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A comparison of methods for estimating parameters of the type I discrete Weibull distribution

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The type I discrete Weibull distribution can be used in reliability problems for modeling discrete failure data, such as the number of shocks, cycles, or runs a component or structure can overcome before failing. This paper refines and compares some existing methods for estimating its parameters and proposes and evaluates approximate confidence intervals for large samples. A Monte Carlo simulation study was performed in order to assess the statistical performance of the methods for different parameter combinations and sample sizes and then give some indication for their mindful use. Examples are considered as a practical application of the proposed procedures. A software implementation of the model is provided as a contributed package in the R programming environment, which reveals a useful and friendly tool for the researcher who has to handle discrete Weibull data.

Keywords and phrases: Failure data, Maximum likelihood, Method of moments, Method of proportion, Reliability.

1. INTRODUCTION

Almost all reliability studies assume that time is continuous and continuous probability distributions, such as exponential, gamma, Weibull, normal, and lognormal, are commonly used to model the lifetime of a component or a structure. These distributions and the methods for estimating their parameters are well-known. In many practical situations, however, lifetime is not measured with the calendar time, for example, when the equipment works in cycles or on demands and the number of cycles or demands prior to failure is observed, or when the regular operation of the equipment is monitored once per period, and the number of time periods successfully completed is observed. Moreover, reliability data are often grouped into classes or truncated according to some censoring criterion. In all these situations, lifetime is modeled as a discrete random variable (r.v.). Indeed, not so much work has been done in reliability for discrete data. Reference [1] provided an exhaustive survey on discrete lifetime concepts and distributions. Generally, most reliability concepts for continuous lifetimes have been adapted to the discrete context; in particular, discrete counterparts of continuous distributions have been introduced; see for example [2]. In this context, geometric and negative binomial distributions are known to be the corresponding discrete alternatives for the exponential and gamma distributions, respectively. Yet, discrete lifetime distributions can also be defined without any reference to a continuous counterpart.

As a discrete alternative to the Weibull distribution, three main forms have been introduced. The first one, which this article takes into consideration, was introduced in [3] and is referred to as type I discrete Weibull; its use has been recently extended to further fields of application, that is, for modeling the distribution of pathogen counts in treated water over time [4]. The second one was proposed and studied in [5], the third in [6]. From a different perspective and with a different objective, Roy and Dasgupta [7] proposed a discretization method for continuous r.v. for the computation of reliability in complex stress-strength models, with a specific application to the Weibull r.v., thus deriving another alternative discrete Weibull distribution.

Type I discrete Weibull (henceforth simply discrete Weibull) r.v. was originally defined in [3] by the following probability mass function:

$$P(X=x;q,\beta)=q^{x^\beta}-q^{(x+1)^\beta}\quad x=0,1,\dots$$

with $0<q<1$ and $\beta>0$. If one would confine the support of this discrete r.v. to the positive integers only (see [8]), in order to model discrete lifetimes that cannot assume zero value, then the probability mass function becomes

(1)
$$P(X=x;q,\beta)=\phi(x;q,\beta)=q^{(x-1)^\beta}-q^{x^\beta}\quad x=1,2,\dots$$

and the corresponding cumulative distribution function is

$$F(x;q,\beta)=1-q^{x^\beta}\quad x=1,2,\dots$$

again with $0<q<1$ and $\beta>0$. Note that for $\beta=1$, the discrete Weibull r.v. reduces to the geometric r.v. with parameter $p=1-q$.

Its failure rate is

$$r(x;q,\beta)=P(X=x)/P(X\geq x)=1-q^{x^\beta-(x-1)^\beta}$$

for $x = 1, 2, \ldots$, which is a monotone increasing function of x for $\beta > 1$, a monotone decreasing function for $\beta < 1$, and constant (equal to $1 - q$) for $\beta = 1$.

The expected value is given by the following infinite convergent series:

$$E(X) = \sum_{x=1}^{+\infty} x(q^{(x-1)^\beta} - q^{x^\beta}) = \sum_{x=0}^{+\infty} q^{x^\beta}; \tag{2}$$

its second-order moment is given by this other infinite convergent series:

$$\begin{aligned} E(X^2) &= \sum_{x=1}^{+\infty} x^2(q^{(x-1)^\beta} - q^{x^\beta}) = 2\sum_{x=1}^{+\infty} xq^{x^\beta} + \sum_{x=0}^{+\infty} q^{x^\beta} \\ &= 2\sum_{x=1}^{+\infty} xq^{x^\beta} + E(X). \end{aligned} \tag{3}$$

The expected value of the reciprocal of the discrete Weibull r.v. is computed as

$$E(1/X) = \sum_{x=1}^{+\infty} \frac{1}{x}(q^{(x-1)^\beta} - q^{x^\beta}) = 1 - \sum_{x=1}^{+\infty} \frac{q^{x^\beta}}{x(x+1)}$$

Note that for $\beta = 1$ (geometric distribution) $E(X) = (1-q)/q$ and, being $\sum_{x=1}^{+\infty} q^x/x = \log[1/(1-q)]$ and $\sum_{x=1}^{+\infty} q^{x-1}/x = q^{-1}\log[1/(1-q)]$, $E(1/X) = (1-q)/q \log[1/(1-q)]$.

This study first describes, refines, and discusses existing procedures for the point estimation of parameters q and β of the discrete Weibull r.v. and then suggests and examines large-sample interval estimators for the same parameters (Section 2). An extensive Monte Carlo study assesses and compares the performance of these estimators, for different combinations of the parameters and sample sizes; a software implementation of the model in the `R` programming environment is also briefly presented (Section 3). The estimation procedures are applied to two datasets taken from the literature (Section 4). Finally, some summarizing indications and remarks conclude the article (Section 5).

2. POINT AND INTERVAL ESTIMATION OF THE PARAMETERS

Focusing on the point estimation of the parameters of the discrete Weibull r.v., three techniques discussed in the literature are now described: the method of proportion, which is strictly related to the specific features of the distribution function of the discrete Weibull r.v.; the classical method of moments; and the maximum likelihood method. Moreover, an alternative method of moments will be introduced. All the methods assume that both parameters of the distribution are unknown.

Method of proportion The method, introduced in [8] and extended, with the appropriate modifications, to the estimation of the parameters of a discrete inverse Weibull distribution, see [9], relies on the following equality, valid for the discrete Weibull r.v.: $P(X = 1) = 1 - q$, by which an estimate of q is $\hat{q}_P = 1 - \sum_{i=1}^n I_{x_i=1}/n = 1 - y/n$, where I_A is the indicator function, which equals 1 if A is true, and 0 if A is false, so that $y = \sum_{i=1}^n I_{x_i=1}$ denotes the number of 1s in the sample. Following similar arguments, an estimate of β is provided:

$$\hat{\beta}_P = \log\{\log(1 - y/n - z/n)/\log(1 - y/n)\}/\log 2 \tag{4}$$

where z denotes the number of 2s in the sample: $z = \sum_{i=1}^n I_{x_i=2}$. The method fails if there are no 1s in the sample: in this case, $\hat{q}_P = 1$, which is a boundary value for q, and, more importantly, $\hat{\beta}_P$ cannot be computed. This is particularly frequent for small samples and for high values of q. The method fails in computing an estimate for β also if the sample contains only 1s and 2s ($y + z = n$). Furthermore, if some 1s are present in the sample, but there are no 2s (that means $y \neq 0$ and $z = 0$), then by formula (4), the estimate $\hat{\beta}_P$ reduces to 0, which is a boundary value for β.

Maximum likelihood method Having defined the log-likelihood function as $l(q, \beta; x_1, \ldots, x_n) = \log \prod_{i=1}^n \phi(x_i; q, \beta) = \sum_{i=1}^n \log(q^{(x_i-1)^\beta} - q^{x_i^\beta})$, the maximum likelihood estimates of q and β are obtained by maximizing $l(q, \beta; x_1, \ldots, x_n)$ with respect to q and β. The solution to the maximization of $l(q, \beta; x_1, \ldots, x_n)$ (with the constraints that q and β belong to their natural parameter spaces) can be obtained only numerically, for example, by using the packages `nlm` or `Rsolnp` in `R` [10], which allow the user to solve nonlinearly constrained minimization/maximization problems. It is worth noting that even this method cannot be applied to every possible sample; in particular, the method fails to provide a solution if $y + z = n$. In this case, in fact, it can be easily proved that the first-order partial β derivative of the log-likelihood function is never null; the log-likelihood function does not have an absolute maximum in the parameter space. Let $(\hat{q}_{ML}, \hat{\beta}_{ML})$ denote the maximum likelihood estimates of (q, β), when they exist.

Method of moments The parameter estimates are obtained solving the equations $E(X) = \mu_1 = m_1$ and $E(X^2) = \mu_2 = m_2$ in terms of β and q, where m_1 and m_2 are the first- and second-order sample moments: $m_1 = \frac{1}{n}\sum_{i=1}^n x_i$, $m_2 = \frac{1}{n}\sum_{i=1}^n x_i^2$. Since they cannot be solved analytically, as suggested in [8], one can minimize with respect to q and β the quadratic loss function $\mathcal{L}(q, \beta; x_1, \ldots, x_n) = (m_1 - \mu_1)^2 + (m_2 - \mu_2)^2$. The task can be carried out, for example, using again the packages `nlm` or `Rsolnp` in the `R` environment. The solution is the couple $(\hat{q}_M, \hat{\beta}_M)$.

A modified version of the method of moments can be provided, following somehow a proposal for the Birnbaum-Saunders distribution [11], by considering the expected value of the reciprocal of X, $\mu_{-1} = E(1/X)$ (instead of the second moment of X), and equating it to the sample analogous, $m_{-1} = 1/n \sum_{i=1}^n 1/x_i$. The parameter estimates can be obtained by minimizing $\mathcal{L}^*(q, \beta; x_1, \ldots, x_n) = (m_1 - \mu_1)^2 +$

$(m_{-1}-\mu_{-1})^2$. The solution of this modified method of moments is denoted as $(\hat{\beta}_{M*}, \hat{q}_{M*})$.

As starting values of β and q for the iterative algorithm of the minimization procedure, one can set $\beta^{(0)}=1$, then, by substituting this value to β in the expression of $E(X)$, one obtains $E(X)=\sum_0^{+\infty} q^x = 1/(1-q)$ and equating $E(X)$ to the first sample moment m_1, a launch value for q is derived: $q^{(0)}=(m_1-1)/m_1$.

When the sample contains only 1s and 2s, the method of moments, as well as its modified version, is not applicable. To see this, first, consider the probability mass function of the discrete Weibull r.v. (1), and note that letting β tend to $+\infty$, it degenerates into a r.v. that takes only two values: 1 with probability $1-q$ and 2 with probability q. It is then clear that if the sample is made up of a fraction r of 1s and a fraction $(1-r)$ of 2s, the equality of both first and second moments computed on the sample and on the original r.v. holds for $q=1-r$ and $\beta\to+\infty$. In this case, the loss function $\mathcal{L}(q,\beta;x_1,\ldots,x_n)$ does not admit an absolute minimum but only an inferior limit.

Given the complexity of the estimators listed in this section (only the method of proportion provides an analytical expression for them), not so much can be analytically derived about their statistical properties for finite sample size. When the method of proportion can be applied, it provides an unbiased and consistent estimator for q; $\hat{\beta}_P$ has been shown to be consistent as well, but nothing can be said about its unbiasedness [8]. For large samples, the general properties of the estimators derived from the maximum likelihood and moments methods can be recalled. The method of proportion and the method of moments have been empirically explored and compared for several parameter configurations by [8], where their MC means and variances were computed. The maximum likelihood method was not taken into consideration there; it was later investigated through simulation by [12], where it was applied to discrete Weibull censored data, and concluded that the method performed well for large values of q and $\beta>1$. The simulation results in [8], even if not completely reliable since they are based on 100 MC replications only and on few parameter combinations, at least highlight that the variance of $\hat{\beta}_P$ is often far greater than that derived by $\hat{\beta}_M$. The bias (in absolute value) of the estimators of q and β derived by both methods is rather high for $q=0.5, \beta=0.5$, even for large sample size ($n=100$). A more detailed study on maximum likelihood estimators was performed in [1]. Summarizing the results, $\hat{q}_{ML}$ overestimates q when $q<0.9$ and underestimates it when $q\geq 0.9$; its bias is independent of β; $\hat{\beta}_{ML}$, slightly biased, shows more variability than $\hat{q}_{ML}$. The maximum likelihood method gives good results for large values of q and less reliable results for small values of q. These findings confirm those of [12] for censored data.

With regard to the interval estimation of the parameters β and q, little work has been done so far. In [12], the (approximate) maximum likelihood estimators of β and q for type I censored data were considered, and their joint distribution was proved to be asymptotically bivariate normal so that customary confidence intervals were easily derived. In fact, even if the maximum likelihood estimators for β and q do not have a close expression, the Fisher information matrix can be at least numerically derived, and the following well-known asymptotic result holds for complete samples:

$$(5)\quad \sqrt{n}\left(\begin{pmatrix}\hat{q}_{ML}\\ \hat{\beta}_{ML}\end{pmatrix}-\begin{pmatrix}q\\ \beta\end{pmatrix}\right)\to N\left(\begin{pmatrix}0\\ 0\end{pmatrix},\ I^{-1}(q,\beta)\right)$$

where $I^{-1}(q,\beta)$ is the inverse of the Fisher information matrix

$$(6)\qquad I(q,\beta)=-E\begin{pmatrix}\frac{\partial^2\log\phi(x_i;q,\beta)}{\partial q^2} & \frac{\partial^2\log\phi(x_i;q,\beta)}{\partial q\partial\beta}\\ \frac{\partial^2\log\phi(x_i;q,\beta)}{\partial\beta\partial q} & \frac{\partial^2\log\phi(x_i;q,\beta)}{\partial\beta^2}\end{pmatrix}$$

Alternatively, the result in (5) can be expressed as

$$\begin{pmatrix}\hat{q}_{ML}\\ \hat{\beta}_{ML}\end{pmatrix}\approx N\left(\begin{pmatrix}q\\ \beta\end{pmatrix},\ I^{-1}(q,\beta)/n\right)$$

Letting $\hat{I}(\hat{q}_{ML},\hat{\beta}_{ML})$ denote the matrix obtained by substituting the sample means to the expected values and the maximum likelihood estimates to the unknown parameters into (6), $\hat{I}^{-1}(\hat{q}_{ML},\hat{\beta}_{ML})$ its inverse, and $\hat{I}^{-1}_{11}(\hat{q}_{ML},\hat{\beta}_{ML})$, $\hat{I}^{-1}_{22}(\hat{q}_{ML},\hat{\beta}_{ML})$ its diagonal elements, then approximate confidence intervals for q and β at $1-\alpha$ level can be provided:

$$(q_L,q_U)=(\hat{q}_{ML}\mp z_{1-\alpha/2}\sqrt{\hat{I}^{-1}_{11}(\hat{q}_{ML},\hat{\beta}_{ML})/n})$$
$$(\beta_L,\beta_U)=(\hat{\beta}_{ML}\mp z_{1-\alpha/2}\sqrt{\hat{I}^{-1}_{22}(\hat{q}_{ML},\hat{\beta}_{ML})/n})$$

Note that the CI are built independently for q and β, even if independence between $\hat{q}_{ML}$ and $\hat{\beta}_{ML}$ does not hold.

Here, the exact expression of the Fisher information matrix $I(q,\beta)$ is not presented since its derivation could be cumbersome and its expression quite complex; nevertheless, its sample analogous $\hat{I}(\hat{q}_{ML},\hat{\beta}_{ML})$ can be computed by well-known statistical software programs, such as R and Mathematica.

In the next section, a more complete and extensive simulation study is presented, which has been performed in order to investigate the performance of the estimation methods discussed so far and outline some practical advice for their employment.

3. SIMULATION STUDY

In this section, the design of a Monte Carlo simulation study is first outlined, and then the results are presented and discussed.

3.1 Simulation design

The Monte Carlo simulation study investigated the estimators presented in the previous section; the point estimators were compared in terms of percentage Monte Carlo relative bias (RB), defined as $RB(\hat{\theta})=(E_{MC}(\hat{\theta})-\theta)/\theta\cdot 100\%$,

Table 1. Parameter combinations and corresponding expected value, standard deviation, and 0.99 quantile for the simulation study

q	β	$E(X)$	$SD(X)$	$F^{-1}(0.99)$	q	β	$E(X)$	$SD(X)$	$F^{-1}(0.99)$
0.3	0.5	2.08	3.01	15	0.7	0.8	4.65	5.16	25
0.3	0.8	1.53	1.07	6	0.7	1	3.33	2.79	13
0.3	1	1.43	0.78	4	0.7	1.2	2.74	1.86	9
0.5	0.5	4.79	9.24	45	0.7	1.5	2.30	1.24	6
0.5	0.8	2.37	2.21	11	0.9	1	10	9.49	44
0.5	1	2	1.41	7	0.9	1.2	6.64	5.14	24
0.5	1.2	1.82	1.06	5	0.9	1.5	4.55	2.76	13
					0.9	2	3.23	1.46	7

where $\hat{\theta}$ is an estimator of the parameter θ (here, q or β), and standard deviation (SD); the performance of the 95% confidence intervals was stated in terms of coverage (C) and average length (AL).

Several parameter combinations and sample sizes ($n = 10, 20, 50, 100$) were considered. The values of the pair (q, β) were chosen in order to explore a large spectrum of the discrete Weibull distribution, in particular, to comprise increasing, constant, and decreasing failure rates. At the same time, the parameters were set in order to keep the discrete nature of the distribution reasonable: values entailing a nonnegligible probability for a large number of integers were deliberately excluded (in this case, a continuous r.v. should be preferred to model failure data). Analogously, parameter values ensuring a nonnegligible probability for just the first integers were avoided (one assumes the component that is monitored is likely to last for more than one or two cycles). Table 1 shows the combinations of q and β explored in the simulation study, along with the corresponding expected value, standard deviation, and 99% quantile of the distribution. A note about the computation of the expected value and standard deviation is due: they are calculated numerically (see formulas 2 and 3), considering the first $n_{\max}$ integers, with $n_{\max}$ as large as possible ($F^{-1}(1-\epsilon)$, with ϵ sufficiently small).

The Monte Carlo simulation study was based on 5,000 replications for each scenario and was carried out under the `R` programming environment. The type I discrete Weibull model has been implemented in the `R` environment through a contributed package, `DiscreteWeibull` [15], freely available on the CRAN website, which comprises several functions, implementing the probability mass function, the cumulative distribution function, the quantile function, the random generation, the computation of the first- and second-order moments, and the point and interval estimation. The package also provides analogous routines for the competitor type III discrete Weibull model.

The package is an easy-to-use tool for any researcher or user who has to handle this distribution, either for analyzing discrete data that can be fitted by it or for assessing methods or techniques of statistical analysis that require a massive simulation of artificial data. Further details can be found in the accompanying package manual.

3.2 Simulation results

Tables 2 and 3 show the relative bias and standard deviation for each estimator derived from the four methods, under each combination of the two parameters and with $n = 10, 20$, and $n = 50, 100$, respectively. The study took into account the drawback of the methods discussed above, computing the MC quantities over the feasible samples only. With this respect, the worst scenarios were those characterized by $q = 0.3, \beta = 1$, and $q = 0.9, \beta = 2$, both for $n = 10$. In the first case, 1,843 samples out of 5,000 (36.9%) were nonfeasible in all the methods (they contained only 1s and 2s); in the second case, 1,721 samples were not feasible for the method of proportion (34.4%; they did not contain any 1). Increasing the sample size strongly reduces the presence of nonfeasible samples: for $n = 50$, just a few parameter configurations are affected by this drawback, with a percentage of nonfeasible samples in any case smaller than 1%; for $n = 100$ and for each parameter configuration, all the samples are feasible.

As to the estimators of q, as said in Section 2, $\hat{q}_P$ is an unbiased estimator for q. This is empirically confirmed by the simulation study results; in fact, the Monte Carlo percentage relative bias is almost negligible under each combination of parameters, especially for larger samples (in absolute value, always smaller than 2.65%). The other three estimators of q show a larger relative bias; in particular, $\hat{q}_M$ looks the most biased: for $n = 10$, its relative bias reaches 36%, while $\hat{q}_{ML}$ never exceeds 18% and $\hat{q}_{M*}$ 17%. As a general trend, the relative bias of $\hat{q}_M$, $\hat{q}_{ML}$, and $\hat{q}_{M*}$ is larger for smaller values of q. Surprisingly, $\hat{q}_{M*}$ shows a relative bias always smaller than $\hat{q}_M$ for small samples ($n = 10$). For $n = 100$, $\hat{q}_M$, $\hat{q}_{M*}$, and $\hat{q}_{ML}$ are nearly unbiased. The standard deviation of the estimator derived from the method of proportion is overall greater than or close to those of its competitors. For large samples and for small values of both q and β (here, $q = 0.3$ and $\beta = 0.5$), $\hat{q}_P$ performance improves, and $\hat{q}_P$ shows to be even better than $\hat{q}_M$ and comparable with the other two estimators. The degradation of $\hat{q}_P$ as q increases is easily justified since it is based on the number of 1s in the sample, whose expected value is proportional to $1 - q$, and neglects all the other information contained in the sample. The greater the q, the smaller the number of 1s, and the

Table 2. Simulation results for point estimators (1). Legend: P = method of proportion, M = method of moments, M = modified method of moments, ML = maximum likelihood method, RB = percentage relative bias, SD = standard deviation*

q		P	M	M*	ML	β		P	M	M*	ML
						n = 10					
0.9	RB	-0.04	0.31	-0.18	0.29	2	RB	-21.42	16.57	19.36	18.50
	SD	0.095	0.071	0.080	0.074		SD	0.798	0.701	0.883	0.825
0.9	RB	-0.04	0.65	-0.24	0.25	1.5	RB	-26.09	16.75	16.78	16.92
	SD	0.095	0.067	0.078	0.073		SD	0.770	0.484	0.594	0.549
0.9	RB	-0.04	1.08	-0.28	0.24	1.2	RB	-29.68	18.01	15.53	16.67
	SD	0.095	0.064	0.077	0.073		SD	0.723	0.371	0.446	0.434
0.9	RB	-0.04	1.51	-0.37	0.23	1	RB	-31.59	18.83	14.06	16.50
	SD	0.095	0.060	0.076	0.072		SD	0.682	0.293	0.348	0.358
0.7	RB	0.00	2.64	0.77	2.01	1.5	RB	8.03	19.92	16.51	19.59
	SD	0.144	0.128	0.136	0.134		SD	0.788	0.594	0.671	0.686
0.7	RB	0.00	3.69	0.73	1.88	1.2	RB	8.06	22.30	16.31	19.03
	SD	0.144	0.125	0.134	0.134		SD	0.740	0.456	0.502	0.502
0.7	RB	0.00	4.92	0.69	1.75	1	RB	8.43	24.94	16.10	18.53
	SD	0.144	0.122	0.133	0.133		SD	0.692	0.374	0.401	0.404
0.7	RB	0.00	7.35	0.75	1.73	0.8	RB	7.39	30.15	16.18	18.23
	SD	0.144	0.118	0.132	0.133		SD	0.651	0.295	0.307	0.315
0.5	RB	0.14	7.91	4.24	5.71	1.2	RB	5.17	21.48	14.57	17.86
	SD	0.158	0.144	0.149	0.151		SD	0.649	0.464	0.493	0.508
0.5	RB	0.14	8.82	3.57	4.89	1	RB	7.25	27.27	17.73	20.72
	SD	0.158	0.143	0.149	0.152		SD	0.620	0.409	0.432	0.454
0.5	RB	0.14	11.22	3.00	4.09	0.8	RB	8.60	34.13	19.97	22.17
	SD	0.158	0.143	0.149	0.153		SD	0.579	0.337	0.353	0.376
0.5	RB	0.14	22.91	2.74	3.36	0.5	RB	10.75	55.45	24.07	23.57
	SD	0.158	0.144	0.150	0.155		SD	0.479	0.223	0.226	0.245
0.3	RB	2.65	24.10	16.91	17.88	1	RB	-13.26	17.42	6.87	8.47
	SD	0.138	0.128	0.132	0.137		SD	0.546	0.369	0.399	0.430
0.3	RB	2.65	25.01	14.75	15.30	0.8	RB	-8.00	30.36	15.69	16.43
	SD	0.138	0.128	0.132	0.137		SD	0.508	0.326	0.353	0.386
0.3	RB	2.65	35.85	12.55	11.93	0.5	RB	-1.33	61.55	31.93	29.02
	SD	0.138	0.135	0.131	0.141		SD	0.430	0.250	0.271	0.300
						n = 20					
0.9	RB	-0.17	0.06	-0.22	0.08	2	RB	-1.06	7.23	7.97	7.63
	SD	0.068	0.051	0.057	0.053		SD	0.768	0.430	0.529	0.447
0.9	RB	-0.17	0.24	-0.26	0.04	1.5	RB	-2.01	7.63	7.00	7.14
	SD	0.068	0.050	0.056	0.052		SD	0.738	0.309	0.365	0.323
0.9	RB	-0.17	0.48	-0.28	0.04	1.2	RB	-3.28	8.57	6.60	7.05
	SD	0.068	0.049	0.055	0.052		SD	0.702	0.245	0.279	0.255
0.9	RB	-0.17	0.72	-0.33	0.03	1	RB	-4.30	9.29	6.10	6.98
	SD	0.068	0.047	0.054	0.052		SD	0.671	0.199	0.222	0.212
0.7	RB	-0.21	1.21	0.11	0.81	1.5	RB	5.26	9.04	6.83	8.35
	SD	0.102	0.093	0.097	0.096		SD	0.581	0.353	0.378	0.369
0.7	RB	-0.21	1.96	0.16	0.80	1.2	RB	5.31	10.40	6.69	8.03
	SD	0.102	0.091	0.096	0.095		SD	0.541	0.277	0.291	0.287
0.7	RB	-0.21	2.84	0.16	0.75	1	RB	5.72	12.26	6.72	7.91
	SD	0.102	0.091	0.095	0.095		SD	0.503	0.234	0.237	0.238
0.7	RB	-0.21	4.60	0.20	0.75	0.8	RB	5.85	15.77	6.86	7.81
	SD	0.102	0.090	0.094	0.094		SD	0.472	0.193	0.185	0.188
0.5	RB	-0.02	2.98	0.84	1.66	1.2	RB	4.63	12.68	8.64	10.37
	SD	0.111	0.107	0.109	0.110		SD	0.455	0.322	0.335	0.341
0.5	RB	-0.02	4.12	0.84	1.56	1	RB	4.50	14.71	8.95	10.36
	SD	0.111	0.106	0.108	0.109		SD	0.426	0.271	0.279	0.288
0.5	RB	-0.02	6.49	0.96	1.51	0.8	RB	4.43	18.52	9.58	10.47
	SD	0.111	0.106	0.107	0.109		SD	0.395	0.221	0.222	0.232
0.5	RB	-0.02	16.91	1.21	1.51	0.5	RB	4.64	32.61	11.05	10.27
	SD	0.111	0.106	0.107	0.109		SD	0.321	0.147	0.136	0.144
0.3	RB	0.30	8.98	5.22	5.84	1	RB	-1.35	14.03	7.62	8.89
	SD	0.102	0.097	0.098	0.099		SD	0.403	0.296	0.314	0.327
0.3	RB	0.30	10.04	3.99	4.33	0.8	RB	2.11	21.89	12.55	13.10
	SD	0.102	0.098	0.098	0.100		SD	0.383	0.262	0.276	0.291
0.3	RB	0.30	20.46	3.52	3.11	0.5	RB	5.37	39.68	19.13	17.03
	SD	0.102	0.105	0.099	0.101		SD	0.320	0.187	0.195	0.208

Table 3. Simulation results for point estimators (2). Legend: P = method of proportion, M = method of moments, M = modified method of moments, ML = maximum likelihood method, RB = percentage relative bias, SD = standard deviation*

q		P	M	M*	ML	β		P	M	M*	ML
					n = 50						
0.9	RB	-0.01	0.08	-0.03	0.11	2	RB	5.76	2.90	3.02	3.10
	SD	0.042	0.032	0.036	0.033		SD	0.670	0.251	0.293	0.256
0.9	RB	-0.01	0.13	-0.05	0.09	1.5	RB	6.95	3.11	2.71	2.92
	SD	0.042	0.032	0.034	0.032		SD	0.626	0.186	0.206	0.186
0.9	RB	-0.01	0.19	-0.07	0.09	1.2	RB	7.56	3.50	2.57	2.90
	SD	0.042	0.033	0.034	0.032		SD	0.587	0.154	0.159	0.147
0.9	RB	-0.01	0.25	-0.10	0.09	1	RB	8.11	3.82	2.38	2.86
	SD	0.042	0.033	0.033	0.032		SD	0.555	0.129	0.129	0.122
0.7	RB	-0.02	0.58	0.11	0.42	1.5	RB	2.15	3.71	2.72	3.37
	SD	0.063	0.059	0.061	0.060		SD	0.341	0.210	0.218	0.212
0.7	RB	-0.02	0.95	0.13	0.42	1.2	RB	2.14	4.42	2.68	3.26
	SD	0.063	0.059	0.060	0.059		SD	0.320	0.170	0.170	0.167
0.7	RB	-0.02	1.41	0.13	0.41	1	RB	2.17	5.39	2.71	3.22
	SD	0.063	0.061	0.059	0.059		SD	0.300	0.147	0.139	0.138
0.7	RB	-0.02	2.42	0.17	0.42	0.8	RB	1.81	7.40	2.82	3.22
	SD	0.063	0.064	0.059	0.059		SD	0.277	0.127	0.109	0.110
0.5	RB	0.08	1.38	0.37	0.75	1.2	RB	2.07	5.58	3.68	4.43
	SD	0.070	0.068	0.069	0.069		SD	0.281	0.199	0.201	0.202
0.5	RB	0.08	2.01	0.40	0.73	1	RB	2.09	6.48	3.67	4.25
	SD	0.070	0.069	0.068	0.068		SD	0.259	0.168	0.164	0.166
0.5	RB	0.08	3.40	0.46	0.74	0.8	RB	2.11	8.53	3.88	4.24
	SD	0.070	0.070	0.068	0.068		SD	0.236	0.140	0.130	0.133
0.5	RB	0.08	10.69	0.61	0.73	0.5	RB	2.10	17.48	4.61	4.16
	SD	0.070	0.077	0.068	0.068		SD	0.192	0.100	0.081	0.082
0.3	RB	-0.06	2.43	0.69	0.99	1	RB	3.17	9.56	6.41	7.03
	SD	0.064	0.063	0.063	0.064		SD	0.288	0.225	0.231	0.235
0.3	RB	-0.06	3.91	0.68	0.85	0.8	RB	2.89	11.89	6.88	7.10
	SD	0.064	0.064	0.064	0.064		SD	0.254	0.182	0.183	0.189
0.3	RB	-0.06	12.16	1.09	0.87	0.5	RB	2.48	20.68	7.98	6.81
	SD	0.064	0.069	0.063	0.064		SD	0.201	0.121	0.113	0.116
					n = 100						
0.9	RB	-0.02	0.01	-0.03	0.04	2	RB	2.83	1.37	1.49	1.50
	SD	0.030	0.023	0.025	0.023		SD	0.443	0.172	0.200	0.174
0.9	RB	-0.02	0.02	-0.04	0.03	1.5	RB	3.40	1.42	1.32	1.41
	SD	0.030	0.023	0.025	0.023		SD	0.416	0.128	0.141	0.126
0.9	RB	-0.02	-0.01	-0.06	0.02	1.2	RB	3.75	1.46	1.23	1.39
	SD	0.030	0.024	0.024	0.023		SD	0.388	0.108	0.110	0.101
0.9	RB	-0.02	-0.03	-0.08	0.02	1	RB	3.90	1.46	1.13	1.37
	SD	0.030	0.024	0.024	0.023		SD	0.366	0.090	0.089	0.084
0.7	RB	-0.07	0.22	-0.01	0.14	1.5	RB	1.00	1.79	1.33	1.61
	SD	0.047	0.043	0.044	0.043		SD	0.242	0.146	0.152	0.146
0.7	RB	-0.07	0.39	0.00	0.13	1.2	RB	1.01	2.10	1.29	1.52
	SD	0.047	0.043	0.044	0.043		SD	0.228	0.118	0.118	0.114
0.7	RB	-0.07	0.64	0.02	0.14	1	RB	0.87	2.64	1.33	1.54
	SD	0.047	0.045	0.043	0.043		SD	0.212	0.104	0.096	0.095
0.7	RB	-0.07	1.18	0.02	0.13	0.8	RB	0.89	3.72	1.35	1.51
	SD	0.047	0.049	0.043	0.043		SD	0.194	0.091	0.076	0.076
0.5	RB	0.04	0.65	0.17	0.34	1.2	RB	1.13	2.67	1.78	2.09
	SD	0.050	0.049	0.049	0.049		SD	0.193	0.135	0.135	0.135
0.5	RB	0.04	0.96	0.17	0.32	1	RB	1.22	3.19	1.82	2.06
	SD	0.050	0.050	0.049	0.049		SD	0.180	0.117	0.111	0.112
0.5	RB	0.04	1.71	0.20	0.32	0.8	RB	1.29	4.30	1.91	2.05
	SD	0.050	0.052	0.049	0.049		SD	0.165	0.100	0.088	0.089
0.5	RB	0.04	6.67	0.26	0.32	0.5	RB	1.26	10.16	2.26	2.03
	SD	0.050	0.062	0.048	0.049		SD	0.134	0.076	0.055	0.055
0.3	RB	-0.18	0.96	0.06	0.21	1	RB	1.88	4.74	3.11	3.39
	SD	0.046	0.046	0.046	0.046		SD	0.201	0.155	0.156	0.157
0.3	RB	-0.18	1.92	0.14	0.22	0.8	RB	1.56	5.86	3.12	3.19
	SD	0.046	0.046	0.045	0.045		SD	0.176	0.125	0.121	0.122
0.3	RB	-0.18	7.43	0.34	0.23	0.5	RB	1.46	11.59	3.66	3.05
	SD	0.046	0.052	0.045	0.045		SD	0.139	0.088	0.075	0.076

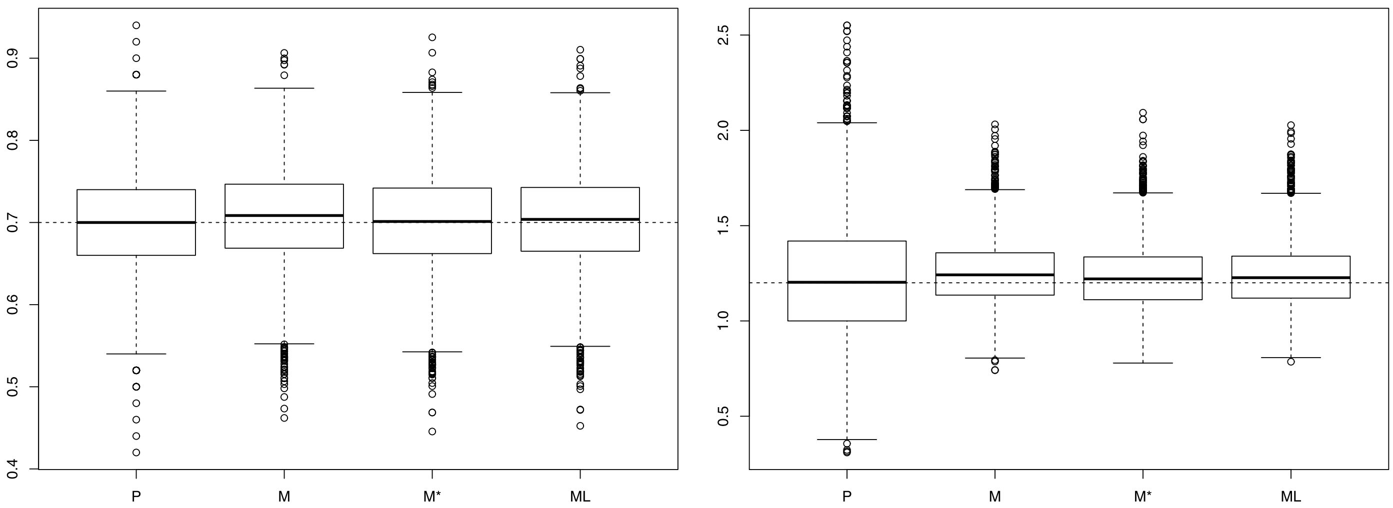

Figure 1. Boxplot of $\hat{q}$ (left) and $\hat{\beta}$ (right) ($q = 0.7$, $\beta = 1.2$, $n = 50$).

less reliable its estimate $\hat{q}_P$. The standard deviation of $\hat{q}_{M*}$ is overall larger than that of $\hat{q}_M$, especially for $n = 10$; the difference diminishes, increasing the sample size.

As to the estimators of β, their relative bias is not so negligible: even for $n = 100$, it is never less than 0.8% under the scenarios considered. All the estimators but $\hat{\beta}_P$ always overestimate β. For $n = 10$ and $q = 0.3, \beta = 0.5$, the relative bias of $\hat{\beta}_M$, $\hat{\beta}_{M*}$, and $\hat{\beta}_{ML}$ gets up to 62%, 32%, and 29%, thus making the estimates very unreliable. $\hat{\beta}_P$ in this sense is preferable since its relative bias in absolute value is kept down, yet for some combinations of the parameters, it underestimates β. Performances get better for larger values of n; $\hat{\beta}_P$ and, as a second choice, $\hat{\beta}_{M*}$ are overall preferable.

The standard deviations of all the four estimators of β decrease as β decreases, for fixed q and n. For fixed β and n, a clear monotonic relationship between the standard deviations of the $\hat{\beta}$s and q is not apparent. Obviously, under the same combination of parameters q and β, the standard deviations decrease as the sample size increases. The standard deviations of $\hat{\beta}_M$ and $\hat{\beta}_{M*}$ look quite similar to each other and overall greater than but comparable to that of $\hat{\beta}_{ML}$. For all sample sizes, and even for the largest one, $\hat{\beta}_M$, $\hat{\beta}_{M*}$, and $\hat{\beta}_{ML}$ present far smaller standard deviations than $\hat{\beta}_P$, except for a very few scenarios.

Note that the general findings about the maximum likelihood method agree with the previous ones in [12], [1].

To better visualize the simulation results, the boxplots of the MC sample distribution of the four estimators are displayed for $q = 0.7$, $\beta = 1.2$, and $n = 50$ (see Figure 1): they show a similar behavior for the estimators of q, and a higher variability, partially compensated by a lower bias, of β_P versus its competitors.

From this discussion, it is clear that there is not a "best" estimator (in terms of bias or variability) for the parameters q and β, a fortiori for the couple (q, β). As a general recommendation, one should use the method of proportion — if applicable — for small sample sizes (say smaller than 50) as long as the sample contains a decent number of 1s and 2s, and one of the methods of moments or the maximum likelihood method for larger sample sizes (say greater than 50) or when the sample does not contain many 1s or 2s.

Table 4 shows the simulation results (coverage and average length) for the 95% confidence intervals based on the maximum likelihood estimates and Fisher approximation. The performance of the confidence intervals for q and β based on maximum likelihood estimates are overall satisfactory in terms of coverage even for moderate samples: the coverage is never smaller than 87% for q with $n \geq 20$ and always close to 95% for β. When $n = 10$, the confidence interval for q shows indeed a poor performance, especially for high values of q; on the contrary, with the same sample size, the confidence interval for β achieves coverage rates sensibly larger than the nominal level (up to 97.7%).

Even if q and β take values on different ranges, and then a direct comparison cannot be carried out, one should observe that the confidence intervals for β are on average much larger than those for q.

4. APPLICATIONS

In this section, we provide two examples of applications of the inferential methods proposed so far. The first example, based on a small-size dataset, allows us to illustrate also the criticality of estimation procedures; the second example, based on a larger sample, allows us a deeper statistical analysis.

4.1 Lifetimes of electronic components

The methods that have been illustrated and empirically investigated in the previous sections are applied to a dataset

Table 4. Simulation results for interval estimators. Legend: C = coverage rate, AL = average length of the confidence interval

q	β		q	β	q	β	q	β	q	β
			n = 10		n = 20		n = 50		n = 100	
0.9	2	C	0.816	0.971	0.873	0.953	0.916	0.954	0.929	0.951
		AL	0.227	2.622	0.187	1.172	0.126	0.964	0.090	0.668
0.9	1.5	C	0.811	0.953	0.873	0.953	0.916	0.951	0.932	0.952
		AL	0.226	1.835	0.187	1.172	0.124	0.706	0.089	0.490
0.9	1.2	C	0.813	0.952	0.872	0.951	0.920	0.952	0.930	0.950
		AL	0.225	1.450	0.187	0.930	0.124	0.561	0.088	0.389
0.9	1	C	0.813	0.951	0.874	0.950	0.922	0.952	0.930	0.950
		AL	0.225	1.202	0.187	0.772	0.123	0.466	0.088	0.323
0.7	1.5	C	0.873	0.975	0.907	0.962	0.934	0.949	0.938	0.945
		AL	0.481	2.211	0.363	1.352	0.234	0.801	0.166	0.554
0.7	1.2	C	0.872	0.966	0.911	0.956	0.937	0.951	0.943	0.947
		AL	0.478	1.693	0.360	1.059	0.232	0.632	0.165	0.437
0.7	1	C	0.875	0.962	0.908	0.954	0.936	0.951	0.943	0.945
		AL	0.477	1.384	0.359	0.876	0.231	0.524	0.164	0.363
0.7	0.8	C	0.872	0.960	0.910	0.953	0.935	0.950	0.939	0.945
		AL	0.476	1.125	0.357	0.697	0.230	0.417	0.163	0.289
0.5	1.2	C	0.890	0.977	0.931	0.966	0.938	0.947	0.944	0.949
		AL	0.576	1.971	0.420	1.302	0.269	0.762	0.190	0.523
0.5	1	C	0.885	0.974	0.925	0.958	0.938	0.945	0.943	0.949
		AL	0.574	1.676	0.418	1.073	0.267	0.628	0.190	0.432
0.5	0.8	C	0.878	0.965	0.927	0.955	0.938	0.942	0.942	0.948
		AL	0.572	1.354	0.417	1.024	0.267	0.500	0.189	0.344
0.5	0.5	C	0.874	0.960	0.927	0.954	0.939	0.944	0.942	0.946
		AL	0.570	0.856	0.415	0.529	0.266	0.311	0.188	0.214
0.3	1	C	0.937	0.969	0.958	0.969	0.940	0.952	0.942	0.947
		AL	0.552	1.834	0.396	1.397	0.251	0.874	0.178	0.587
0.3	0.8	C	0.949	0.972	0.953	0.967	0.939	0.949	0.941	0.946
		AL	0.547	1.589	0.394	1.162	0.250	0.688	0.177	0.461
0.3	0.5	C	0.960	0.970	0.947	0.959	0.939	0.947	0.941	0.947
		AL	0.540	1.120	0.391	0.752	0.250	0.424	0.177	0.285

of 20 lifetimes of electronic components taken from dataset 13.1 [13]:

$$2, 3, 6, 6, 7, 9, 9, 10, 10, 11, 12, 12, 12, 13, 13, 13, 15, 16, 16, 18$$

If the data are assumed to follow a type I discrete Weibull distribution, its parameters can be estimated through one of the methods described in Section 2. Note that since the sample does not contain any 1, the method of proportion for the point estimation of β cannot be applied. The estimates from the methods of moments and of maximum likelihood, and the asymptotic confidence interval relying on the Fisher information matrix are reported in Table 5. The estimates of q are all very close to 1; the estimates of β are less close to each other: the two methods of moments, in particular, provide quite different values from the maximum likelihood method.

Table 5. Estimates of the parameters of the discrete Weibull for the dataset of Section 4.1

point estimators					
$\hat{q}_M$	$\hat{\beta}_M$	$\hat{q}_{M*}$	$\hat{\beta}_{M*}$	$\hat{q}_{ML}$	$\hat{\beta}_{ML}$
0.9919	2.008	0.9903	1.902	0.998	2.636
large sample CI					
q_L	q_U	β_L	β_U		
0.9942	1.0000	1.655	3.618		

4.2 Repair times for an airborne communications receiver

Here, we consider a dataset containing the repair times (in hours) for 46 failures of an airborne communications receiver (example 4.8 in [14]). Such data are indeed reported as numbers with one decimal value; to adapt them to our case, we round them up to the closer integer, in order to get positive integer values. If the data are assumed to follow a type I discrete Weibull distribution, its parameters can be estimated through one of the methods described in Section 2. In this case, differently from the previous example, all the estimation techniques can be applied. The estimates from the methods of proportion, of moments, and of maximum likelihood and the asymptotic confidence interval relying on the (observed) Fisher information matrix are reported in

Table 6. Estimates of the parameters of the discrete Weibull for the dataset of Section 4.2

Point Estimators							
$\hat{q}_P$	$\hat{\beta}_P$	$\hat{q}_M$	$\hat{\beta}_M$	$\hat{q}_{M*}$	$\hat{\beta}_{M*}$	$\hat{q}_{ML}$	$\hat{\beta}_{ML}$
0.6304	0.7652	0.6090	0.7062	0.6251	0.7339	0.6213	0.7289
large-sample CI							
q_L	q_U	β_L	β_U				
0.4908	0.7518	0.5315	0.9263				

Table 6. All the couples of point estimates are (pairwise) very close to each other. This fact may let us expect that the data can be fitted well by the discrete Weibull distribution.

In order to check the null hypothesis H_0, "The repair times come from a dicrete Weibull random variable," versus the alternative hypothesis H_1, "The repair times do not come from a dicrete Weibull random variable," we resort to the classical goodness-of-fit χ^2 test. To this aim, plugging the maximum likelihood estimates of q and β into the probability density function, we compute the probabilities for each integer value $0 < x \leq \max\{x_i, i = 1, \ldots, n\} = 25$. Then we group the ordered observed values in order to build classes whose theoretical absolute frequencies (denoted as np_i, where p_i is the probability of the i-th class) are greater than 5. Such classes and the corresponding observed and expected frequencies are reported in Table 7. Finally, we compute the usual χ^2 statistic

$$\chi^2 = \sum_{i=1}^{k} (n_i - np_i)^2/(np_i) = 0.0766$$

If the discrete Weibull model holds, that is, under H_0, χ^2 is asymptotically distributed as a chi-squared r.v. with $k-1-p$ degrees of freedom, where p is the number of parameters to be estimated. In this case, $k - 1 - p = 2$, and the p-value associated to the observed value χ^2 is 0.9624. This means that we accept H_0 at any significance level α smaller than or equal to 0.9624; in other words, the discrete Weibull model fits the data very well.

Table 7. Observed and theoretical distribution for the dataset of Section 4.2

class	1	2	3−4	5−8	>8
n_i	17	8	9	7	5
np_i	17.42	7.68	8.46	7.17	5.27

5. CONCLUSIONS

This paper examined several estimators for the parameters of the type I discrete Weibull r.v. Due to the particular form of its probability mass function, only one of the presented methods provides a closed form for both the estimators, while the others (maximum likelihood and method of moments, in its original and modified versions) provide the estimates as numerical solution to a minimization/maximization problem. It follows that not so much can be said about the statistical properties of the estimators for a finite sample size; then an extensive Monte Carlo simulation study was carried out in order to assess their behavior. Far from giving a definitive solution to the problem, the study highlighted that the method of proportion provides an unbiased and reliable estimate for the first parameter, even for small sample sizes, whereas the estimator of the second parameter is empirically biased and suffers from an excessive variability. The other methods provide estimates for the first parameter affected by bias, which is nonnegligible for small and moderate sample sizes and under some specific scenarios and is on the contrary negligible for larger sample sizes and under complementary configurations; the estimators of the second parameter provided by these methods are biased as well but overall more reliable in terms of precision than the analogous provided by the method of proportion. The study stressed that small size samples may make the method of proportion and, less often, the method of moments and of maximum likelihood unusable. The approximate confidence intervals based on maximum likelihood estimates reveal themselves quite satisfactory even for moderate sample sizes. The coverage rates are overall close to the nominal level (here, set at 95%), especially for the interval estimator of the second parameter (yet paid in terms of a large average length); the interval estimator of the first parameter may provide a poor coverage for values close to 1 and small samples.

Although the type I discrete Weibull model has met some obstacles to its diffusion within the scientific community, presumably due to the burdensome tractability of its probability mass function, nevertheless, it can be fruitfully used and easily handled, as proved by the applications and software implementation here presented.

A potential point of future research is further refining the estimators presented in this work, improving their statistical performance. In this sense, the application of some kind of resampling procedure (e.g., parametric bootstrap) may provide some useful prompts.

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