Supplementary material to "Likelihood ratio tests in the Rasch model for item response data when the number of persons and items goes to infinity"

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This is a supplementary material that contains the proofs of Propositions 1–3 in Yan et, al. (2015).

1 Proof of Proposition 1

Proposition 1. *If $M_{rt} = o(\rho_{rt}/\log\rho_{rt})$ and $\lambda_{rt}/\rho_{rt} \to c$ as r and t go to infinity, where c is a constant hereafter, then P(Condition A holds) $\to 1$.*

Proof. Denote the probability that Condition A fails by P_{rt}. Note that $M_{rt} \ge 1$ and

$$\max_{i,j=0,\cdots,t} p_{ij} = \max_{i,j=0,\cdots,t} \frac{1}{1+u_j/u_i} \le \frac{1}{1+1/M_{rt}} \le (\frac{1}{2})^{1/M_{rt}}.$$

Consider a particular partition of Ω_1 into two nonempty subsets $F_{1,i}$ and $F_{2,r-i}$ and Ω_2 into two nonempty subsets $F_{3,j}$ and $F_{4,t-j}$, where the second subscript i denotes the number of subjects in $F_{k,i}$, $k=1,\ldots,4$. The probability that $a_{kl}=1$ for all $k \in F_{1,i}, l \in F_{3,j}$ and $a_{kl}=0$ for all $k \in F_{2,r-i}, l \in F_{4,t-j}$ is bounded above by $(\frac{1}{2})^{[ij+(r-i)(t-j)]/M_{rt}}$. Note that when $1 \le i \le [r/2], 1 \le j \le [t/2]$ or $[r/2]+1 \le i \le r, [t/2]+1 \le j \le t$,

$$\begin{aligned} & ij + (r-i)(t-j) - [i(t-j) + j(r-i)] \\ = \; & (r-i)(t-2j) - i(t-2j) = (r-2i)(t-2j) > 0. \end{aligned}$$

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It follows that

$$
\begin{aligned}
P_{rt} &\leq \sum_{i=1}^{r} \sum_{F_{1,i}\subset\{1,\cdots,r\}} \sum_{j=1}^{t} \sum_{F_{3,j}\subset\{1,\cdots,t\}} (\frac{1}{2})^{[ij+(r-i)(t-j)]/M_{rt}} \\
&= (\sum_{i=1}^{[r/2]}\sum_{j=1}^{[t/2]} + \sum_{i=[r/2]+1}^{r}\sum_{j=[t/2]+1}^{t} + \sum_{i=1}^{[r/2]}\sum_{j=[t/2]+1}^{t} \\
&\quad + \sum_{i=[r/2]+1}^{r}\sum_{j=1}^{[t/2]+1}) \binom{r}{i}\binom{t}{j}(\frac{1}{2})^{[ij+(r-i)(t-j)]/M_{rt}} \\
&\leq \sum_{i=1}^{[r/2]}\sum_{j=1}^{[t/2]} \binom{r}{i}\binom{t}{j}(\frac{1}{2})^{[i(t-j)+j(r-i)]/M_{rt}} \\
&\quad + \sum_{i=[r/2]+1}^{r}\sum_{j=[t/2]+1}^{t} \binom{r}{i}\binom{t}{j}(\frac{1}{2})^{[i(t-j)+j(r-i)]/M_{rt}} \\
&\quad + \sum_{i=1}^{[r/2]}\sum_{j=[t/2]+1}^{t} \binom{r}{i}\binom{t}{j}(\frac{1}{2})^{[ij+(r-i)(t-j)]/M_{rt}} \\
&\quad + \sum_{i=[r/2]+1}^{r}\sum_{j=1}^{[t/2]+1} \binom{r}{i}\binom{t}{j}(\frac{1}{2})^{[ij+(r-i)(t-j)]/M_{rt}} \\
&\leq \sum_{i=1}^{[r/2]}\sum_{j=1}^{[t/2]} \binom{r}{i}\binom{t}{j}(\frac{1}{2})^{[it/2+jr/2]/M_{rt}} \\
&\quad + \sum_{i=[r/2]+1}^{r}\sum_{j=[t/2]+1}^{t} \binom{r}{i}\binom{t}{j}(\frac{1}{2})^{[r(t-j)/2+t(r-i)/2]/M_{rt}} \\
&\quad + \sum_{i=1}^{[r/2]}\sum_{j=[t/2]+1}^{t} \binom{r}{i}\binom{t}{j}(\frac{1}{2})^{[it/2+r(t-j)/2]/M_{rt}} \\
&\quad + \sum_{i=[r/2]+1}^{r}\sum_{j=1}^{[t/2]+1} \binom{r}{i}\binom{t}{j}(\frac{1}{2})^{[rj/2+(r-i)t/2]/M_{rt}} \\
&\leq 4[(1+(\frac{1}{2})^{r/(2M_{rt})})^t - 1)(1+(\frac{1}{2})^{t/(2M_{rt})})^r - 1)].
\end{aligned}
$$

Consequently, when $M_{rt} = o(\rho_{rt}/\log\rho_{rt})$ and $\lambda_{rt}/\rho_{rt} \to c$, the above term goes to zero as $\rho_{rt} \to \infty$. □

2 Proof of Proposition 2

Proposition 2. *Let $S = (s_{ij})_{i,j=2,\dots,r+t}$ be the matrix with*

$$s_{ij} = \frac{\delta_{ij}}{v_{ii}} + \frac{1}{v_{11}}$$

where δ_{ij} is the Kroneck delta function. The upper bound of the approximation error using S to approximate the inverse of V takes:

$$||V^{-1} - S|| \leq O(\frac{M_{rt}^4}{\rho_{rt}^2}) \quad as \quad \rho_{rt} \to \infty, \tag{1}$$

where $||A|| = \max_{i,j} |a_{ij}|$ for a general matrix.

Proof. For convenience, we introduce a nonnegative array $\{d_{ij}\}_{i,j=2}^{r+t}$ defined in terms of V as follows:

$d_{ij} = -v_{ij}$ for $i \neq j$ and $d_{ii} = \sum_{j=2}^{r+t} v_{ij} = -v_{i1}$.

Moreover, let

$$d_* := \max_{i,j} d_{ij} \leq \frac{1}{4}, \quad d_{**} := \min_{(i,j)\in\{(i,j):d_{ij}>0\}} d_{ij} \geq \frac{M_{rt}}{(1+M_{rt})^2}.$$

Then we have

$$d_{ij} \geq 0,\ d_{ij} = d_{ji},\ v_{ij} = -d_{ij} \text{ for } i,j = 2,\dots,r+t; i \neq j,$$

$$d_* n_i \geq v_{ii} = \sum_{k=1}^{r+t} d_{ik} \geq d_{**} n_i, \quad i = 2,\dots,r+t.$$

Note that $V^{-1} - S = (V^{-1} - S)(I - VS) + S(I - VS)$, where I is the indentity matrix with dimension $(r+t-1)\times(r+t-1)$. Letting $X = I - VS$ and $Y = SX$ and $Z = V^{-1} - S$, we have the recursion

$$Z = ZX + Y. \tag{2}$$

Thus, the objective of the proof is to show

$$||Z|| \leq O(\frac{M_{rt}^4}{\rho_{rt}^2}).$$

Direct calculations give that

$$\begin{aligned} x_{ij} &= (1-\delta_{ij})\frac{d_{ij}}{v_{jj}} - \frac{d_{ii}}{v_{11}} \\ y_{ij} &= \frac{(1-\delta_{ij})d_{ij}}{v_{ii}v_{jj}} - \frac{d_{ii}}{v_{ii}v_{11}} - \frac{d_{ij}}{v_{jj}v_{11}}. \end{aligned} \tag{3}$$

According the definitions of d_* and d_{**}, we have

$$0 \leq \frac{d_{ij}}{v_{ii}v_{jj}} \leq \frac{d_*}{d_{**}^2\rho_{rt}^2}, \quad \text{and} \quad 0 \leq \frac{d_{ii}}{v_{ii}v_{11}} \leq \frac{d_*}{d_{**}^2\rho_{rt}^2},$$

so that for all different i, j, k,

$$|y_{ij}| \le a \quad \text{and} \quad |y_{ij} - y_{ik}| \le a,$$

where $a = 2d_*/(d_{**}^2 \rho_{rt}^2)$.

By (2) and (3), we have

$$z_{ij} = \sum_{k=2}^{r+t} z_{ik}(1-\delta_{kj})\frac{d_{kj}}{v_{jj}} - \sum_{k=2}^{r+t} z_{ik}\frac{d_{kk}}{v_{11}} + y_{ij}, \quad i, j = 2, \ldots, r+t. \tag{4}$$

Since the index i plays no essential role in (4), we fix i. Let α and β be such that $z_{i\alpha} = \max_{2\le k\le r+t} z_{ik}$ and $z_{i\beta} = \min_{2\le k\le r+t} z_{ik}$. Without loss of generality, assume $z_{i\alpha} \ge |z_{i\beta}|$. Otherwise, we may reverse the signs of z_{ik}'s and proceed analogously. First, we show $z_{i\beta} \le 0$. By taking the summation for $j = 2, \ldots, r+t$ after multiplying v_{jj} to both sides of (4), it yields,

$$\begin{aligned} \sum_{j=2}^{r+t} v_{jj} z_{ij} &= \sum_{k=2}^{r+t}\sum_{j=2}^{r+t} z_{ik}(1-\delta_{kj})d_{kj} - \sum_{k=2}^{r+t}\sum_{j=2}^{r+t} z_{ik}\frac{d_{kk}v_{jj}}{v_{11}} \\ &\quad + \sum_{j=2}^{r+t} v_{jj}\left(\frac{(1-\delta_{ij})d_{ij}}{v_{ii}v_{jj}} - \frac{d_{ii}}{v_{ii}v_{11}} - \frac{d_{jj}}{v_{jj}v_{11}}\right). \end{aligned}$$

It follows that

$$\sum_{k=2}^{r+t} z_{ik}\frac{d_{kk}}{v_{11}} = -\frac{d_{ii}}{v_{ii}v_{11}}, \tag{5}$$

so that

$$z_{i\beta} \le \sum_{k=2}^{r+t} z_{ik}\frac{d_{kk}}{v_{11}} = -\frac{d_{ii}}{v_{ii}v_{11}} \le 0.$$

There are four cases to for α and β to be considered.

Case I: $\alpha, \beta \in \Omega_1 = \{1, \ldots, r\}$. Since $\sum_{k=1}^{r+t} \frac{d_{k\alpha}}{v_{\alpha\alpha}} = 1$ and $d_{k\alpha} = 0$ when $k \in \Omega_1$, by (4) and (5), we have

$$\sum_{k\in\Omega_2} [z_{i\alpha} - z_{ik}(1-\delta_{k\alpha})]\frac{d_{k\alpha}}{v_{\alpha\alpha}} = \frac{d_{ii}}{v_{ii}v_{11}} + y_{i\alpha} = \frac{(1-\delta_{ij})d_{ij}}{v_{ii}v_{jj}} - \frac{d_{ij}}{v_{jj}v_{11}} \le a. \tag{6}$$

It follows

$$a \quad \ge \quad \sum_{k\in\Omega_2} [z_{i\alpha} - z_{ik}]\frac{d_{**}}{td_*}. \tag{7}$$

Similar to (7), we have

$$a \ge \sum_{k\in\Omega_2} [z_{ik} - z_{i\beta}]\frac{d_{**}}{td_*}. \tag{8}$$

Combining (7) and (8), it yields

$$z_{i\alpha} - z_{i\beta} \quad \leq \quad \frac{2ad_*}{d_{**}} = \frac{4d_*^2}{d_{**}^3 \rho_{rt}^2} \leq O(\frac{M_{rt}^3}{\rho_{rt}^2}). \tag{9}$$

Case II: $\alpha, \beta \in \Omega_2$. Similar to the proof of Case I, we also have the inequality (9).

Case III: $\alpha \in \Omega_1, \beta \in \Omega_2$. Let $\beta_1 \in \Omega_1$ be such that $z_{i\beta_1} = \min_{k\in\Omega_1} z_{ik}$. Since $\sum_{k=2}^{r+t} \frac{d_{k\beta}}{v_{\beta\beta}} = 1$ and $d_{k\beta} = 0$ when $k \in \Omega_2$, by (4) and (5), we have

$$\sum_{k\in\Omega_1} [z_{ik} - z_{i\beta}] \frac{d_{k\beta}}{v_{\beta\beta}} = \frac{d_{ii}}{v_{ii}v_{11}} + y_{i\alpha} = \frac{(1-\delta_{ij})d_{ij}}{v_{ii}v_{jj}} - \frac{d_{ij}}{v_{jj}v_{11}} \leq a. \tag{10}$$

Thus, we have

$$\frac{d_{**}(z_{i\beta_1} - z_{i\beta})}{d_*} \leq a.$$

Similar to (7), we have

$$\sum_{k\in\Omega_2} [z_{ik} - z_{i\beta})] \frac{d_{**}}{d_* t} \leq a + (z_{i\beta_1} - z_{i\beta}) \frac{d_*}{d_{**}}. \tag{11}$$

Combining (7) and (11), it yields

$$\frac{d_{**}(z_{i\alpha} - z_{i\beta})}{d_*} \quad \leq \quad 2a + (z_{i\beta_1} - z_{i\beta}) \frac{d_*}{d_{**}}.$$

Consequently, we have

$$z_{i,\alpha} - z_{i,\beta} \leq \frac{2ad_*}{d_{**}} + \frac{ad_*^2}{d_{**}^2} = \frac{4d_*^2}{d_{**}^3 \rho_{rt}^2} + \frac{2d_*^3}{d_{**}^4 \rho_{rt}^2} \leq O(\frac{M_{rt}^4}{\rho_{rt}^2}). \tag{12}$$

Case IV: $\alpha \in \Omega_2, \beta \in \Omega_1$. Similar to the proof of Case III, we have also have (12). This completes the proof. □

3 Proof of Proposition 3

Proposition 3. *Let $\Delta u_i = (\hat{u}_i/u_i) - 1$. If $M_{rt} = o(\log \rho_{rt})$ and $\lambda_{rt}/\rho_{rt} \to c$ as r and t go to infinity, then with probability approaching 1,*

$$\max_{i=1,\ldots,r+t} |\Delta u_i| \leq \max_{i,j=1,\ldots,r+t} |\Delta u_i - \Delta u_j| \leq \frac{\theta_{rt}}{1-\theta_{rt}} \to 0, \tag{13}$$

where

$$\theta_{rt} = \left[\frac{2(1+M_{rt})^2}{M_{rt}} + \frac{e^{c^* M_{rt}}(1+M_{rt})^4}{4M_{rt}^2} \right] \sqrt{\frac{\lambda_{rt} \log \lambda_{rt}}{\rho_{rt}^2}}, \tag{14}$$

and c^* is a constant.

Proposition 3 is a direct conclusion from the following four lemmas. The proof of Lemma 4 is very close to the Erdős-Galli graph condition.

Lemma 1. *If $M_{rt} = o(\rho_{rt}/\log \rho_{rt})$ and $\lambda_{rt}/\rho_{rt} \to c$ as r and t go to infinity, with probability approaching* 1*, then*

$$\begin{array}{l} \max_{i=1,\dots,r} |a_i - E(a_i)| = \max_{i=1,\dots,r} |\sum_j (\frac{n_{ij}\hat{u}_i}{\hat{u}_i+\hat{u}_j} - \frac{n_{ij}u_i}{u_i+u_j})| \le \sqrt{t\log t} \\ \max_{i=r+1,\dots,r+t} |a_i - E(a_i)| = \max_{i=r+1,\dots,r+t} |\sum_j (\frac{n_{ij}\hat{u}_i}{\hat{u}_i+\hat{u}_j} - \frac{n_{ij}u_i}{u_i+u_j})| \le \sqrt{r\log r}. \end{array} \tag{15}$$

Proof. For $i = 1, \dots, r$, $a_i = \sum_{j=r+1}^{r+t} a_{ij}$ is a sum of t independent Bernoulli random variables. Therefore by Hoeffding's (**?**)inequality,

$$P(|a_i - E(a_i)| > x) \le 2e^{-2x^2/t}.$$

Taking $x = \sqrt{t\log t}$, it yields

$$P(|a_i - E(a_i)| > \sqrt{t\log t}) \le 2e^{-2\log t} = \frac{2}{t^2}.$$

Therefore, we have

$$P(\max_{i=1,\dots,r} |a_i - E(a_i)| > \sqrt{t\log t}) \le \frac{2r}{t^2}.$$

Similarly,

$$P(\max_{i=r+1,\dots,r+t} |a_i - E(a_i)| > \sqrt{r\log r}) \le \frac{2t}{r^2}.$$

Therefore,

$$P(F_1 \bigcap F_2) \ge 1 - P(F_1^c) - P(F_2^c) \ge 1 - \frac{2r}{t^2} - \frac{2t}{r^2},$$

where

$$F_1 = \{\max_{i=1,\dots,r} |a_i - E(a_i)| \le \sqrt{t\log t}\}, \quad F_2 = \{\max_{i=r+1,\dots,r+t} |a_i - E(a_i)| \le \sqrt{r\log r}\}. \tag{16}$$

Thus, by Proposition 1, if $M_{rt} = o(\rho_{rt}/\log \rho_{rt})$ and $\lambda_{rt}/\rho_{rt} \to c$, then the MLE satisfies the inequality (15) with probability approaching 1 as $\rho_{rt} \to \infty$. This completes the proof. □

Lemma 2. *If $M_{rt} = o(\log \rho_{rt})$, then there exist $c_1 = (M_{rt} + c_1^*)/(1 + M_{rt} + c_1^*)$ and $c_2 = 1/(1 + M_{rt} + c_2^*)$ such that as r and t go to infinity, with probability approaching 1,*

$$c_2 n_i \le a_i \le c_1 n_i, \quad i = 1, \dots, r+t, \tag{17}$$

where c_1^* $(0 < c_1^* < 1)$ and c_2^* are constants and for any nonempty $B \subseteq \{1, \dots, r+t\}$,

$$\begin{array}{rl} & \sum_{j\in B^c} \min\{a_j, \sum_{i\in B} n_{ij}\} + \sum_{i,j\in B} n_{ij} - \sum_{i\in B} a_i \\ \ge & \frac{1}{1+M_{rt}} \min\{\sum_{i,j\in B^c} n_{ij}, \sum_{i\in B, j\in B^c} n_{ij}\} - t\sqrt{t\log t} - r\sqrt{r\log r}. \end{array} \tag{18}$$

Proof. The following calculations are based on the event $F_1 \bigcap F_2$, where F_1 and F_2 defined in (16).

Since

$$\frac{tM_{rt}}{1+M_{rt}} \ge E(a_i) = \sum_{j=r+1}^{r+t} \frac{u_i/u_j}{u_i/u_j+1} \ge \frac{t}{1+M_{rt}}, \quad i=1,\ldots,r,$$

for $i=1,\ldots,r$, we have

$$\begin{aligned} a_i &\le E(a_i) + \sqrt{t\log t} \le t[\tfrac{M_{rt}}{1+M_{rt}} + \sqrt{\tfrac{\log t}{t}}] \\ a_i &\ge E(a_i) - \sqrt{t\log t} \ge t(\tfrac{1}{1+M_{rt}} - \sqrt{\tfrac{\log t}{t}}). \end{aligned}$$

If $M_{rt} = o(\log \rho_{rt})$, then there exist constants c_{1a}^* and c_{2a}^* $(0 < c_{2a}^*, c_{1a}^* < 1)$ such that when t is large enough,

$$\frac{t(M_{rt}+c_{1a}^*)}{1+M_{rt}+c_{1a}^*} \ge a_i \ge \frac{t}{1+M_{rt}+c_{2a}^*}, \quad i=1,\ldots,r.$$

Similarly, we have that for $i=r+1,\ldots,r+t$,

$$\frac{r(M_{rt}+c_{1b}^*)}{1+M_{rt}+c_{1b}^*} \ge a_i \ge \frac{r}{1+M_{rt}+c_{2b}^*}$$

by noting

$$\frac{rM_{rt}}{1+M_{rt}} \ge E(a_i) = \sum_{j=1}^{r} \frac{u_i/u_j}{u_i/u_j+1} \ge \frac{r}{1+M_{rt}}, \quad i=r+1,\ldots,r+t.$$

Thus, when t is sufficiently large, we can choose

$$c_1 = \frac{M_{rt}+c_1^*}{1+M_{rt}+c_1^*}, \quad c_2 = \frac{1}{1+M_{rt}+c_2^*},$$

where $c_1^* = \max\{c_{1a}^*, c_{1b}^*\}$ and $c_2^* = \max\{c_{2a}^*, c_{2b}^*\}$. This shows (17).

For any nonempty $B \subseteq \Omega_1 \bigcup \Omega_2 = \{1,\ldots,r+t\}$, define

$$g(a_1,\ldots,a_t,B) = \sum_{j\in B^c} \min\{a_j, \sum_{i\in B} n_{ij}\} + \frac{1}{2}\sum_{i,j\in B} n_{ij} - \sum_{i\in B} a_i.$$

We have

$$\begin{aligned} & |g(a_1,\cdots,a_t,B) - g(E(a_1),\ldots,E(a_t),B)| \\ = & \ |\sum_{j\in B^c} \min\{a_j, \sum_{i\in B} n_{ij}\} - \sum_{j\in B^c} \min\{E(a_j), \sum_{i\in B} n_{ij}\} + \sum_{i\in B} E(a_i) - \sum_{i\in B} a_i| \\ \le & \sum_{j\in B^c} |(\min\{a_j, \sum_{i\in B} n_{ij}\} - \min\{E(a_j), \sum_{i\in B} n_{ij}\})| + \sum_{i\in B} |E(a_i) - a_i| \\ \le & \sum_{i=1}^{r} |a_i - E(a_i)| + \sum_{k=r+1}^{r+t} |a_i - E(a_i)| \\ \le & \ r \max_{i=1,\ldots,r} |a_i - E(a_i)| + t \max_{k=r+1,\ldots,r+t} |a_i - E(a_i)| \\ \le & \ r\sqrt{r\log r} + t\sqrt{t\log t}. \end{aligned} \tag{19}$$

Since $p_{ij} + p_{ji} = 1$,

$$
\begin{aligned}
& g(E(a_1), \ldots, E(a_t), B) \\
= & \sum_{j \in B^c} \min\{E(a_j), \sum_{i \in B} n_{ij}\} + \frac{1}{2} \sum_{i,j \in B} n_{ij} - \sum_{i \in B} E(a_i) \\
= & [\frac{1}{2} \sum_{i,j \in B} n_{ij} - \sum_{i,j \in B} n_{ij} p_{ij}] + [\sum_{j \in B^c} (\min\{E(a_j), \sum_{i \in B} n_{ij}\} - \sum_{i \in B} p_{ij})] \\
= & \sum_{j \in B^c} [\min\{E(a_j) - \sum_{i \in B} p_{ij}, \sum_{i \in B} n_{i,j} - \sum_{i \in B} p_{ij}\}] \\
= & \sum_{j \in B^c} [\min\{\sum_{i \in B^c} n_{i,j} p_{ij}, \sum_{i \in B} n_{i,j}(1 - p_{ij})\}] \\
\geq & \frac{1}{1 + M_{rt}} \min\{\sum_{i,j \in B^c} n_{ij}, \sum_{j \in B^c} \sum_{i \in B} n_{ij}\}. \qquad (20)
\end{aligned}
$$

Combining (19) and (20) yields (18). By Lemma 1, if $M_{rt} = o(\log \rho_{rt})$ and $\lambda_{rt}/\rho_{rt} \to c$, then $P(F_1 \bigcap F_2) \to 1$ as ρ_{rt} goes to infinity. This completes the proof of the lemma. □

Lemma 3. *If* (15), (17) *and* (18) *hold, then*

$$\hat{u}_{\max}/\hat{u}_{\min} \leq e^{c^* M_{rt}^4}, \qquad (21)$$

where $\hat{u}_{\max} = \max_{i=1,\ldots,r+t} \hat{u}_i$ *and* $\hat{u}_{\min} = \min_{i=1,\ldots,r+t} \hat{u}_i$.

Proof. Let E be the event that $c_2 n_i \leq a_i \leq c_1 n_i$ for all i, where c_1, c_2 are defined in Lemma 2 and F be the event that for any nonempty $B \subseteq \{1, \cdots, r+t\}$, the inequality (18) holds and G be the event that Condition A holds. By Lemmas 1 and 2 and Proposition 1, we know that if $M_{rt} = o(\log \rho_{rt})$ and $\lambda_{rt}/\rho_{rt} \to c$, then

$$P(E \bigcap F \bigcap G) \to 1 \quad \text{as} \quad \rho_{rt} \to \infty.$$

The following calculations are based on the event $E \bigcap F \bigcap G$. The argument repeatedly uses the monotonicity of $e^{x+y}/(1+e^{x+y})$ in x for each y.

First, we assume that $\hat{\beta}_{\min} := \min_k \hat{\beta}_k = 0$. Let $i_1, i_2 \in \Omega_1$ such that $\hat{\beta}_{i_1} = \max_{i \in \Omega_1} \hat{\beta}_i$, $\hat{\beta}_{i_2} = \min_{i \in \Omega_1} \hat{\beta}_i$ and $j_1, j_2 \in \Omega_2$ such that $\hat{\beta}_{j_1} = \max_{j \in \Omega_2} \hat{\beta}_j$, $\hat{\beta}_{j_2} = \min_{j \in \Omega_2} \hat{\beta}_j$. There are four cases to be considered.

Case I: $\hat{\beta}_{i_1} \geq \hat{\beta}_{j_1} \geq \hat{\beta}_{j_2} \geq \hat{\beta}_{i_2} = 0$. If $\hat{\beta}_{i_1} = 0$, then there is nothing to prove. Without loss of generality, assume that $\hat{\beta}_{i_1} > 0$. Let

$$m_1 = |\{j : \hat{\beta}_j \geq \hat{\beta}_{i_1}/5\} \bigcap \Omega_2|.$$

By (17), we have

$$c_2 n_1 \leq a_1 = \sum_{j \in \Omega_2} \frac{1}{1 + e^{\hat{\beta}_j - \hat{\beta}_1}} < \frac{m_1}{1 + e^{\hat{\beta}_{i_1}/5 - \hat{\beta}_1}} + n_1 - m_1 = n_1 - \frac{m_1 e^{\hat{\beta}_{i_1}/5 - \hat{\beta}_1}}{1 + e^{\hat{\beta}_{i_1}/5 - \hat{\beta}_1}},$$

such that

$$c_2 n_1 - \frac{n_1}{1 + e^{\hat{\beta}_{i_1}/5 - \hat{\beta}_1}} < (n_1 - m_1)\frac{e^{\hat{\beta}_{i_1}/5 - \hat{\beta}_1}}{1 + e^{\hat{\beta}_{i_1}/5 - \hat{\beta}_1}}.$$

Thus, if $(\hat{\beta}_{i_1}/5) - \hat{\beta}_1 = \hat{\beta}_{i_1}/5$ ($\hat{\beta}_1 = 0$) is so large (we assume that this is true), then

$$\begin{aligned} n_1 - m_1 &> \frac{c_2(1 + e^{\hat{\beta}_{i_1}/5})n_1}{e^{\hat{\beta}_{i_1}/5}} - \frac{n_1}{e^{\hat{\beta}_{i_1}/5}} = c_2 n_1 - \frac{n_1(1 - c_2)}{e^{\hat{\beta}_{i_1}/5}} \\ &> \frac{n_1}{1 + M_{rt} + c_{31}}, \end{aligned}$$

where c_{31} is a constant. Thus, we have $|A_1| \geq \frac{n_1}{1+M_{rt}+c_{31}}$, where

$$A_1 := \{j : \hat{\beta}_j < \hat{\beta}_{i_1}/5\} \bigcap \Omega_2.$$

Let

$$m_{i_1} = |\{j : \hat{\beta}_j < \hat{\beta}_{j_1}/2\} \bigcap \Omega_2|.$$

By (17), we have

$$\begin{aligned} c_1 n_{i_1} &\geq a_{i_1} = \sum_{j \in \Omega_2} \frac{e^{\hat{\beta}_{i_1}}}{e^{\hat{\beta}_{i_1}} + e^{\hat{\beta}_j}} > \frac{m_{i_1} e^{\hat{\beta}_{i_1}}}{e^{\hat{\beta}_{i_1}} + e^{\hat{\beta}_{j_1}/2}} + \frac{n_{i_1} - m_{i_1}}{2} \\ &= \frac{(m_{i_1} - n_{i_1})e^{\hat{\beta}_{i_1}}}{e^{\hat{\beta}_{i_1}} + e^{\hat{\beta}_{j_1}/2}} + \frac{n_{i_1} - m_{i_1}}{2} + \frac{n_{i_1} e^{\hat{\beta}_{i_1}}}{e^{\hat{\beta}_{i_1}} + e^{\hat{\beta}_{j_1}/2}}, \end{aligned}$$

such that

$$(n_{i_1} - m_{i_1})[\frac{e^{\hat{\beta}_{i_1}}}{e^{\hat{\beta}_{i_1}} + e^{\hat{\beta}_{j_1}/2}} - \frac{1}{2}] > n_{i_1}[\frac{e^{\hat{\beta}_{i_1}}}{e^{\hat{\beta}_{i_1}} + e^{\hat{\beta}_{j_1}/2}} - c_1].$$

Note that $\hat{\beta}_{i_1} \geq \hat{\beta}_{j_1}$. Thus, if $\hat{\beta}_{i_1}$ is large enough (assume that this is true), then

$$\begin{aligned} n_{i_1} - m_{i_1} &> n_{i_1} \frac{e^{\hat{\beta}_{i_1}} - c_1(e^{\hat{\beta}_{i_1}} + e^{\hat{\beta}_{j_1}/2})}{e^{\hat{\beta}_{i_1}} - \frac{1}{2}(e^{\hat{\beta}_{i_1}} + e^{\hat{\beta}_{j_1}/2})} = n_{i_1} \frac{1 - c_1(1 + e^{\hat{\beta}_{j_1}/2 - \hat{\beta}_{i_1}})}{1 - \frac{1}{2}(1 + e^{\hat{\beta}_{j_1}/2 - \hat{\beta}_{i_1}})} \\ &\geq \frac{2n_{i_1}}{1 + M_{rt} + c_{32}}, \end{aligned}$$

where c_{32} is a constant. Thus, we have $|A_2| \geq \frac{2n_{i_1}}{1+M_{rt}+c_{32}}$, where

$$A_2 = \{j : \hat{\beta}_j \geq \hat{\beta}_{j_1}/2\} \bigcap \Omega_2.$$

Choose a point $j_3 \in A_2$. Let

$$m_{j_3} = |\{i : \hat{\beta}_i \geq \hat{\beta}_{i_1}/4\} \bigcap \Omega_1|.$$

By (17), we have

$$\begin{aligned} c_2 n_{j_3} &\le a_{j_3} = \sum_{i\in\Omega_1} \frac{1}{1+e^{\hat\beta_i-\hat\beta_{j_3}}} < \frac{m_{j_3}}{1+e^{\hat\beta_{i_1}/4-\hat\beta_{j_3}}} + n_{j_3} - m_{j_3} \\ &< \frac{m_{j_3}}{1+e^{\hat\beta_{i_1}/4-\hat\beta_{j_1}/5}} + n_{j_3} - m_{j_3} = n_{j_3} - \frac{m_{j_3} e^{\hat\beta_{i_1}/20}}{1+e^{\hat\beta_{i_1}/20}}, \end{aligned}$$

such that

$$c_2 n_{j_3} - \frac{n_{j_3}}{1+e^{\hat\beta_{i_1}/20}} < \frac{(n_{j_3}-m_{j_3})e^{\hat\beta_{i_1}/20}}{1+e^{\hat\beta_{i_1}/20}}.$$

Thus, if $\hat\beta_{i_1}$ is large enough (we assume that this is true), then

$$\begin{aligned} n_{j_3} - m_{j_3} &> \frac{c_2(1+e^{\hat\beta_{i_1}/20})n_{j_3}}{e^{\hat\beta_{i_1}/20}} - \frac{n_{j_3}}{e^{\hat\beta_{i_1}/20}} = c_2 n_{j_3} - \frac{n_{j_3}(1-c_2)}{e^{\hat\beta_{i_1}/20}} \\ &> \frac{n_{j_3}}{1+M_{rt}+c_{33}}, \end{aligned}$$

where c_{33} is a constant. Thus, we have $|A_3| \ge \frac{n_{j_3}}{1+M_{rt}+c_{33}}$, where

$$A_3 = \{i : \hat\beta_i < \hat\beta_{i_1}/4\} \bigcap \Omega_1.$$

Let $h = \hat\beta_{i_1}^{1/2} = \hat\beta_{\max}^{1/2}$ and for each integer k between 0 and $(h/8)-1$ (assume that $(h/8)-1>1$, otherwise there is nothing to prove), define

$$D_k = \{j : (\hat\beta_{i_1}/4) + kh \le \hat\beta_i < (\hat\beta_{i_1}/4) + (k+1)h\}.$$

Note that $\sum_k |D_k| \le r+t$. Thus, exists such a k such that $|D_k| \le (r+t)/(h/8-1)$. Fix such k and let

$$B = \{j : \hat\beta_j \ge (\hat\beta_{i_1}/4) + (k+1)h\}. \tag{22}$$

Therefore we have $A_1 \subset B^c$, $A_2 \subset B$ and $A_3 \subset B^c$ such that

$$\begin{aligned} &\textstyle\sum_{k\in B, l\in B^c} n_{kl} \ge \sum_{i\in A_2, j\in A_1} n_{ij} \ge \frac{2n_1 n_2}{(1+M_{rt}+c_{31})(1+M_{rt}+c_{32})}, \\ &\textstyle\sum_{k\in B^c, l\in B^c} n_{kl} \ge \sum_{i\in A_2, j\in A_3} n_{ij} \ge \frac{2n_1 n_{j_3}}{(1+M_{rt}+c_{32})(1+M_{rt}+c_{33})}. \end{aligned} \tag{23}$$

Observe that

$$\tfrac{1}{2}\textstyle\sum_{i,j\in B} n_{ij} - \sum_{i,j\in B} n_{ij}\hat p_{ij} \;=\; 0,$$

where $\hat p_{ij} = e^{\hat\beta_i}/(e^{\hat\beta_i}+e^{\hat\beta_j})$. Consider any $j \notin B$. There are only two cases for $j \notin B$:
Case A: $\hat\beta_j \le \hat\beta_{i_1}/4 + kh$. For each $l \in B$, we have

$$\hat\beta_l - \hat\beta_j \ge \hat\beta_{i_1}/4 + (k+1)h - (\hat\beta_{i_1}/4 + kh) = h.$$

Therefore,

$$\begin{aligned}\min\{a_j, \sum_{i\in B} n_{ij}\} - \sum_{i\in B} n_{ij}\hat{p}_{ij} &\le \sum_{i\in B} n_{ij} - \sum_{i\in B} n_{ij}\hat{p}_{ij} = \sum_{i\in B} n_{ij}(\frac{1}{1+e^{\hat{\beta}_j-\hat{\beta}_i}}) \\ &\le \frac{\sum_{i\in B} n_{ij}}{1+e^h}.\end{aligned}$$

Case B: $j \in D_k$. Thus

$$\min\{a_j, \sum_{i\in B} n_{ij}\} - \sum_{i\in B} n_{ij}\hat{p}_{ij} \le \sum_{i\in B} n_{ij} - \sum_{i\in B} n_{ij}\hat{p}_{ij} \le \sum_{i\in B} n_{ij}.$$

Combining the above two cases, it yields

$$\begin{aligned}&\sum_{j\in B^c} \min\{a_j, \sum_{i\in B} n_{ij}\} + \frac{1}{2}\sum_{i,j\in B} n_{ij} - \sum_{i\in B} a_i \\ =& [\frac{1}{2}\sum_{i,j\in B} n_{ij} - \sum_{i,j\in B} n_{ij}\hat{p}_{ij}] + [\sum_{j\in B^c}(\min\{a_j, \sum_{i\in B} n_{ij}\} - \sum_{i\in B} n_{ij}\hat{p}_{ij})] \\ =& \sum_{j\in B^c}(\min\{a_j, \sum_{i\in B} n_{ij}\} - \sum_{i\in B} n_{ij}\hat{p}_{ij}) \\ \le& \frac{\sum_{i\in B, j\in B^c} n_{ij}}{1+e^h} + \sum_{i\in B}\sum_{j\in D_k} n_{ij} \\ \le& \frac{\sum_{i\in B, j\in B^c} n_{ij}}{1+e^h} + \frac{|B|(r+t)}{h/8-1}.\end{aligned}$$

By (18), we have

$$\begin{aligned}&\tfrac{\sum_{i\in B, j\in B^c} n_{ij}}{1+e^h} + \tfrac{|B|t}{\hat{\beta}_{\max}^{1/2}} \\ \ge& \tfrac{1}{1+M_{rt}} \min\{\sum_{i,j\in B^c} n_{ij}, \sum_{i\in B, j\in B^c} n_{ij}\} - t\sqrt{t\log t} - r\sqrt{r\log r}.\end{aligned} \tag{24}$$

Since $\sum_{i\in B, j\in B^c} n_{ij} \le rt$ and $|B| < r+t$, by (23), we have

$$\begin{aligned}\tfrac{rt}{1+e^h} + \tfrac{(r+t)^2}{h/8-1} \ge& \min\{\tfrac{2n_1n_2}{(1+M_{rt}+c_{31})(1+M_{rt}+c_{32})}, \tfrac{2n_1n_{j_3}}{(1+M_{rt}+c_{32})(1+M_{rt}+c_{33})}\} \\ &-t\sqrt{t\log t} - r\sqrt{r\log r}.\end{aligned}$$

Note that $e^{\hat{\beta}_{\max}^{1/2}} \gg \hat{\beta}_{\max}^{1/2}$ if $\hat{\beta}_{\max}$ is sufficiently large. Thus if $M_{rt} = o(\log \rho_{rt})$, then when r and t are sufficient large, we have

$$\frac{(\lambda_{rt}/\rho_{rt})^2}{h} \ge \frac{c_{34}}{M_{rt}^2},$$

where c_{34} is constant. By note that $h = \hat{\beta}_{\max}^{1/2}$, consequently, we have

$$\hat{u}_{\max} = e^{\hat{\beta}_{\max}} \le e^{c^* M_{rt}^4},$$

where c^* is a constant.

On the other hand, if $\hat{\beta}_{\min} \neq 0$, reparametrize $\tilde{\beta}_i = \hat{\beta}_i - \hat{\beta}_{\min}$. Repeat the above arguments, we have $e^{\tilde{\beta}_{\max}} \le e^{c^* M_{rt}^4}$, where $\tilde{\beta}_{\max} = \max_i \tilde{\beta}_i$. Therefore, $\hat{u}_{\max}/\hat{u}_{\min} = e^{\hat{\beta}_{\max} - \hat{\beta}_{\min}} \le e^{c^* M_{rt}^4}$.

For other cases: Case II. $\beta_{j_1} \ge \beta_{i_1} \ge \beta_{i_2} \ge \beta_{j_2}$; Case III. $\beta_{i_1} \ge \beta_{i_2} \ge \beta_{j_1} \ge \beta_{j_2}$; Case IV. $\beta_{j_1} \ge \beta_{j_2} \ge \beta_{i_1} \ge \beta_{i_2}$, the arguments are similar and we omit their proofs.

□

Lemma 4. *If (15) and (21) hold, then*

$$\max_{i=1,\dots,r+t} |\Delta u_i| \le \max_{i,j=1,\dots,r+t} |\Delta u_i - \Delta u_j| = \frac{\theta_{rt}}{1-\theta_{rt}}, \tag{25}$$

where θ_{rt} is defined in Proposition 3.

Proof. Let k_1 and k_2 be two indices such that $\hat{\alpha}_* := \hat{u}_{k_1}/u_{k_1} = \max\limits_{1\le k\le r+t} \hat{u}_k/u_k$ and $\hat{\alpha}_{**} := \hat{u}_{k_2}/u_{k_2} = \min\limits_{1\le k\le r+t} \hat{u}_k/u_k$. There are four cases to consider for k_1 and k_2.

Case I: $k_1, k_2 \in \Omega_1$. A direct calculation gives

$$\begin{aligned} \frac{\hat{u}_{k_1}}{\hat{u}_{k_1}+\hat{u}_l} - \frac{u_{k_1}}{u_{k_1}+u_l} &= \frac{\hat{u}_{k_1}/u_{k_1} - \hat{u}_l/u_l}{(\frac{\hat{u}_{k_1}}{u_{k_1}} + \frac{\hat{u}_l}{u_l}\frac{u_l}{u_{k_1}})(1+\frac{u_{k_1}}{u_l})} \ge \frac{\hat{u}_{k_1}/u_{k_1} - \hat{u}_l/u_l}{\hat{\alpha}_*(1+\frac{u_l}{u_{k_1}})(1+\frac{u_{k_1}}{u_l})} \\ &\ge \frac{(\hat{\alpha}_* - \hat{u}_l/u_l)M_{rt}}{\hat{\alpha}_*(1+M_{rt})^2}. \end{aligned} \tag{26}$$

Therefore, we have $(n_{k_1,l} = 0, l = 1, \cdots, r; n_{k_1,l} = 1, l = r+1, \cdots, r+t)$

$$a_{k_1} - E(a_{k_1}) = \sum_{l=1}^{r+t} n_{ij}\Big(\frac{\hat{u}_{k_1}}{\hat{u}_{k_1}+\hat{u}_l} - \frac{u_{k_1}}{u_{k_1}+u_l}\Big) \ge \sum_{l=r+1}^{r+t} \frac{(\hat{\alpha}_* - \hat{u}_l/u_l)M_{rt}}{\hat{\alpha}_*(1+M_{rt})^2}. \tag{27}$$

Similarly, we have

$$E(a_{k_2}) - a_{k_2} \ \ge\ \sum_{l=r+1}^{r+t} \frac{(\hat{u}_l/u_l - \hat{\alpha}_{**})M_{rt}}{\hat{\alpha}_*(1+M_{rt})^2}. \tag{28}$$

Combining the above two inequality, it yields

$$2\sqrt{t\log t} \ \ge\ 2\max_{k=1,\cdots,r} |a_k - E(a_k)| \ge a_{k_1} - E(a_{k_1}) + E(a_{k_2}) - a_{k_2} \ge \frac{t(\hat{\alpha}_* - \hat{\alpha}_{**})M_{rt}}{\hat{\alpha}_*(1+M_{rt})^2}.$$

It follows

$$\frac{\hat{\alpha}_* - \hat{\alpha}_{**}}{\hat{\alpha}_*} \le \frac{2(1+M_{rt})^2}{M_{rt}}\sqrt{\frac{t}{\log t}}.$$

Case II: $k_1, k_2 \in \Omega_2$. Similar to the proof of Case I, we have

$$\frac{\hat{\alpha}_* - \hat{\alpha}_{**}}{\hat{\alpha}_*} \le \frac{2(1+M_{rt})^2}{M_{rt}}\sqrt{\frac{r}{\log r}}.$$

Case III: $k_1 \in \Omega_1, k_2 \in \Omega_2$. Let k_3 be an index such that $\hat{u}_{k_3}/u_{k_3} = \min_{k\in\Omega_1} \hat{u}_k/u_k$. Similar to the proof of (27), we have

$$\begin{aligned}\sqrt{r\log r} &\geq E(a_{k_2}) - a_{k_2} = \sum_{l\in\Omega_1} (\frac{u_{k_2}}{u_{k_2}+u_l} - \frac{\hat{u}_{k_2}}{\hat{u}_{k_2}+\hat{u}_l}) \geq \sum_{l\in\Omega_1} \frac{(\hat{u}_l/u_l - \hat{\alpha}_{**})M_{rt}}{\hat{\alpha}_*(1+M_{rt})^2} \\ &\geq \frac{t(\hat{u}_{k_3}/u_{k_3} - \hat{\alpha}_{**})M_{rt}}{\hat{\alpha}_*(1+M_{rt})^2}.\end{aligned}$$

Therefore, we have

$$\frac{\hat{u}_{k_3}}{u_{k_3}} - \hat{\alpha}_{**} \leq \frac{\hat{\alpha}_*(1+M_{rt})^2}{tM_{rt}}\sqrt{r\log r}.$$

Since

$$\sqrt{t\log t} \geq E(a_{k_3}) - a_{k_3} = \sum_{l\in\Omega_2} \frac{\hat{u}_l/u_l - \hat{\alpha}_{**} + \hat{\alpha}_{**} - \hat{u}_{k_3}/u_{k_3}}{(\frac{\hat{u}_{k_1}}{u_{k_1}} + \frac{\hat{u}_l}{u_l}\frac{u_l}{u_{k_1}})(1+\frac{u_{k_1}}{u_l})}$$

and

$$\begin{aligned}\sum_{l\in\Omega_2} \frac{\hat{u}_{k_3}/u_{k_3} - \hat{\alpha}_{**}}{(\frac{\hat{u}_{k_1}}{u_{k_1}} + \frac{\hat{u}_l}{u_l}\frac{u_l}{u_{k_1}})(1+\frac{u_{k_1}}{u_l})} &\leq \sum_{l\in\Omega_2} \frac{\hat{u}_{k_3}/u_{k_3} - \hat{\alpha}_{**}}{\hat{\alpha}_{**}(1+\frac{u_l}{u_{k_1}})(1+\frac{u_{k_1}}{u_l})} \\ &\leq \frac{t(1+M_{rt})^2}{M_{rt}} \times (\frac{\hat{u}_{k_3}}{u_{k_3}} - \hat{\alpha}_{**}),\end{aligned}$$

we have

$$\begin{aligned}\sqrt{t\log t} + \frac{\hat{\alpha}_*(1+M_{rt})^2\sqrt{r\log r}}{4M_{rt}\hat{\alpha}_{**}} &\geq \sqrt{t\log t} + \sum_{l\in\Omega_2} \frac{\hat{u}_{k_3}/u_{k_3} - \hat{\alpha}_{**}}{(\frac{\hat{u}_{k_1}}{u_{k_1}} + \frac{\hat{u}_l}{u_l}\frac{u_l}{u_{k_1}})(1+\frac{u_{k_1}}{u_l})} \\ &\geq \sum_{l\in\Omega_2} \frac{\hat{u}_l/u_l - \hat{\alpha}_{**}}{(\frac{\hat{u}_{k_1}}{u_{k_1}} + \frac{\hat{u}_l}{u_l}\frac{u_l}{u_{k_1}})(1+\frac{u_{k_1}}{u_l})} \\ &\geq \sum_{l\in\Omega_2} \frac{(\hat{u}_l/u_l - \hat{\alpha}_{**})M_{rt}}{\hat{\alpha}_*(1+M_{rt})^2}.\end{aligned}$$

On the other hand,

$$\sqrt{t\log t} \geq a_{k_1} - E(a_{k_1}) \geq \sum_{l\in\Omega_2} \frac{(\hat{\alpha}_* - \hat{u}_l/u_l)M_{rt}}{\hat{\alpha}_*(1+M_{rt})^2}.$$

Consequently,

$$\frac{(\hat{\alpha}_* - \hat{\alpha}_{**})}{\hat{\alpha}_*} \leq \frac{2(M_{rt}+1)^2}{M_{rt}}\sqrt{\frac{\log t}{t}} + \frac{\hat{\alpha}_*(1+M_{rt})^4}{4M_{rt}^2\hat{\alpha}_{**}}\sqrt{\frac{r\log r}{t^2}}.$$

Case IV: $k_1 \in \Omega_2, k_2 \in \Omega_1$. Similar to the proof of Case III, we have

$$\frac{(\hat{\alpha}_* - \hat{\alpha}_{**})}{\hat{\alpha}_*} \leq \frac{2(M_{rt}+1)^2}{M_{rt}}\sqrt{\frac{\log r}{r}} + \frac{\hat{\alpha}_*(1+M_{rt})^4}{4M_{rt}^2\hat{\alpha}_{**}}\sqrt{\frac{t\log t}{r^2}}.$$

In view of the above four cases, by Lemma 4, we have

$$\begin{aligned}
\frac{(\hat{\alpha}_* - \hat{\alpha}_{**})}{\hat{\alpha}_*} &\leq \frac{2(M_{rt}+1)^2}{M_{rt}} \times \max\{\sqrt{\frac{\log t}{t}}, \sqrt{\frac{\log r}{r}}\} \\
&\quad + \frac{e^{c_* M_{rt}}(1+M_{rt})^4}{4M_{rt}^2} \times \max\{\sqrt{\frac{t\log t}{r^2}}, \sqrt{\frac{r\log r}{t^2}}\} \\
&\leq [\frac{2(M_{rt}+1)^2}{M_{rt}} + \frac{e^{c_* M_{rt}}(1+M_{rt})^4}{4M_{rt}^2}]\sqrt{\frac{\lambda_{rt}\log\lambda_{rt}}{\rho_{rt}^2}}.
\end{aligned}$$

This completes the proof. □

References

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