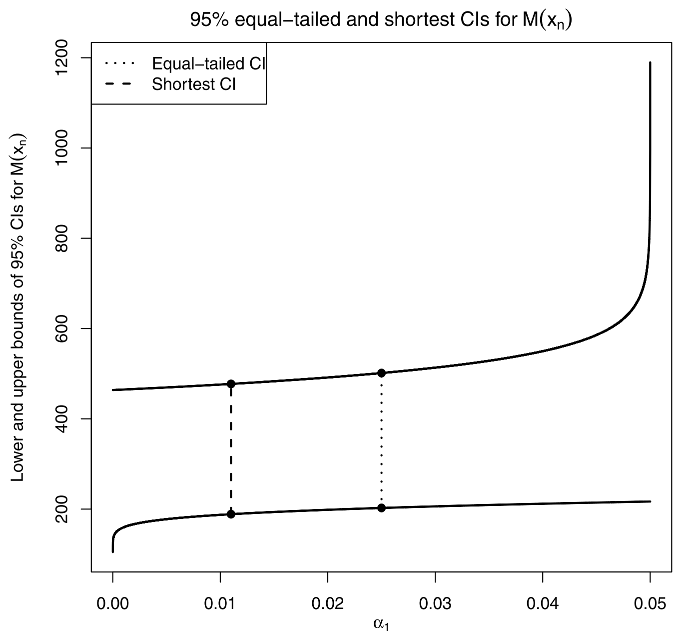

Figure 5. The lower and upper bounds of 95% CIs for $M(x_n)$ obtained by the sampling-based method versus α_1. The equal-tailed and shortest CIs are denoted by the dotted and dashed lines, respectively.

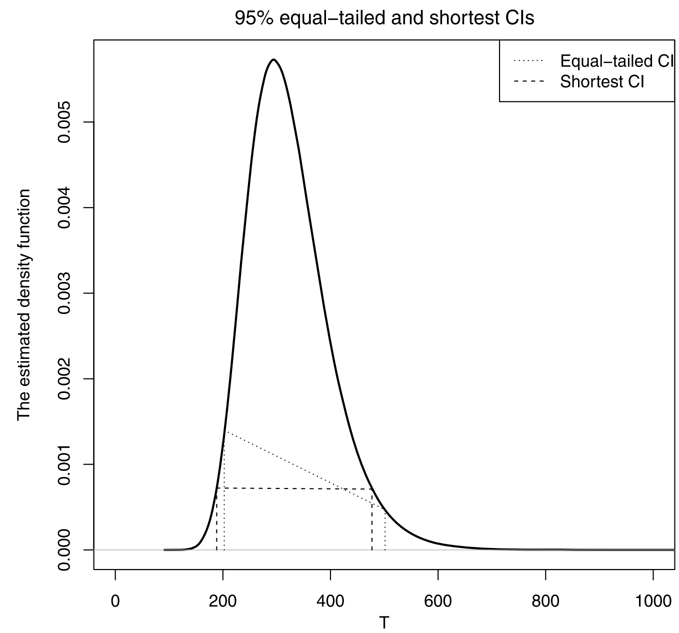

Figure 6. The estimated density function of the IG$(72, 80)$ distribution and the 95% equal-tailed and shortest CIs for $M(x_n)$ obtained by the sampling-based method.

When $\alpha_1 = 0.01090$, the width of the corresponding 95% CI for $M(x_n)$ is the shortest.

6.1.2 Testing hypothesis on $M(x_n)$ by the critical region approach

Consider to test the null hypothesis

(6.1)

$$H_0\colon M(x_n) = M_0 = 200 \quad \text{against} \quad H_1\colon M(x_n) \neq M_0$$

by the density-based method presented in Section 5.2.1(a) at the 0.05 level of significance. Using the grid method, from (5.13), we obtain $\alpha_1 = 0.011397$. From (5.12), we have $k_1 = 3251.067$ and $k_2 = 8360.43$. Since $Z_{0,\,\text{obs}} = 8825.24 > 8360.43 = k_2$, from (5.11), we reject the null hypothesis H_0 at the 0.05 level of significance.

Now, we consider the same hypothesis testing problem in (6.1) by the sampling-based method. With $J = 10{,}000{,}000$, we generate i.i.d. samples $\{Z^{(j)}\}_{j=1}^J$ from $G(72, 80)$. From Step 2 in Section 5.2.1(b), we obtain $k_1 = 3321.5815072$ and $k_2 = 8422.4377422$ with $\alpha_1 = \Pr(Z_0 \leqslant k_1|H_0) \approx (1/J)\sum_{j=1}^J I(Z^{(j)} \leqslant k_1) = 0.01409$. Therefore, the null hypothesis H_0 is rejected at the 0.05 level of significance.

6.1.3 Testing hypothesis on $M(x_n)$ by the p-value approach

We consider the same hypothesis testing problem in (6.1) by the density-based method presented in Section 5.2.2(a) at the 0.05 level of significance. By solving the equation (3.8), we can obtain the mode of the density function of the $G(72, 80)$ distribution, i.e., $\omega = 5386.71$. In addition, we observe that $Z_{0,\,\text{obs}} = 8825.24 > \omega$. Hence, $b_2 = 8825.24$. From (5.15), we have $b_1 = 3145.1847 < \omega$. From (5.14), we have

$$\begin{aligned} p &= p_1 + p_2 \\ &= \Pr(Z_0 \leqslant 3145.1847|H_0) + \Pr(Z_0 \geqslant 8825.24|H_0) \\ &\approx 0.008028815 + 0.02175188 = 0.0297807 < 0.05, \end{aligned}$$

which implies that the H_0 must be rejected. Figure 7 shows the density function of $G(72, 80)$ and p-value for the hypothesis testing problem in (6.1) by the density-based method.

Now, consider the same hypothesis testing problem in (6.1) by the sampling-based method presented in Section 5.2.2(b). Based on the i.i.d. samples $\{Z^{(j)}\}_{j=1}^J$ from $G(72, 80)$ with $J = 10{,}000{,}000$, we find the mode $\omega = 5346.6732799 < 8825.24 = Z_{0,\,\text{obs}}$. Let $b_2 = 8825.24$, we can determine b_1 by solving (5.16) subject to $0 < b_1 < \omega$ and obtain $b_1 = 3142.0088478$. Thus, from (5.17), we have $p = p_1 + p_2 \approx 0.0079030 + 0.0217985 = 0.0297015 < 0.05$, implying that the H_0 must be rejected. Figure 8 gives the estimated density function of $G(72, 80)$ and p-value for the hypothesis testing problem in (6.1) by the sampling-based method.

6.2 Electron failure data

The following life data from two electron systems are given by Crow [2] and Bain [1]. Assume that the two systems are observed in the time interval (0, 200 h], and their successive failure times are

Table 2. The equal-tailed and shortest CIs for $M_2(y_m)/M_1(x_n) = M_2(190.8)/M_1(197.2)$

Type of CI	α_1	Lower bound	Upper bound	Width
Equal-tailed CI	0.02500	0.1793489	1.926728	1.747379
Shortest CI	0.00221	0.1004948	1.620268	1.519774

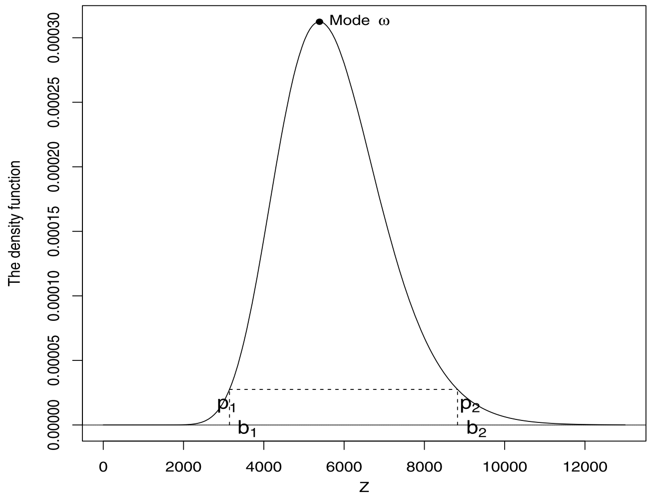

Figure 7. The density function of the $G(72, 80)$ distribution and the p-value for the hypothesis testing problem in (6.1) obtained by the density-based method.

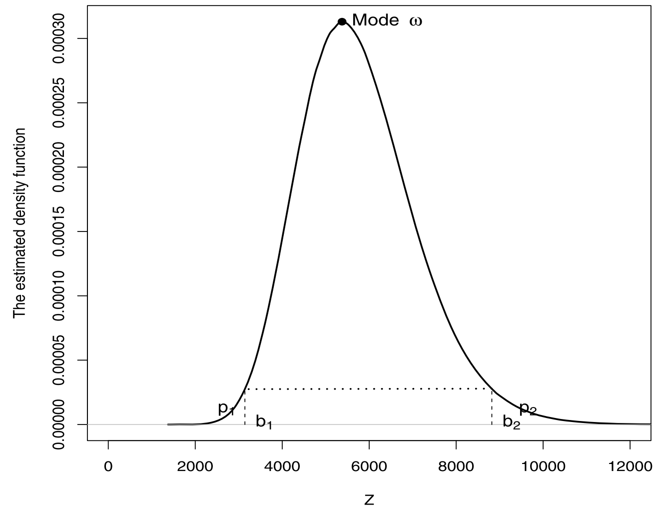

Figure 8. The estimated density function of the $G(72, 80)$ distribution and the p-value for the hypothesis testing problem in (6.1) obtained by the sampling-based method.

System 1:	4.3,	4.4,	10.2,	23.5,	23.8,
	26.4,	74.0,	77.1,	92.1,	197.2;
System 2:	0.1,	5.6,	18.6,	19.5,	24.2,
	26.7,	45.1,	45.6,	75.7,	79.7,
	98.6,	120.1,	161.8,	180.6,	190.8.

Crow [2] showed that the two life data sets obey the same NHPP with the intensity function (1.1). The numbers of the failures for the first and second systems are $n = 10$ and $m = 15$, respectively. Since these observations are complete, we have $r_1 = r_2 = 1$. The MLE of the achieved MTBF for the first system at $x_{10} = 197.2$ is $\hat{M}_1(x_{10}) = 38.77768$ and the MLE of the achieved MTBF for the second system at $y_{15} = 190.8$ is $\hat{M}_2(y_{15}) = 25.01278$.

6.2.1 Shortest confidence interval for $M_2(y_m)/M_1(x_n)$

We first generate i.i.d. samples $\{V^{(j)}\}_{j=1}^{J}$ with $J = 10{,}000{,}000$ from $\mathrm{RG}(18, 20; 28, 30)$ as shown in Step 1 of Section 5.3. Figure 9 shows the histogram of the samples $\{V^{(j)}\}_{j=1}^{J}$, indicating that the density of the $\mathrm{RG}(18, 20; 28, 30)$ distribution is unimodal.

Based on the i.i.d. samples $\{V^{(j)}\}_{j=1}^{J}$, the 95% equal-tailed and shortest CIs for the MTBF ratio $M_2(y_m)/M_1(x_n) = M_2(190.8)/M_1(197.2)$ are reported in Table 2. Figure 10 shows lower and upper bounds of 95% CIs for $M_2(y_m)/M_1(x_n)$ obtained by the sampling-based method versus α_1. Figure 11 shows the estimated density function of $\mathrm{RG}(18, 20; 28, 30)$, the equal-tailed and shortest CIs for $M_2(190.8)/M_1(197.2)$. When $\alpha_1 = 0.00221$, the width of the corresponding CI for $M_2(190.8)/M_1(197.2)$ is shortest, as shown in Figures 10 and 11.

6.2.2 Testing hypothesis on $M_2(y_m)/M_1(x_n)$ by the critical region approach

Consider the following hypothesis testing problem

$$H_0'\colon \frac{M_2(y_m)}{M_1(x_n)} = \rho_0 = 1 \quad \text{against} \quad H_1'\colon \frac{M_2(y_m)}{M_1(x_n)} \neq \rho_0 \tag{6.2}$$

by the critical region approach with the sampling-based method introduced in Section 5.4.1 at the 0.05 level of significance. With $J = 10{,}000{,}000$, we generate i.i.d. samples $\{V^{(j)}\}_{j=1}^{J}$ from $\mathrm{RG}(18, 20; 28, 30)$, and obtain $k_1 = 0.1004626$ and $k_2 = 1.6224078$ with $\alpha_1 = \Pr(V_0 \leqslant k_1 | H_0') \approx (1/J)\sum_{J=1}^{J} I(V^{(j)} \leqslant k_1) = 0.00219$. We calculate the observed value of the test statistic V_0 as $V_{0,\,\mathrm{obs}} = 0.6890287$. Since $k_1 < V_{0,\,\mathrm{obs}} < k_2$, we cannot reject the null hypothesis H_0'.

6.2.3 Testing hypothesis on $M_2(y_m)/M_1(x_n)$ by the p-value approach

Now, we consider the same hypothesis testing problem in (6.2) by the p-value approach with the sampling-based method introduced in Section 5.4.2 at the 0.05 level

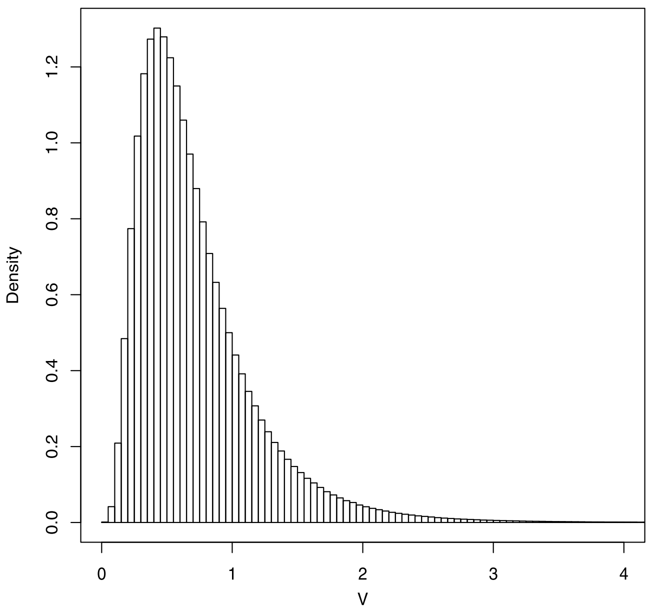

Figure 9. Histogram of the i.i.d. samples $\{V^{(j)}\}_{i=1}^{J}$ *with* $J =$ *10,000,000 from RG*$(18, 20; 28, 30)$.

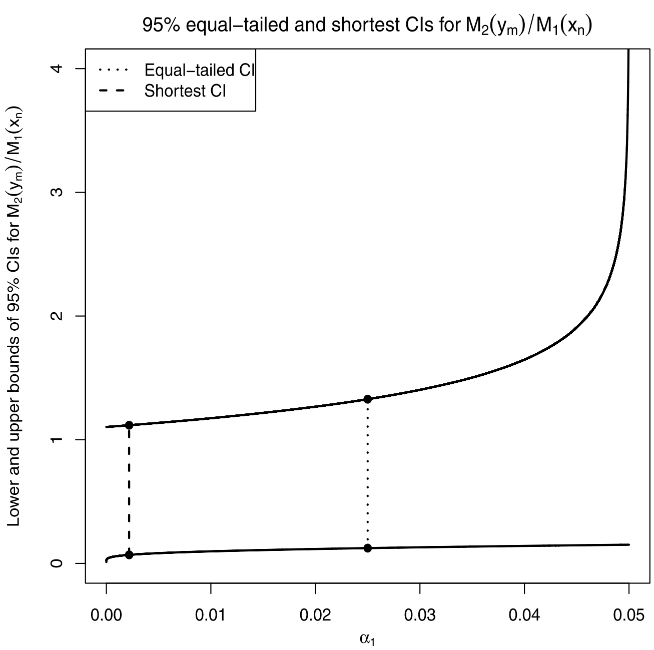

Figure 10. The lower and upper bounds of 95% CIs for $M_2(y_m)/M_1(x_n)$ *obtained by the sampling-based method versus* α_1*. The equal-tailed and shortest CIs are denoted by the dotted and dashed lines, respectively.*

of significance. Based on the i.i.d. samples $\{V^{(j)}\}_{j=1}^{J}$ with $J = 10{,}000{,}000$ from $\text{RG}(18, 20; 28, 30)$, we find mode as $\omega = 0.4350391$. Since $V_{0,\,\text{obs}} = 0.6890287 > \omega$, we let $b_2 = 0.6890287$. Then, we can determine b_1 by solving (5.23) subject to $0 < b_1 < \omega$ and obtain $b_1 = 0.2494748$. From

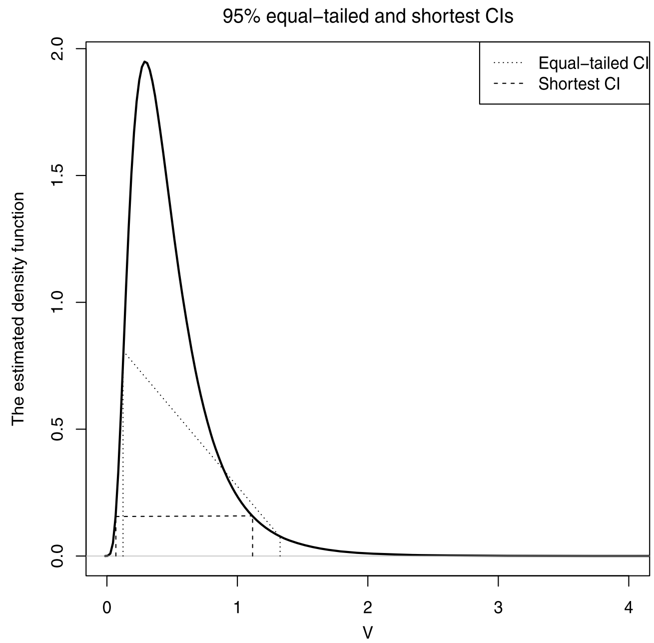

Figure 11. The estimated density function of the RG$(18, 20; 28, 30)$ *distribution, the 95% equal-tailed and shortest CIs for* $M_2(y_m)/M_1(x_n)$ *obtained by the sampling-based method.*

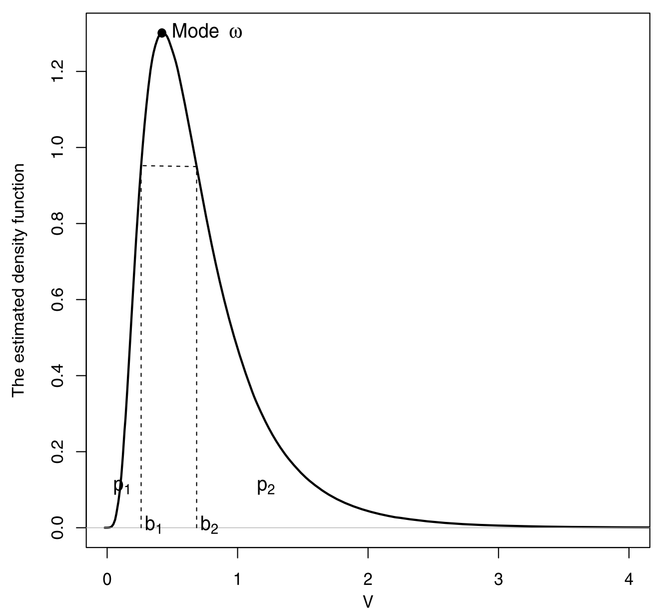

Figure 12. The estimated density function of the RG$(18, 20; 28, 30)$ *distribution and the* p*-value obtained by the sampling-based method.*

(5.24), the p-value is given by $p = p_1 + p_2 \approx 0.4119490 + 0.0751200 = 0.4870690 > 0.05$. Hence, the H_0' cannot be rejected. Figure 12 gives the estimated density function of $\text{RG}(18, 20; 28, 30)$ and p-value obtained by the sampling-based method.

Table 3. The average CP, average CIW and time of computation for the equal-tailed and shortest CIs of $M(x_n)$ by the quantile-based and sampling-based methods

$(\alpha,\beta)=(0.5,1)$	Method	Type of CI	CP	CIW	Time
$(n,r)=(20,1)$	Quantile-based	Equal-tailed CI	0.9385	2.7969	57139.502
		Shortest CI	0.9357	2.6201	
	Sampling-based	Equal-tailed CI	0.9398	2.9249	40225.770
		Shortest CI	0.9331	2.7381	
$(n,r)=(30,4)$	Quantile-based	Equal-tailed CI	0.9393	2.2449	58532.575
		Shortest CI	0.9354	2.1428	
	Sampling-based	Equal-tailed CI	0.9420	2.3369	39820.835
		Shortest CI	0.9363	2.2295	
$(n,r)=(40,2)$	Quantile-based	Equal-tailed CI	0.9484	1.8498	61583.070
		Shortest CI	0.9465	1.7888	
	Sampling-based	Equal-tailed CI	0.9467	1.7995	38481.816
		Shortest CI	0.9440	1.7394	
$(\alpha,\beta)=(0.2,0.8)$	Method	Type of CI	CP	CIW	Time
$(n,r)=(20,1)$	Quantile-based	Equal-tailed CI	0.9354	29.2953	55266.837
		Shortest CI	0.9294	27.4425	
	Sampling-based	Equal-tailed CI	0.9400	27.2867	40835.591
		Shortest CI	0.9331	25.5442	
$(n,r)=(30,4)$	Quantile-based	Equal-tailed CI	0.9390	24.0813	63615.449
		Shortest CI	0.9337	22.9880	
	Sampling-based	Equal-tailed CI	0.9389	24.8602	40654.883
		Shortest CI	0.9325	23.7186	
$(n,r)=(40,2)$	Quantile-based	Equal-tailed CI	0.9426	22.3758	60419.262
		Shortest CI	0.9406	21.6385	
	Sampling-based	Equal-tailed CI	0.9478	22.0755	40077.030
		Shortest CI	0.9437	21.3385	

NOTE: CP = Average coverage probability; CIW = Average confidence interval widths; Time = The “elapsed time” in `R` in seconds.

7. SIMULATION STUDIES

In this section, several simulation studies are performed to compare the proposed methods.

7.1 Comparison of CIs for the MTBF $M(x_n)$

Let $X_1,\ldots,X_n$ be the arrival times of the NHPP with intensity function $\lambda(t)$ specified by (1.1), denoted by $X_1,\ldots,X_n \sim \text{NHPP}(\alpha,\beta)$. Let $x_1,\ldots,x_n$ be the corresponding realizations of $X_1,\ldots,X_n$. For the failure-truncated case, the first $r-1$ observations $x_1,\ldots,x_{r-1}$ are missing, so the observed data are denoted by $Y_{\text{obs}}^{\text{ft}} = \{x_r,\ldots,x_n\}$. Based on $Y_{\text{obs}}^{\text{ft}}$, the MLE of the achieved MTBF at x_n is

$$\hat{M}(x_n) = x_n^{1-\hat{\beta}}/(\hat{\alpha}\hat{\beta}), \tag{7.1}$$

where $(\hat{\alpha},\hat{\beta})$ are MLEs of (α,β) given by (1.2), and the resulting CIs of $M(x_n)$ can be obtained by using the proposed three methods (Lagrange multiplier, quantile-based and sampling-based). Since the Lagrange multiplier method involves the computation of both the quantile and pdf, we only compare the quantile-based method with the sampling-based method introduced in Sections 5.1.2 and 5.1.3 for constructing the equal-tailed and shortest CIs of $M(x_n)$.

In the first experiment, we set $(\alpha,\beta) = (0.5,1),(0.2,0.8)$ and $(n,r) = (20,1),(30,4),(40,2)$. For a given combination of (α,β,n,r),

Step 1. We generate $X_1,\ldots,X_n \sim \text{NHPP}(\alpha,\beta)$ and obtain $Y_{\text{obs}}^{\text{ft}} = \{x_r,\ldots,x_n\}$. Based on $Y_{\text{obs}}^{\text{ft}}$, we calculate the MLEs $(\hat{\alpha},\hat{\beta})$ according to (1.2).

Step 2. We generate $X_1^*,\ldots,X_n^* \sim \text{NHPP}(\hat{\alpha},\hat{\beta})$ and obtain $Y_{\text{obs}}^{\text{ft}*} = \{x_r^*,\ldots,x_n^*\}$. Based on $Y_{\text{obs}}^{\text{ft}*}$, we calculate the MLEs $(\hat{\alpha}^*,\hat{\beta}^*)$ according to (1.2), the MLE of the achieved MTBF at x_n as $\hat{M}^*(x_n) = x_n^{1-\hat{\beta}^*}/(\hat{\alpha}^*\hat{\beta}^*)$, and four CIs of $M(x_n) = x_n^{1-\beta}/(\alpha\beta)$.

Step 3. By repeating Step 2 for L ($L = 100$) times, we can obtain the empirical *coverage probability* (CP) and the average *confidence interval width* (CIW).

Step 4. Finally, by repeating Steps 1–3 for G ($G = 100$) times, we can obtain the average CP and the average CIW.

The resulting average CP and average CIW are reported in Table 3. From Table 3, we can see that the shortest CI of $M(x_n)$ has a smaller coverage probability than the equal-

Table 4. The average CP and average CIW and time of computation for the equal-tailed and shortest CIs of the MTBF ratio $M_2(y_m)/M_1(x_n)$ by the sampling-based method with $(\alpha_1, \beta_1; \alpha_2, \beta_2) = (0.5,\ 1;\ 0.2, 1.5)$

True values	Method	Type of CI	CP	CIW	Time
$(n, r_1; m, r_2) = (40, 3; 30, 2)$	Sampling-based	Equal-tailed CI	0.9356	0.5022	57997.121
		Shortest CI	0.9330	0.4731	
$(n, r_1; m, r_2) = (30, 1; 30, 1)$	Sampling-based	Equal-tailed CI	0.9383	0.5658	58197.716
		Shortest CI	0.9368	0.5310	

NOTE: CP = Average coverage probability; CIW = Average confidence interval widths; Time = The "elapsed time" in R in seconds.

tailed CI since the former has a narrower width. In most situations, the CIs calculated by the sampling-based method have a relative larger coverage probability than those calculated by the quantile-based method, while the quantile-based method is much more time-consuming compared with the sampling-based method.

7.2 Comparison of CIs for the MTBF ratio

Let $X_1, \ldots, X_n \sim \text{NHPP}(\alpha_1, \beta_2)$, $Y_1, \ldots, Y_m \sim \text{NHPP}(\alpha_2, \beta_2)$, and they are independent. Let $\{x_i\}_{i=1}^n$ be the corresponding realizations of $\{X_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^m$ be the corresponding realizations of $\{Y_j\}_{j=1}^m$. For the failure-truncated case, the first $r_1 - 1$ observations $\{x_i\}_{i=1}^{r_1-1}$ for System 1 are missing and the first $r_2 - 1$ observations $\{y_j\}_{j=1}^{r_2-1}$ for System 2 are also missing, so the observed data are denoted by $Y_{\text{obs}}^{\text{ft}} = \{Y_{\text{obs},1}^{\text{ft}}, Y_{\text{obs},2}^{\text{ft}}\}$, where $Y_{\text{obs},1}^{\text{ft}} = \{x_i\}_{i=r_1}^n$ and $Y_{\text{obs},2}^{\text{ft}} = \{y_j\}_{j=r_2}^m$. Based on $Y_{\text{obs}}^{\text{ft}}$, the MLE of the MTBF ratio $M_2(y_m)/M_1(x_n)$ is $\hat{M}_2(y_m)/\hat{M}_1(x_n)$, where

(7.2)
$$\hat{M}_1(x_n) = x_n^{1-\hat{\beta}_1}/(\hat{\alpha}_1\hat{\beta}_1), \qquad \hat{M}_2(y_m) = y_m^{1-\hat{\beta}_2}/(\hat{\alpha}_2\hat{\beta}_2),$$

and $(\hat{\alpha}_1, \hat{\beta}_1; \hat{\alpha}_2, \hat{\beta}_2)$ are respective MLEs of $(\alpha_1, \beta_1; \alpha_1, \beta_1)$ given by (1.2) similarly. The resulting equal-tailed and shortest CIs of the MTBF ratio $M_2(y_m)/M_1(x_n)$ can be obtained by using the proposed sampling-based method in Section 5.3.

In the second experiment, we set $(\alpha_1, \beta_1; \alpha_2, \beta_2) = (0.5,\ 1;\ 0.2, 1.5)$ and $(n, r_1; m, r_2) = (40, 3; 30, 2)$, $(30, 1; 30, 1)$. For a given combination of $(\alpha_1, \beta_1, n, r_1; \alpha_2, \beta_2, m, r_2)$,

Step 1. We generate $X_1, \ldots, X_n \sim \text{NHPP}(\alpha_1, \beta_1)$ and independently generate $Y_1, \ldots, Y_m \sim \text{NHPP}(\alpha_2, \beta_2)$. Based on $Y_{\text{obs}}^{\text{ft}} = \{x_{r_1}, \ldots, x_n;\ y_{r_2}, \ldots, y_m\}$, we calculate the MLEs $(\hat{\alpha}_1, \hat{\beta}_1;\ \hat{\alpha}_2, \hat{\beta}_2)$ according to (1.2).

Step 2. We generate $X_1^*, \ldots, X_n^* \sim \text{NHPP}(\hat{\alpha}_1, \hat{\beta}_1)$ and independently generate $Y_1^*, \ldots, Y_m^* \sim \text{NHPP}(\hat{\alpha}_2, \hat{\beta}_2)$. Based on $Y_{\text{obs}}^{\text{ft}*} = \{x_{r_1}^*, \ldots, x_n^*;\ y_{r_2}^*, \ldots, y_m^*\}$, we calculate the MLEs $(\hat{\alpha}_1^*, \hat{\beta}_1^*;\ \hat{\alpha}_2^*, \hat{\beta}_2^*)$ according to (1.2), the MLE of the MTBF ratio as $\hat{M}_2^*(y_m)/\hat{M}_1^*(x_n) = \hat{\alpha}_1^*\hat{\beta}_1^* y_m^{1-\hat{\beta}_2^*}/(\hat{\alpha}_2^*\hat{\beta}_2^* x_n^{1-\hat{\beta}_1^*})$, and two CIs of $M_2(y_m)/M_1(x_n)$.

Step 3. By repeating Step 2 for L ($L = 100$) times, we can obtain the empirical *coverage probability* (CP) and the average *confidence interval width* (CIW).

Step 4. Finally, by repeating Steps 1–3 for G ($G = 100$) times, we can obtain the average CP and the average CIW.

The resulting average CP and average CIW are reported in Table 4. From Table 4, we can see that the shortest CI of $M_2(y_m)/M_1(x_n)$ has a smaller coverage probability than the equal-tailed CI.

7.3 Comparison of p-values for testing hypotheses on $M(x_n)$

In Section 5.2, we developed both the critical region approach and the p-value approach to test H_0: $M(x_n) = M_0$ against H_1: $M(x_n) \neq M_0$, where M_0 is a given positive constant. Since the p-value approach is more straightforward than the critical region approach, we only consider the former and compare the corresponding p-values calculated by the proposed density- and sampling-based methods.

Let $X_1, \ldots, X_n \sim \text{NHPP}(\alpha, \beta)$ and the realizations be $x_1, \ldots, x_n$, where $x_1, \ldots, x_{r-1}$ are missing. For each combination of (α, β, n, r), we test H_0: $M(x_n) = M_0$ ($M_0 = 20, 2, 50$) by considering two cases. The corresponding p-values calculated by the density-based method and sampling-based method are summarized in Table 5.

Note that the p-values calculated via the sampling-based method is very close to those calculated via the density-based method, but the former is much more time-consuming.

8. DISCUSSION

It is of considerable interest and practical significance to construct the shortest CIs for the MTBF in a single repairable system (whose failures follow a Weibull process) and for the MTBF ratio in two independent repairable systems (whose failures follow the same Weibull process). In addition, reliability engineers are also interested in establishing a correct critical region by employing the equal-height (instead of the equal-tail) method and in obtaining an exact (rather than an approximate) p-value when a hypothesis testing on MTBF or MTBF ratio is performed. In this paper, we developed three methods (i.e., the Lagrange multiplier,

Table 5. The p-values for testing H_0: $M(x_n) = M_0$ against H_1: $M(x_n) \neq M_0$ by using the density-based and sampling-based methods

	H_0: $M(x_n) = 20$ against H_1: $M(x_n) \neq 20$			
$(\alpha, \beta, n, r) = (0.2, 0.8, 20, 2)$	Method	p-value	Time†	Time‡
$\hat{M}(x_n) = 26.20303$	Density-based	0.166656	0.02	0.01
	Sampling-based	0.166257	0.17	5.90
$\hat{M}(x_n) = 16.85696$	Density-based	0.955431	0.02	0.01
	Sampling-based	0.966014	0.25	5.95
	H_0: $M(x_n) = 2$ against H_1: $M(x_n) \neq 2$			
$(\alpha, \beta, n, r) = (0.5, 1, 30, 1)$	Method	p-value	Time†	Time‡
$\hat{M}(x_n) = 1.823796$	Density-based	0.916960	0.02	0.01
	Sampling-based	0.923513	0.26	5.88
$\hat{M}(x_n) = 2.288272$	Density-based	0.336492	0.01	0.01
	Sampling-based	0.338066	0.29	5.84
	H_0: $M(x_n) = 50$ against H_1: $M(x_n) \neq 50$			
$(\alpha, \beta, n, r) = (0.3, 0.6, 40, 4)$	Method	p-value	Time†	Time‡
$\hat{M}(x_n) = 76.80926$	Density-based	0.020815	0.02	0.01
	Sampling-based	0.020680	0.26	5.82
$\hat{M}(x_n) = 97.83368$	Density-based	0.000552	0.00	0.00
	Sampling-based	0.000552	0.24	5.79

NOTE: Time† = The "elapsed time" in `R` in seconds; Time‡ = The "system time" in `R` in seconds.

quantile-based and sampling-based methods) to construct the shortest CIs for the MTBF in a single repairable system and for the MTBF ratio in two independent repairable systems; and also develop two methods (i.e., the density-based and sampling-based methods) within the framework of the critical region and p-value approaches to test hypotheses on the MTBF and the MTBF ratio. The implementation of the five methods involve the application of the G, inverse G and RG distributions. Some important properties on the three distributions have been studied in the paper. It is noted that the developed sampling-based method is very useful to practical users since it does not require the expressions of the pdf, cdf and the mode of the relevant distributions.

In fact, the G distribution also finds its application in wavelet analysis and wireless communication (Ge, [5]; Tsai, [13]; Lee and Cho, [9]), where many quantities are the form of product of two or more complex numbers. For example, let $Z_k = X_k + \mathrm{i}Y_k$, $k = 1, 2$, be two complex random variables, where X_1, X_2, Y_1, Y_2 are independent standard normal random variables. Note that $X_1^2 + Y_1^2$, $X_2^2 + Y_2^2 \overset{\text{iid}}{\sim} \chi^2(2)$, thus the squared magnitude of $Z_1 Z_2$ defined by $|Z_1 Z_2|^2 = (X_1^2 + Y_1^2)(X_2^2 + Y_2^2)$ is the product of two independent chi-squared random variables.

It is worthwhile to generalize the present paper from the failure-truncated case to the time-truncated case. Finally, it would be convenient to extend the frequentist methods to those situations in which the prior knowledge on parameters is available. `R` codes are available once requested.

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