Supplementary Material for "Transformed Linear Quantile Regression With Censored Survival Data" by Rui Miao, Liuquan Sun and Guo-Liang Tian

Appendix: Proofs of Theorems

Proof of Theorem 1. For a fixed γ, let $\boldsymbol{\beta}(\tau;\gamma)$ be the solution to the following equation:

$$E\Big\{ZN(H_\gamma^{-1}(\mathbf{Z}^T\boldsymbol{\beta}(\tau)) - \int_0^\tau ZI\{X \geq H_\gamma^{-1}(\mathbf{Z}^T\boldsymbol{\beta}(u))\}\,\mathrm{d}G(\mu)\Big\} = 0. \tag{A.1}$$

In particular, $\boldsymbol{\beta}(\tau;\gamma_0) = \boldsymbol{\beta}_0(\tau)$. To show the consistency of $\hat{\gamma}$ and $\hat{\boldsymbol{\beta}}(\tau)$, we mainly take the following three steps.

Step A1. We first show the uniform consistency of $\hat{\boldsymbol{\beta}}(\tau;\gamma)$; that is,

$$\sup_{\tau\in[\nu,\tau_U],\ \gamma\in\Omega_{\gamma_0}} \|\hat{\boldsymbol{\beta}}(\tau;\gamma) - \boldsymbol{\beta}(\tau;\gamma)\| \xrightarrow{\mathrm{P}} 0. \tag{A.2}$$

Note that $\beta(\tau;\gamma)$ is the solution to the equation (A.1) for any given γ, and $\hat{\beta}(\tau;\gamma)$ is the estimator of $\beta(\tau;\gamma)$. Define $\mathcal{G}_1 = \{Z_iI\{X_i \leq H_\gamma^{-1}(Z_i^Tb)\}\delta_i : b \in R^p, \gamma \in \Omega_{\gamma_0}\}$, and $\mathcal{G}_2 = \{Z_iI\{X_i \geq H_\gamma^{-1}(Z_i^Tb)\} : b \in R^p, \gamma \in \Omega_{\gamma_0}\}$, where Ω_{γ_0} is a neighborhood of γ_0. Both $\mathcal{G}_1$ and $\mathcal{G}_1$ are Glivenko-Cantelli (van der Vaart and Wellner, 1996). Then with some modification of the proof of Theorem 1 in Peng and Huang (2008) and using the Glivenko-Cantelli theorem, we can show that **Step A1** holds. The similar arguments are also used in the proof of Theorem 1 of Qian and Peng (2010).

Step A2. We should prove that

$$\hat{\gamma} \xrightarrow{\mathrm{P}} \gamma_0. \tag{A.3}$$

Note that $\hat{\gamma} = \arg\min_\gamma R_n(\gamma)$. Define

$$R(\gamma) = \int_\nu^{\tau_U} E[W(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3;\ \tau, \gamma)]\,\mathrm{d}\tau,$$

where

$$\begin{aligned} W(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3;\ \tau, \gamma) \;&=\; I(\mathbf{Z}_2 \le \mathbf{Z}_1) I(\mathbf{Z}_3 \le \mathbf{Z}_1) \\ &\quad \times \Big\{ N_2\{H_\gamma^{-1}(\mathbf{Z}_2^T \boldsymbol{\beta}(\tau;\gamma))\} - \int_0^\tau I\{X_2 \ge H_\gamma^{-1}(\mathbf{Z}_2^T \boldsymbol{\beta}(u;\gamma))\}\,\mathrm{d}G(\mu) \Big\} \\ &\quad \times \Big\{ N_3\{H_\gamma^{-1}(\mathbf{Z}_3^T \boldsymbol{\beta}(\tau;\gamma))\} - \int_0^\tau I\{X_3 \ge H_\gamma^{-1}(\mathbf{Z}_3^T \boldsymbol{\beta}(u;\gamma))\}\,\mathrm{d}G(\mu) \Big\}. \end{aligned}$$

To prove (A.3), according to Amemiya (1985, p.106–108), we need to verify the following three conditions:

(i) $R(\gamma)$ is continuous on Ω_{γ_0}, where Ω_{γ_0} is a neighborhood of γ_0;

(ii) $R(\gamma)$ is uniquely minimized at γ_0;

(iii) $\sup_{\gamma \in \Omega_{\gamma_0}} \|R_n(\gamma) - R(\gamma)\| = o_p(1)$.

Now, we verify the above three conditions step by step. First, we have

$$\begin{aligned} R(\gamma) \;&=\; E\Big[\int_\nu^{\tau_U} E^2 \Big\{ I(\mathbf{Z}_2 \le \mathbf{Z}_1) \Big(N_2\{H_\gamma^{-1}(\mathbf{Z}_2^T \boldsymbol{\beta}(\tau;\gamma))\} \\ &\quad - \int_0^\tau I\{X_2 \ge H_\gamma^{-1}(\mathbf{Z}_2^T \boldsymbol{\beta}(u;\gamma))\}\,\mathrm{d}G(\mu) \Big) \Big| \mathbf{Z}_1 \Big\}\,\mathrm{d}\tau \Big]. \end{aligned}$$

By Conditions (R2) and (R4), it is easy to see that $R(\gamma)$ is continuous in Ω_{γ_0}.

Second, note that by (A.1), γ_0 is a minimizer of $R(\gamma)$. Suppose that γ^* ($\neq \gamma_0$) is another minimizer of $R(\gamma)$. Then we have $R(\gamma^*) = R(\gamma_0) = 0$. That is,

$$E\Big\{ N_2\{H_{\gamma^*}^{-1}(\mathbf{Z}_2^T \boldsymbol{\beta}(\tau;\gamma^*))\} - \int_0^\tau I\{X_2 \ge H_{\gamma^*}^{-1}(\mathbf{Z}_2^T \boldsymbol{\beta}(u;\gamma^*))\}\,\mathrm{d}G(\mu) \Big\} = 0,$$

and

$$E\Big\{ N_2\{H_{\gamma_0}^{-1}(\mathbf{Z}_2^T \boldsymbol{\beta}(\tau;\gamma_0))\} - \int_0^\tau I\{X_2 \ge H_{\gamma_0}^{-1}(\mathbf{Z}_2^T \boldsymbol{\beta}(u;\gamma_0))\}\,\mathrm{d}G(\mu) \Big\} = 0.$$

This implies that for $\tau \in [\nu, \tau_U]$,

$$\tilde{F}\{H_{\gamma^*}^{-1}(\mathbf{Z}_2^T\boldsymbol{\beta}(\tau;\gamma^*))\} - \int_0^\tau \left[\bar{F}\{H_{\gamma^*}^{-1}(\mathbf{Z}_2^T\boldsymbol{\beta}(u;\gamma^*))\}\right] \mathrm{d}G(\mu) = 0, \tag{A.4}$$

and

$$\tilde{F}\{H_{\gamma_0}^{-1}(\mathbf{Z}_2^T\boldsymbol{\beta}(\tau;\gamma_0))\} - \int_0^\tau \left[\bar{F}\{H_{\gamma_0}^{-1}(\mathbf{Z}_2^T\boldsymbol{\beta}(u;\gamma_0))\}\right] \mathrm{d}G(\mu) = 0, \tag{A.5}$$

where $\bar{F}(t|\mathbf{Z}) = 1 - F(t|\mathbf{Z})$. Then by (A.4) and (A.5), both $\tilde{F}\{H_{\gamma^*}^{-1}(\mathbf{Z}_2^T\boldsymbol{\beta}(\tau;\gamma^*))\}$ and $\tilde{F}\{H_{\gamma_0}^{-1}(\mathbf{Z}_2^T\boldsymbol{\beta}(\tau;\gamma_0))\}$ are solutions to the following nonlinear integral equation of the second kind for $h(\tau)$:

$$h(\tau) - \int_0^\tau [\bar{F}\{\tilde{F}^{-1}\{h(\mu)\}\}] \, \mathrm{d}G(\mu) = 0.$$

Since the above integral equation has a unique solution (Polyanin and Manzhirov, 2008, p.426), we have $\tilde{F}\{H_{\gamma_0}^{-1}(\mathbf{Z}_2^T\boldsymbol{\beta}(\tau;\gamma_0))\} = \tilde{F}\{H_{\gamma^*}^{-1}(\mathbf{Z}_2^T\boldsymbol{\beta}(\tau;\gamma^*))\}$. Hence it follows from Condition (R5)(a) that $H_{\gamma_0}^{-1}(\mathbf{Z}_2^T\boldsymbol{\beta}(\tau;\gamma_0)) = H_{\gamma^*}^{-1}(\mathbf{Z}_2^T\boldsymbol{\beta}(\tau;\gamma^*))$. From Condition (R3), we obtain $\gamma^* = \gamma_0$. This completes the proof of (ii).

Third, we need to verify (iii). Note that

$$\sup_{\gamma\in\Omega_{\gamma_0}} |R_n(\gamma) - R(\gamma)| \;\; \le \;\; \sup_{\gamma\in\Omega_{\gamma_0}} |R_n(\gamma) - R_n^0(\gamma)| + \sup_{\gamma\in\Omega_{\gamma_0}} |R_n^0(\gamma) - R(\gamma)|, \tag{A.6}$$

where

$$R_n^0(\gamma) = \frac{1}{n}\int_\nu^{\tau_U} \sum_{i=1}^n D_n^0(\mathbf{Z}_i, \tau, \gamma)^2 \, \mathrm{d}\tau,$$

and

$$D_n^0(\boldsymbol{z}, \tau, \gamma) = \frac{1}{n}\sum_{i=1}^n I(\mathbf{Z}_i \le \boldsymbol{z})\Big\{N_i(H_\gamma^{-1}(\mathbf{Z}_i^T\boldsymbol{\beta}(\tau;\gamma))) - \int_0^\tau I[X_i \ge H_\gamma^{-1}(\mathbf{Z}_i^T\boldsymbol{\beta}(\mu;\gamma))]\mathrm{d}G(\mu)\Big\}.$$

Obviously, $R_n^0(\gamma)$ is a V-statistic. Let $U_n^0(\gamma)$ be the corresponding U-statistic. Following the same argument as that of Lemma 1 in the Chapter 2 of Mu (2005) and using the properties of Vapnik–Chervonenkis (VC) class from Lemma 2.5 of Pakes and Pollard (1989), we can prove that the class of functions $\{W(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3;\, \tau, \gamma) : \gamma \in \Omega_{\gamma_0}, \tau \in [\nu, \tau_U]\}$

is Euclidean with constant envelop. Then it follows from Corollary 3.5 of Arcones and Giné (1993) that $\sup_{\gamma\in\Omega_{\gamma_0}} \|U_n^0(\gamma)-R(\gamma)\| = o_p(1)$. Thus, Theorem 3.11 of Arcones and Giné (1993) yields

$$\sup_{\gamma\in\Omega_{\gamma_0}} |R_n^0(\gamma)-R(\gamma)| = o_p(1). \tag{A.7}$$

In the follows, we will prove that $\sup_{\gamma\in\Omega_{\gamma_0}} |R_n(\gamma)-R_n^0(\gamma)| = o_p(1)$. Denote $\theta(\tau) = (\gamma, \boldsymbol{\beta}(\tau))$. Define

$$\psi_{n,i}(\boldsymbol{z};\tau,\theta) = I(\mathbf{Z}_i \le \boldsymbol{z})\Big\{N_i\{H_\gamma^{-1}(\mathbf{Z}_i^T\boldsymbol{\beta}(\tau))\} - \int_0^\tau I[X_i \ge H_\gamma^{-1}(\mathbf{Z}_i^T\boldsymbol{\beta}(\mu))]\,\mathrm{d}G(\mu)\Big\},$$

and

$$\psi(\boldsymbol{z};\tau,\theta) = E\Big[I(\mathbf{Z} \le \boldsymbol{z})\Big\{N\{H_\gamma^{-1}(\mathbf{Z}^T\boldsymbol{\beta}(\tau))\} - \int_0^\tau I[X \ge H_\gamma^{-1}(\mathbf{Z}^T\boldsymbol{\beta}(\mu))]\,\mathrm{d}G(\mu)\Big\}\Big].$$

Similarly, the class of functions $\{\psi_{n,i}(\boldsymbol{z};\gamma,\boldsymbol{\beta})\}$ is Euclidean with constant envelop. Hence for $\theta_\gamma(\tau) = (\gamma, \boldsymbol{\beta}(\tau;\gamma))$, it follows from Lemma 2.8 of Pakes and Pollard (1989) that uniformly in $\boldsymbol{z}\in\mathbb{R}^p$, $\tau\in[\nu,\tau_U]$ and $\gamma\in\Omega_{\gamma_0}$,

$$\Big|\frac{1}{n}\sum_{i=1}^n[\psi_{n,i}(\boldsymbol{z};\tau,\theta) - \psi_{n,i}(\boldsymbol{z};\tau,\theta_\gamma) - \psi(\boldsymbol{z};\tau,\theta) + \psi(\boldsymbol{z};\tau,\theta_\gamma)]\Big| = o_p(1). \tag{A.8}$$

Applying Conditions (R1), (R2) and (R4), we obtain

$$\begin{aligned}
&|\psi(\boldsymbol{z};\tau,\theta) - \psi(\boldsymbol{z};\tau,\theta_\gamma)|\\
\le\ & \Big|E\Big\{I(\mathbf{Z}\le\boldsymbol{z})\Big[\tilde{F}\{H_\gamma^{-1}(\mathbf{Z}_i^T\boldsymbol{\beta}(\tau))\} - \tilde{F}\{H_\gamma^{-1}(\mathbf{Z}_i^T\boldsymbol{\beta}(\tau;\gamma))\}\Big]\Big\}\Big|\\
&+\Big|E\Big\{I(\mathbf{Z}\le\boldsymbol{z})\int_0^\tau\Big[\bar{F}\{H_\gamma^{-1}(\mathbf{Z}_i^T\boldsymbol{\beta}(\mu))\} - \bar{F}\{H_\gamma^{-1}(\mathbf{Z}_i^T\boldsymbol{\beta}(\mu;\gamma))\}\Big]\mathrm{d}G(\mu)\Big\}\Big|\\
\le\ & C_0\Big[\sup_{u\in[\nu,\tau]}\|\boldsymbol{\beta}(u)-\boldsymbol{\beta}(u;\gamma)\| + \nu\Big],
\end{aligned} \tag{A.9}$$

where C_0 is a positive constant. Denote $\hat{\theta}_\gamma(\tau) = (\gamma, \hat{\boldsymbol{\beta}}(\tau;\gamma))$. Then it follows from (A.8) and (A.9) that

$$\sup_{\gamma\in\Omega_{\gamma_0}} |R_n(\gamma) - R_n^0(\gamma)|$$

$$\leq \sup_{\gamma\in\Omega_{\gamma_0}} \frac{1}{n}\int_{\nu}^{\tau_U} \sum_{i=1}^{n} \Big|\{\frac{1}{n}\sum_{i=1}^{n}\psi_{n,i}(\mathbf{Z}_i;\tau,\gamma,\boldsymbol{\beta}_1)\}^2 - \{\frac{1}{n}\sum_{i=1}^{n}\psi_{n,i}(\mathbf{Z}_i;\tau,\gamma,\boldsymbol{\beta}_2)\}^2\Big| \,\mathrm{d}\tau$$
$$\leq C_1\Big[\sup_{\tau\in[\nu,\tau_U],\gamma\in\Omega_{\gamma_0}} \|\hat{\boldsymbol{\beta}}(\tau;\gamma)-\boldsymbol{\beta}(\tau;\gamma)\| + \nu\Big] + o_p(1),$$

where C_1 is a positive constant. Note that ν can be arbitrarily small. Then by (A.2), we have

$$\sup_{\gamma\in\Omega_{\gamma_0}} |R_n(\gamma) - R_n^0(\gamma)| = o_p(1). \tag{A.10}$$

Thus, by combining (A.6), (A.7) and (A.10), we completed the proof of (iii), and hence the proof of (A.3).

Step A3. we show that $\sup_{\tau\in[\nu,\tau_U]} \|\hat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}_0(\tau)\| \xrightarrow{\text{P}} 0.$

Let $\dot{\boldsymbol{\beta}}(\tau;\gamma) = \partial\boldsymbol{\beta}(\tau;\gamma)/\partial\gamma$. First, we need to show that $\dot{\boldsymbol{\beta}}(\tau;\gamma)$ is uniformly bounded in $\tau\in[\nu,\tau_U]$ and $\gamma\in\Omega_{\gamma_0}$. Define

$$\mathbf{B}\{b,\gamma\} = E\Big\{\mathbf{Z}^{\otimes 2}\tilde{f}[H_\gamma^{-1}(\mathbf{Z}^T b)|\mathbf{Z}]\frac{\partial H_\gamma^{-1}(t)}{\partial t}|_{t=\mathbf{Z}^T b}\Big\},$$
$$\mathbf{J}\{b,\gamma\} = E\Big\{\mathbf{Z}^{\otimes 2}\bar{f}[H_\gamma^{-1}(\mathbf{Z}^T b)|\mathbf{Z}]\frac{\partial H_\gamma^{-1}(t)}{\partial t}|_{t=\mathbf{Z}^T b}\Big\}, \tag{A.11}$$

where $\bar{f}(t|\mathbf{Z}) = \mathrm{d}\bar{F}(t|\mathbf{Z})/\mathrm{d}t$. Thus, it follows from (A.1) that

$$\begin{aligned}&\mathbf{B}\{\boldsymbol{\beta}(\tau;\gamma),\gamma\}^{-1}\mathbf{Q}\{\boldsymbol{\beta}(\tau;\gamma),\gamma\}\\ =\;& \dot{\boldsymbol{\beta}}(\tau;\gamma) - \int_0^\tau \mathbf{B}\{\boldsymbol{\beta}(\tau;\gamma),\gamma\}^{-1}\mathbf{J}\{\boldsymbol{\beta}(\mu;\gamma),\gamma\}\dot{\boldsymbol{\beta}}(\mu;\gamma)\,\mathrm{d}G(\mu),\end{aligned}$$

where

$$\begin{aligned}\mathbf{Q}\{\boldsymbol{\beta}(\tau;\gamma),\gamma\} \;=\;& E\Big\{\mathbf{Z}\tilde{f}[H_\gamma^{-1}(\mathbf{Z}^T\boldsymbol{\beta}(\tau;\gamma))|\mathbf{Z}]\frac{\partial H_\gamma^{-1}(\mathbf{Z}^T\boldsymbol{\beta}(\tau;\gamma))}{\partial\gamma}\Big\}\\ &-E\Big\{\int_0^\tau \mathbf{Z}\bar{f}[H_\gamma^{-1}(\mathbf{Z}^T\boldsymbol{\beta}(\mu;\gamma))|\mathbf{Z}]\frac{\partial H_\gamma^{-1}(\mathbf{Z}^T\boldsymbol{\beta}(\mu;\gamma))}{\partial\gamma}\,\mathrm{d}G(\mu)\Big\}.\end{aligned}$$

The above equation is a linear Volterra integral equation of the second kind (Polyanin and Manzhirov, 2008). It can be checked that by conditions (R2) and (R4), $\mathbf{B}\{\boldsymbol{\beta}(\tau;\gamma),\gamma\}^{-1}$,

$\mathbf{Q}\{\boldsymbol{\beta}(\tau;\gamma),\gamma\}$ and $J\{\boldsymbol{\beta}(\tau;\gamma),\gamma\}$ are continuous in $\tau\in[\nu,\tau_U]$ and $\gamma\in\Omega_{\gamma_0}$. Therefore, according to Theorem 3.3 in Linz (1985), the above integral equation has a unique solution $\dot{\boldsymbol{\beta}}(\tau;\gamma)$, which is uniformly bounded in $\tau\in[\nu,\tau_U]$ and $\gamma\in\Omega_{\gamma_0}$. Note that

$$\begin{aligned}\|\hat{\boldsymbol{\beta}}(\tau)-\boldsymbol{\beta}_0(\tau)\| &= \|\hat{\boldsymbol{\beta}}(\tau;\hat{\gamma})-\boldsymbol{\beta}(\tau;\gamma_0)\| \\ &\le \|\hat{\boldsymbol{\beta}}(\tau;\hat{\gamma})-\boldsymbol{\beta}(\tau;\hat{\gamma})\|+\|\boldsymbol{\beta}(\tau;\hat{\gamma})-\boldsymbol{\beta}(\tau;\gamma_0)\| \\ &\le \|\hat{\boldsymbol{\beta}}(\tau;\hat{\gamma})-\boldsymbol{\beta}(\tau;\hat{\gamma})\|+\sup_{\tau\in[\nu,\tau_U],\gamma\in\Omega_{\gamma_0}}\|\dot{\boldsymbol{\beta}}(\tau;\gamma)\||\hat{\gamma}-\gamma_0|.\end{aligned}$$

Then by (A.2) and (A.3), we have that $\sup_{\tau\in[\nu,\tau_U]}\|\hat{\boldsymbol{\beta}}(\tau)-\boldsymbol{\beta}_0(\tau)\|\xrightarrow{\mathrm{P}}0$. This completes the proof of Theorem 1.

Proof of Theorem 2. To show the asymptotic distributions of $\hat{\gamma}$ and $\hat{\boldsymbol{\beta}}(\tau)$, we take the following three steps.

Step B1. We prove that $\hat{\gamma}$ is $n^{1/2}$-consistent.

Let $\hat{\theta}(\tau)=(\gamma,\hat{\boldsymbol{\beta}}(\tau;\gamma))$, $\theta_0(\tau)=(\gamma_0,\boldsymbol{\beta}(\tau;\gamma_0))$. In view of (A.2), following the argument in Lai and Ying (1988) and Lemma B.1 in Peng and Huang (2008), we obtain that uniformly in $\boldsymbol{z}\in\mathbb{R}^p$ and $\tau\in[\nu,\tau_U]$, for any sequence $d_n=o(1)$,

$$\sup_{|\gamma-\gamma_0|\le d_n}\Big|n^{-1/2}\sum_{i=1}^{n}[\psi_{n,i}(\boldsymbol{z};\tau,\hat{\theta})-\psi_{n,i}(\boldsymbol{z};\tau,\theta_0)-\psi(\boldsymbol{z};\tau,\hat{\theta})+\psi(\boldsymbol{z};\tau,\theta_0)]\Big|=o_p(1). \quad \text{(A.12)}$$

Denote $\dot{\boldsymbol{\psi}}_\beta(\boldsymbol{z};\tau,\theta)$ and $\dot{\psi}_\gamma(\boldsymbol{z};\tau,\theta)$ to be the partial derivative of $\psi(\boldsymbol{z};\tau,\theta)$ with respect to $\boldsymbol{\beta}$ and γ, respectively, that is,

$$\begin{aligned}\dot{\boldsymbol{\psi}}_\beta(\boldsymbol{z};\tau,\theta) &= E\Big[I(\mathbf{Z}\le\boldsymbol{z})\Big\{\tilde{f}(H_\gamma^{-1}(\mathbf{Z}^T\boldsymbol{\beta}(\tau))|\mathbf{Z})\,\frac{\mathrm{d}H_\gamma^{-1}(t)}{\mathrm{d}t}|_{t=\mathbf{Z}^T\beta(\tau)}\mathbf{Z} \\ &\quad+\int_0^\tau\overline{f}(H_\gamma^{-1}(\mathbf{Z}^T\boldsymbol{\beta}(\mu))|\mathbf{Z})\,\frac{\mathrm{d}H_\gamma^{-1}(t)}{\mathrm{d}t}|_{t=\mathbf{Z}^T\beta(\mu)}\mathbf{Z}\,\mathrm{d}G(\mu)\Big\}\Big], \\ \dot{\psi}_\gamma(\boldsymbol{z};\tau,\theta) &= E\Big[I(\mathbf{Z}\le\boldsymbol{z})\Big\{\tilde{f}(H_\gamma^{-1}(\mathbf{Z}^T\boldsymbol{\beta}(\tau))|\mathbf{Z})\,\frac{\mathrm{d}H_\gamma^{-1}(\mathbf{Z}^T\boldsymbol{\beta}(\tau))}{\mathrm{d}\gamma} \\ &\quad+\int_0^\tau\overline{f}(H_\gamma^{-1}(\mathbf{Z}^T\boldsymbol{\beta}(\mu))|\mathbf{Z})\,\frac{\mathrm{d}H_\gamma^{-1}(\mathbf{Z}^T\boldsymbol{\beta}(\mu))}{\mathrm{d}\gamma}\,\mathrm{d}G(\mu)\Big\}\Big].\end{aligned}$$

Similar to Theorem 2 of Peng and Huang (2008), we get

$$\sup_{\tau\in[\nu,\tau_U],\gamma\in\Omega_{\gamma_0}} \|\hat{\boldsymbol{\beta}}(\tau;\gamma)-\boldsymbol{\beta}(\tau;\gamma)\| = O_p(n^{-1/2}). \tag{A.13}$$

Define $P(\boldsymbol{z},\tau) = \dot{\boldsymbol{\psi}}_\beta(\boldsymbol{z};\tau,\theta_0)^T\dot{\boldsymbol{\beta}}(\tau;\gamma_0)+\dot{\psi}_\gamma(\boldsymbol{z};\tau,\theta_0)$. Thus, it follows from (A.12) and (A.13) that uniformly in $\boldsymbol{z}\in\mathbb{R}^p$, $\tau\in[\nu,\tau_U]$ and $|\gamma-\gamma_0|\le d_n$,

$$\begin{aligned} D_n(\boldsymbol{z},\tau,\gamma) &= D_n^0(\boldsymbol{z},\tau,\gamma_0)+\dot{\boldsymbol{\psi}}_\beta(\boldsymbol{z};\tau,\theta_0)^T[\{\hat{\boldsymbol{\beta}}(\tau;\gamma)-\boldsymbol{\beta}(\tau;\gamma)\}+\{\boldsymbol{\beta}(\tau;\gamma)-\boldsymbol{\beta}(\tau;\gamma_0)\}] \\ &\quad +\dot{\psi}_\gamma(\boldsymbol{z};\tau,\theta_0)(\gamma-\gamma_0)+o_p(n^{-1/2})+O_p(|\gamma-\gamma_0|^2) \\ &= D_n^0(\boldsymbol{z},\tau,\gamma_0)+P(\boldsymbol{z},\tau)(\gamma-\gamma_0)+O_p(n^{-1/2})+o_p(|\gamma-\gamma_0|)). \end{aligned} \tag{A.14}$$

Using the law of large numbers and the central limit theorem, we have

$$\frac{1}{n}\sum_{i=1}^n\int_\nu^{\tau_U} P^2(\mathbf{Z}_i,\tau)\,\mathrm{d}\tau = \Delta+o_p(1),$$

and

$$W_n=\frac{1}{n}\sum_{i=1}^n\int_\nu^{\tau_U} D_n^0(\mathbf{Z}_i,\tau,\gamma_0)P(\mathbf{Z}_i,\tau)\,\mathrm{d}\tau = O_p(n^{-1/2}),$$

where $\Delta = E\{\int_\nu^{\tau_U} P^2(\mathbf{Z}_i,\tau)\,\mathrm{d}\tau\}>0$. Hence (A.3) and (A.14) imply that

$$\begin{aligned} R_n(\hat\gamma)-R_n(\gamma_0) &= \Delta(\hat\gamma-\gamma_0)^2+2W_n(\hat\gamma-\gamma_0)+O_p(n^{-1/2}(\hat\gamma-\gamma_0)) \\ &\quad +O_p(n^{-1})+o_p((\hat\gamma-\gamma_0)^2) \\ &= \Delta(\hat\gamma-\gamma_0)^2+O_p(n^{-1/2}(\hat\gamma-\gamma_0)) \\ &\quad +O_p(n^{-1})+o_p((\hat\gamma-\gamma_0)^2). \end{aligned} \tag{A.15}$$

Note that for sufficiently large n, $o_p((\hat\gamma-\gamma_0)^2)\ge-\Delta(\hat\gamma-\gamma_0)^2/2$. Write $C_n n^{-1/2}(\hat\gamma-\gamma_0)$ for $O_p(n^{-1/2}(\hat\gamma-\gamma_0))$, where $C_n=O_p(1)$. Since $R_n(\hat\gamma)\le R_n(\gamma_0)$, it follows from (A.15) that

$$\frac{\Delta}{2}(\hat\gamma-\gamma_0)^2+C_n n^{-1/2}(\hat\gamma-\gamma_0)=O_p(n^{-1}),$$

that is,

$$\frac{\Delta}{2}\left[(\hat\gamma-\gamma_0)+\frac{n^{-1/2}C_n}{\Delta}\right]^2=\frac{n^{-1}C_n^2}{2\Delta}+O_p(n^{-1})=O_p(n^{-1}).$$

Thus,

$$|(\hat{\gamma}-\gamma_0)+\frac{n^{-1/2}C_n}{\Delta}| = O_p(n^{-1/2}),$$

which implies $|\hat{\gamma}-\gamma_0| = O_p(n^{-1/2})$. Hence $\hat{\gamma}$ is $n^{1/2}$-consistent.

Step B2. We prove that for any sequence $\delta_n = o_p(1)$,

$$\sup_{\tau\in[\nu,\tau_U],\,|\gamma-\gamma_0|\le\delta_n} \|\hat{\boldsymbol{\beta}}(\tau;\gamma)-\boldsymbol{\beta}(\tau;\gamma)+\boldsymbol{\beta}(\tau;\gamma_0)-\hat{\boldsymbol{\beta}}(\tau;\gamma_0)\| = o_p(n^{-1/2}). \tag{A.16}$$

Denote $\boldsymbol{\mu}\{\boldsymbol{\beta}(\tau),\gamma\} = E[\mathbf{Z}N\{H_\gamma^{-1}(\mathbf{Z}^T\boldsymbol{\beta}(\tau))\}]$ and $\tilde{\boldsymbol{\mu}}\{\boldsymbol{\beta}(\tau),\gamma\} = E[\mathbf{Z}I\{X\ge H_\gamma^{-1}(\mathbf{Z}^T\boldsymbol{\beta}(\tau))\}]$. Similar to (A.12), we have that uniformly in $\tau\in[\nu,\tau_U]$,

$$\begin{aligned}
&\sup_{|\gamma-\gamma_0|\le\delta_n}\Big\|n^{1/2}\Big\{\mathbf{S}_n\{\hat{\boldsymbol{\beta}}(\tau;\gamma);\gamma\}-\mathbf{S}_n\{\boldsymbol{\beta}(\tau;\gamma);\gamma\}\Big\}\\
&-n^{1/2}\Big\{[\boldsymbol{\mu}\{\hat{\boldsymbol{\beta}}(\tau;\gamma),\gamma\}-\boldsymbol{\mu}\{\boldsymbol{\beta}(\tau;\gamma),\gamma\}]-\int_0^\tau[\tilde{\boldsymbol{\mu}}\{\hat{\boldsymbol{\beta}}(\mu;\gamma),\gamma\}-\tilde{\boldsymbol{\mu}}\{\boldsymbol{\beta}(\mu;\gamma),\gamma\}]\mathrm{d}G(\mu)\Big\}\Big\|\\
&= o_p(1).
\end{aligned}\tag{A.17}$$

Following the argument in Peng and Huang (2008), we have that $\boldsymbol{\mu}\{\hat{\boldsymbol{\beta}}(\tau),\gamma\}$ converges to $\boldsymbol{\mu}\{\boldsymbol{\beta}_0(\tau),\gamma\}$ in probability uniformly in $\tau\in[\nu,\tau_U]$ and $\gamma\in\Omega_{\gamma_0}$. Let $o_I(\rho_n)$ denote a term that converges to 0 in probability uniformly in $\tau\in I$ after being divided by ρ_n. Because $\lim_{n\to\infty} n^{1/2}\|S_L\| = 0$, it follows from the definition of $\hat{\boldsymbol{\beta}}(\tau;\gamma)$ that $n^{1/2}\mathbf{S}_n(\hat{\boldsymbol{\beta}}(\tau;\gamma);\gamma) = o_{[\nu,\tau_U]}(1)$. Thus, using the similar arguments to the proof of Theorem 2 in Peng and Huang (2008), we have

$$\begin{aligned}
-n^{1/2}\mathbf{S}_n(\boldsymbol{\beta}(\tau;\gamma),\gamma) &= n^{1/2}\Big[\boldsymbol{\mu}\{\hat{\boldsymbol{\beta}}(\tau;\gamma),\gamma\}-\boldsymbol{\mu}\{\boldsymbol{\beta}(\tau;\gamma),\gamma\}\Big]\\
&\quad -\int_0^\tau n^{1/2}\Big[\tilde{\boldsymbol{\mu}}\{\hat{\boldsymbol{\beta}}(\mu;\gamma),\gamma\}-\tilde{\boldsymbol{\mu}}\{\boldsymbol{\beta}(\mu;\gamma),\gamma\}\Big]\,\mathrm{d}G(\mu)+o_{[\nu,\tau_U]}(1)\\
&= n^{1/2}\Big[\boldsymbol{\mu}\{\hat{\boldsymbol{\beta}}(\tau;\gamma),\gamma\}-\boldsymbol{\mu}\{\boldsymbol{\beta}(\tau;\gamma),\gamma\}\Big]\\
&\quad -\int_0^\tau\Big\{\Big[\mathbf{J}\{\boldsymbol{\beta}(\mu;\gamma),\gamma\}\mathbf{B}^{-1}\{\boldsymbol{\beta}(\mu;\gamma),\gamma\}+o_{[\nu,\tau_U]}(1)\Big]\\
&\quad \times n^{1/2}\Big[\boldsymbol{\mu}\{\hat{\boldsymbol{\beta}}(\mu;\gamma),\gamma)\}-\boldsymbol{\mu}\{\boldsymbol{\beta}(\mu;\gamma),\gamma\}\Big]\Big\}\,\mathrm{d}G(\mu)+o_{[\nu,\tau_U]}(1),
\end{aligned}$$

where $\mathbf{J}$ and $\mathbf{B}$ are defined in (A.11). Note that the above equation is a stochastic differential equation about $n^{1/2}[\boldsymbol{\mu}\{\hat{\boldsymbol{\beta}}(\tau;\gamma),\gamma\} - \boldsymbol{\mu}\{\boldsymbol{\beta}(\tau;\gamma),\gamma\}]$. Then it follows from the production integration theory (Gill and Johansen, 1990; Andersen *et al*, 1998) that

$$n^{1/2}[\boldsymbol{\mu}\{\hat{\boldsymbol{\beta}}(\tau),\gamma\} - \boldsymbol{\mu}\{\boldsymbol{\beta}(\tau;\gamma),\gamma\}] = \boldsymbol{\phi}(-n^{1/2}\mathbf{S}_n\{\boldsymbol{\beta}(\tau;\gamma),\gamma\}) + o_{[\nu,\tau_U]}(1),$$

where $\boldsymbol{\phi}$ is a map from $\mathcal{F}$ to $\mathcal{F}$ such that for any $\mathbf{g} \in \mathcal{F}$, $\boldsymbol{\phi}(\mathbf{g})(\tau) = \int_0^\tau I(s,\tau)\,\mathrm{d}\mathbf{g}(s)$ with

$$I(s,t) = \prod_{\mu\in(s,t]} \Big[\mathbf{I}_p + \mathbf{J}\{\boldsymbol{\beta}(\tau;\gamma),\gamma\}\mathbf{B}^{-1}\{\boldsymbol{\beta}(\tau;\gamma),\gamma\}\mathrm{d}G(\mu)\Big],$$

and

$$\mathcal{F} = \{\mathbf{g}\colon [\nu,\tau_U] \to \mathbb{R}^p, \mathbf{g} \text{ is left-continuous with right limit}, \mathbf{g}(0) = \mathbf{0}_p\}.$$

Using the Taylor expansion and the continuous mapping theorem, we have that for any $\gamma \in \Omega_{\gamma_0}$,

$$n^{1/2}(\hat{\boldsymbol{\beta}}(\tau;\gamma) - \boldsymbol{\beta}(\tau;\gamma)) = \mathbf{B}^{-1}\{\boldsymbol{\beta}(\tau;\gamma),\gamma\}\boldsymbol{\phi}(-n^{1/2}\mathbf{S}_n\{\boldsymbol{\beta}(\tau;\gamma),\gamma\}) + o_{[\nu,\tau_U]}(1). \quad \text{(A.18)}$$

Based on the proof of Theorem 1 in Peng and Huang (2008), we have

$$\sup_{\tau\in[\nu,\tau_U]} \|\mathbf{S}_n\{\boldsymbol{\beta}(\tau;\gamma_0),\gamma_0\})\| = O_p(n^{-1/2}),$$

which combining with (A.18) implies

$$\begin{aligned} & n^{1/2}\|\hat{\boldsymbol{\beta}}(\tau;\hat{\gamma}) - \boldsymbol{\beta}(\tau;\hat{\gamma}) + \boldsymbol{\beta}(\tau;\gamma_0) - \hat{\boldsymbol{\beta}}(\tau;\gamma_0)\| \\ & \le C_2\|\boldsymbol{\phi}(-n^{1/2}\mathbf{S}_n\{\boldsymbol{\beta}(\tau;\gamma),\hat{\gamma}\}) - \boldsymbol{\phi}(-n^{1/2}\mathbf{S}_n\{\boldsymbol{\beta}(\tau;\gamma_0),\gamma_0\})\| + o_{[\nu,\tau_U]}(1), \end{aligned} \quad \text{(A.19)}$$

where C_2 is a positive constant. Note that $E\{\mathbf{S}_n(\boldsymbol{\beta}(\tau;\gamma),\gamma)\} = 0$. Similar to (A.17), we have

$$\sup_{\tau\in[\nu,\tau_U],\ |\gamma-\gamma_0|\le\delta_n} \|n^{1/2}\mathbf{S}_n\{\boldsymbol{\beta}(\tau;\gamma),\gamma\} - n^{1/2}\mathbf{S}_n\{\boldsymbol{\beta}(\tau;\gamma_0\},\gamma_0)\| = o_p(1).$$

Because $\boldsymbol{\phi}(.)$ is a linear operator, it follows that

$$\sup_{\tau\in[\nu,\tau_U],\ |\gamma-\gamma_0|\le\delta_n} \|\boldsymbol{\phi}(-n^{1/2}\mathbf{S}_n\{\boldsymbol{\beta}_0(\tau),\gamma\}) - \boldsymbol{\phi}(-n^{1/2}\mathbf{S}_n\{\boldsymbol{\beta}(\tau;\gamma_0),\gamma_0\})\| = o_p(1). \quad \text{(A.20)}$$

Thus, (A.16) follows from (A.19) and (A.20).

Step B3. We show the asymptotic normality of $\hat{\boldsymbol{\beta}}(\tau)$ and $\hat{\gamma}$.

Let γ satisfy that $|\gamma-\gamma_0| \leq \delta n^{-1/2}$, where δ is a positive constant. Note that $\boldsymbol{\beta}(\tau;\gamma_0) = \boldsymbol{\beta}_0(\tau)$. Then it follows from (A.18) that

$$\hat{\boldsymbol{\beta}}(\tau;\gamma_0) - \boldsymbol{\beta}_0(\tau) = n^{-1/2}\mathbf{B}^{-1}\{\boldsymbol{\beta}_0(\tau),\gamma_0\}\boldsymbol{\phi}(-n^{1/2}\mathbf{S}_n\{\boldsymbol{\beta}_0(\tau),\gamma_0\}) + o_{[\nu,\tau_U]}(n^{-1/2}).$$

Therefore, combining (A.14) and (A.16), we obtain

$$\begin{aligned} D_n(\boldsymbol{z},\tau,\gamma) &= D_n^0(\boldsymbol{z},\tau,\gamma_0) + n^{-1/2}\dot{\boldsymbol{\psi}}_\beta(\boldsymbol{z};\tau,\theta_0)^T\mathbf{B}^{-1}\{\boldsymbol{\beta}_0(\tau),\gamma_0\}\boldsymbol{\phi}(-n^{1/2}\mathbf{S}_n\{\boldsymbol{\beta}_0(\tau),\gamma_0)\} \\ &\quad + P(\boldsymbol{z},\tau)(\gamma-\gamma_0) + o_{[\nu,\tau_U]}(n^{-1/2}). \end{aligned}$$

Therefore,

$$\begin{aligned} R_n(\gamma) = \frac{1}{n}\sum_{i=1}^n \int_\nu^{\tau_U} &\Big[D_n^0(\mathbf{Z}_i,\tau,\gamma_0)+\dot{\boldsymbol{\psi}}_\beta(\mathbf{Z}_i;\tau,\theta_0)^T\mathbf{B}^{-1}\{\boldsymbol{\beta}_0(\tau),\gamma_0\}\boldsymbol{\phi}(-n^{1/2}\mathbf{S}_n\{\boldsymbol{\beta}_0(\tau),\gamma_0\})\Big]^2 \mathrm{d}\tau \\ &+ 2H_n(\gamma-\gamma_0) + \Delta(\gamma-\gamma_0)^2 + o_p(n^{-1}), \end{aligned} \tag{A.21}$$

where

$$\begin{aligned} H_n = \frac{1}{n}\sum_{i=1}^n \int_\nu^{\tau_U} &\Big\{P(\mathbf{Z}_i,\tau)[D_n^0(\mathbf{Z}_i,\tau,\gamma_0) \\ &+\dot{\boldsymbol{\psi}}_\beta(\mathbf{Z}_i;\tau,\theta_0)^T\mathbf{B}^{-1}\{\boldsymbol{\beta}_0(\tau),\gamma_0\}\boldsymbol{\phi}(-n^{1/2}\mathbf{S}_n\{\boldsymbol{\beta}_0(\tau),\gamma_0\}]\Big\}\,\mathrm{d}\tau. \end{aligned}$$

Let $\tilde{\gamma} = \gamma_0 - H_n/\Delta$. Then it follows from (A.21) that

$$\begin{aligned} 0 \geq R_n(\hat{\gamma}) - R_n(\tilde{\gamma}) &= 2H_n(\hat{\gamma}-\gamma_0) + \Delta(\hat{\gamma}-\gamma_0)^2 + o_p(n^{-1}) \\ &\quad - 2H_n(\tilde{\gamma}-\gamma_0) - \Delta(\tilde{\gamma}-\gamma_0)^2 - o_p(n^{-1}) \\ &= 2H_n(\hat{\gamma}-\gamma_0) + \Delta(\hat{\gamma}-\gamma_0)^2 + \frac{H_n^2}{\Delta} + o_p(n^{-1}), \end{aligned}$$

which implies that

$$\Delta\Big(\hat{\gamma}-\gamma_0+\frac{H_n}{\Delta}\Big)^2 \leq o_p(n^{-1}).$$

Thus,

$$\hat{\gamma} - \gamma_0 = -\frac{H_n}{\Delta} + o_p(n^{-1/2}). \tag{A.22}$$

Let $Y_i = (\mathbf{Z}_i, X_i, \delta_i)$, and

$$\varphi(Y_i, Y_j) = \int_{\nu}^{\tau_U} \Big[I(\mathbf{Z}_j \leq \mathbf{Z}_i) P(\mathbf{Z}_i, \tau) \{ N_j(H_{\gamma_0}^{-1}(\mathbf{Z}_j^T \boldsymbol{\beta}_0(\tau))) - \int_0^{\tau} I[X_j \geq H_{\gamma_0}^{-1}(\mathbf{Z}_j^T \boldsymbol{\beta}_0(u))] \,\mathrm{d}G(\mu) \}$$

$$+ P(\mathbf{Z}_j, \tau) \dot{\boldsymbol{\psi}}_{\beta}(\mathbf{Z}_j; \tau, \theta_0)^T \mathbf{B}^{-1} \{ \boldsymbol{\beta}_0(\tau), \gamma_0 \} \boldsymbol{\phi}(h_j(\tau)) \Big] \mathrm{d}\tau,$$

where $h_j(\tau) = \mathbf{Z}_j \Big\{ N_j(H_{\gamma_0}^{-1} \{ \mathbf{Z}_j^T \boldsymbol{\beta}_0(\tau) \}) - \int_0^{\tau} I[\, X_j \, \geq \, H_{\gamma_0}^{-1} \{ \mathbf{Z}_j^T \boldsymbol{\beta}_0(\mu) \}] \,\mathrm{d}G(\mu) \}$. Then we have

$$H_n = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \varphi(Y_i, Y_j) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{\varphi}(Y_i, Y_j),$$

which is a V-statistic, where

$$\tilde{\varphi}(Y_i, Y_j) = \frac{1}{2} \Big\{ \varphi(Y_i, Y_j) + \varphi(Y_j, Y_i) \Big\}.$$

Define

$$U_n = \frac{1}{\binom{2}{n}} \sum \sum_{i<j} \tilde{\varphi}(Y_i, Y_j),$$

which is a U-statistic. Since $E\|\varphi(Y_i, Y_j)\|^2 < \infty$, it follows from Lemma 5.7.3 of Serfling (2002) that $E\|H_n - U_n\|^2 = O(n^{-2})$, which implies that $n^{1/2}(H_n - U_n) = o_p(1)$. Hence by Theorem 12.3 of Van der vaart (1998), we get

$$n^{1/2} H_n = n^{1/2} U_n = 2n^{-1/2} \sum_{i=1}^{n} \tilde{\varphi}_1(Y_i) + o_p(1),$$

where $\tilde{\varphi}_1(y) = E\{\tilde{\varphi}(Y_1, Y_2) | Y_1 = y\}$. This combining with (A.22) gives

$$n^{1/2}(\hat{\gamma} - \gamma_0) = -2n^{-1/2} \Delta^{-1} \sum_{i=1}^{n} \tilde{\varphi}_1(Y_i) + o_p(1), \tag{A.23}$$

which means that $n^{1/2}(\hat{\gamma} - \gamma_0)$ is asymptotically normal with mean zero and variance $\sigma^2 = 4\Delta^{-2} E\{\tilde{\varphi}_1(Y_1)^2\}$.

Based on (A.16), (A.18) and (A.23), we obtain

$$\begin{aligned} n^{1/2}\{\hat{\boldsymbol{\beta}}(\tau)-\boldsymbol{\beta}_0(\tau)\} &= n^{1/2}\{\hat{\boldsymbol{\beta}}(\tau;\hat{\gamma})-\boldsymbol{\beta}(\tau;\gamma_0)\} \\ &= n^{1/2}\{\hat{\boldsymbol{\beta}}(\tau;\gamma_0)-\boldsymbol{\beta}(\tau;\gamma_0)\}+\dot{\boldsymbol{\beta}}(\tau;\gamma_0)n^{1/2}(\hat{\gamma}-\gamma_0)+o_{[\nu,\tau_U]}(1) \\ &= n^{-1/2}\sum_{i=1}^{n}\Big\{\mathbf{B}^{-1}\{\boldsymbol{\beta}_0(\tau),\gamma_0\}\boldsymbol{\phi}(h_i(\tau)) \\ &\quad -2\dot{\boldsymbol{\beta}}(\tau;\gamma_0)\Delta^{-1}\tilde{\varphi}_1(Y_i)\Big\}+\ o_{[\nu,\tau_U]}(1), \end{aligned} \tag{A.24}$$

which implies that $n^{1/2}\{\hat{\boldsymbol{\beta}}(\tau)-\boldsymbol{\beta}_0(\tau)\}$ ($\tau\in[\nu,\tau_U]$) converges in finite-dimensional distribution to a zero-mean Gaussian process. Following the arguments in Appendix B in Peng and Huang (2008), it can been shown that $\{\mathbf{B}^{-1}\{\boldsymbol{\beta}_0(\tau),\gamma_0\}\boldsymbol{\phi}(h_i(\tau)),\tau\in[\nu,\tau_U]\}$ is a Donsker class. In addition, $\dot{\boldsymbol{\beta}}(\tau;\gamma_0)$ is a deterministic function and $\tilde{\varphi}_1(Y_i)$ does not involve τ. Thus, by the Donsker theorem, $n^{1/2}\{\hat{\boldsymbol{\beta}}(\tau)-\boldsymbol{\beta}_0(\tau)\}$ converges weakly to a zero-mean Gaussian process for $\tau\in[\nu,\tau_U]$. This completes the proof of Theorem 2.

Justification for the Resampling Method. Since $E(\zeta)=1$ and $\boldsymbol{\phi}(.)$ is a linear operator, using arguments analogous to those in the proof of Theorem 2, we have

$$n^{1/2}(\hat{\gamma}^*-\gamma_0)=-2n^{-1/2}\Delta^{-1}\sum_{i=1}^{n}\zeta_i\tilde{\varphi}_1(Y_i)+o_p(1), \tag{A.25}$$

and uniformly in $\tau\in[\nu,\tau_U]$,

$$\begin{aligned} n^{1/2}\{\hat{\boldsymbol{\beta}}^*(\tau)-\boldsymbol{\beta}_0(\tau)\} &= n^{-1/2}\sum_{i=1}^{n}\zeta_i\Big\{\mathbf{B}^{-1}\{\boldsymbol{\beta}_0(\tau),\gamma_0\}\boldsymbol{\phi}(h_i(\tau)) \\ &\quad -2\dot{\boldsymbol{\beta}}(\tau;\gamma_0)\Delta^{-1}\tilde{\varphi}_1(Y_i)\Big\}+o_p(1). \end{aligned} \tag{A.26}$$

Thus, it follows from (A.23)-(A.26) that

$$n^{1/2}(\hat{\gamma}^*-\hat{\gamma})=-2n^{-1/2}\Delta^{-1}\sum_{i=1}^{n}(\zeta_i-1)\tilde{\varphi}_1(Y_i)+o_p(1), \tag{A.27}$$

and uniformly in $\tau\in[\nu,\tau_U]$,

$$n^{1/2}\{\hat{\boldsymbol{\beta}}^*(\tau)-\hat{\boldsymbol{\beta}}(\tau)\}=n^{-1/2}\sum_{i=1}^{n}(\zeta_i-1)\Big\{\mathbf{B}^{-1}\{\boldsymbol{\beta}_0(\tau),\gamma_0\}\boldsymbol{\phi}(h_i(\tau))$$

$$-2\dot{\boldsymbol{\beta}}(\tau;\gamma_0)\Delta^{-1}\tilde{\varphi}_1(Y_i)\Big\} + o_p(1). \tag{A.28}$$

Note that $\mathrm{var}(\zeta) = 1$. In view of (A.25)-(A.28), by the arguments of Lin, Wei and Ying (1993), we obtain that the conditional distributions of $n^{1/2}(\gamma^* - \hat{\gamma})$ and $n^{1/2}\{\hat{\boldsymbol{\beta}}^*(\tau) - \hat{\boldsymbol{\beta}}(\tau)\}$ given the observed data are asymptotically equivalent to the unconditional distributions of $n^{1/2}(\hat{\gamma} - \gamma_0)$ and $n^{1/2}\{\hat{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}_0(\tau)\}$ for $\tau \in [\nu, \tau_U]$, respectively.