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# An application of stochastic control theory to a bank portfolio choice problem

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This paper presents an application of stochastic control theory to a bank portfolio choice problem. By applying a dynamic programming principle, we find a closed form solution for the CRRA utility function. A case study is given to illustrate our results and analyze the effect of the parameters on the optimal asset allocation strategy.

JEL: G21, C13, C15, C61.
Keywords and phrases: Bank portfolio, Stochastic optimal control, Dynamic programming principle, CRRA utility.

## 1. INTRODUCTION

In this paper, we study an optimal portfolio choice problem for a bank under a stochastic interest rate. Our goal is to present the numerical aspects of the resolution of the Hamilton-Jacobi-Bellman (HJB) equation and focus on the results of the portfolio choice model taking a practical viewpoint. This is motivated by the need of banks to invest in assets with an acceptable level of risk and high returns. For instance, if the returns on a specific loan turn out to be very high at the end of a loan contract period, the bank might regret not having allocated a fairly large portion of its capital to that loan type. A dynamic portfolio position is particularly important in bank risk management, since most banks select an initial loan portfolio at the beginning of a loan period but often do not actively manage their portfolio thereafter unless a possibility of default arises. Another motivation to discuss bank optimal portfolio is the failure of spark risk management strategies and regulatory prescripts to mitigate this risk. One of these prescriptions is the Basel accord on capital adequacy requirements[1], which mandates that all major international banks hold capital in proportion to their perceived risks.

We propose to apply the model in a simplified framework in order to find an analytically tractable solution for the bank portfolio choice problem. In particular, the representative bank dynamically allocates its wealth among the following assets: bank account, securities and loans (i) the asset prices assumed to satisfy the geometric Brownian motion hypothesis which implies that asset prices are stationary and log-normally distributed. All expected asset returns are given as the interest rate plus a constant risk premium, (ii) the interest rates are described by an Ornstein Uhlenbeck process, notably the case of Vasicek model, (iii) and the optimal asset allocation strategy is derived with power utility function. A dynamic programming principle is used to derive the HJB equation. We find a closed form solution for the optimal asset allocation strategy. We try to provide, through a case study on a Tunisian bank, a new insight of the model in terms of practical use.

The rest of the paper is organised as follows: In Section 2, we present the relevant literature. In Section 3, we introduce the bank portfolio model. In Section 4, we define and solve the optimization problem in the power utility case. In Section 5, we numerically illustrate our results and in the last section we draw the conclusion.

## 2. LITERATURE REVIEW

The portfolio choice is one of the most difficult decision problems faced by the banking institutions. The bank managers' objective is to choose an optimal structure of net wealth by allocating assets and liabilities with respect to revenue and cost proportions. In general, they often use Asset Liability Management (ALM) to rebalance the bank's portfolio based on the risk-return trade-off. The earliest approach to solving a portfolio choice problem is the mean-variance approach pioneered by Markowitz (1952) in a one-period decision model. It still has great importance in real-life applications, and is widely applied in the risk management departments of banks. The main reasons for this is being the simplicity with which the algorithm can be implemented, and that it requires no special knowledge on probability. Indeed, the risk is only measured by the variance, the returns are normally distributed and the bank managers utilize risk-averse utility functions. One criticism of the mean-variance criterion is the assumption of static nature of financial market or myopic optimization character. This is an extreme simplification of reality which totally ignores the highly volatile behavior and dynamic nature of prices. However, two main approaches dealing with the dynamic portfolio choice problem use continuous-time models. Stochastic control theory developed by Merton [19, 20] was based on

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[1] Basel III regulation establishes procedures for assessing credit, market, and operational risk (see, BCBS, 2011).

the solution of the HJB equation arising from dynamic programming under the real world probability measure. Several studies related to dynamic portfolio choice problem in banking have recently surfaced in literature (see, for instance, Repullo [27], Hackbarth [14], Mukuddem-Petersen and Petersen [22, 24], Bosch et al. [3], Fouche et al. [12]...). In particular, Mukuddem-Petersen and Petersen [22] suggested an optimal portfolio choice and a rate of bank capital inflow that will keep the loan level as close as possible to an actuarially determined reference process. In a paper [24], the capital adequacy ratio can be optimized in terms of bank equity allocation and the rate at which additional debt and equity is raised. The dynamic programming algorithm for stochastic optimization is employed to verify the results. Afterwards, a general case of maximization problem with Constant Relative Risk Aversion (CRRA) utility function is discussed in [23] that determine an analytical solution for the associated HJB equation in the case where the utility functions are either of power, logarithmic or exponential type. In this case, the control variates are the depository consumption, value of the depository financial institution's investment in loans and provisions for loan losses.

The second approach developed by Pliska [26], Karatzas et al. [15] and Cox-Huang [8] for complete markets relies on martingale theory and convex optimization. These methods frequently appear in research on the optimal asset allocation of a pension fund or life insurance policy (Boulier et al. [4], Campbell and Viceira [6], Brennan and Xia [5], Battocchio and Menoncin [2],...). Furthermore, some recent papers used martingale approach in analyzing the behavior of banks. In Gideon et al. [13], by considering a theoretical quantitative approach for bank liquidity provisioning, the authors used martingale approach to solve a nonlinear stochastic optimal liquidity risk management problem for subprime originators with deposit inflow rates and marketable securities allocation as controls. In this case, they provided an explicit expression for the aggregate liquidity risk when a locally risk minimizing strategy is utilized.

The groundbreaking work of Merton [19, 20] has an impractical assumption of constant interest rate while it is not appropriate to assume a constant interest rate in portfolios with a long horizon, such as banks. Indeed, understanding the term structure of interest rates is essential for appraising the interest rate risk of banks because: banks' interest income is at risk essentially by reason of the continuous movements of interest rates. Future interest rates of borrowing or lending-investing are unknown, if no hedge is contracted before. Banks tend to lend long and borrow short. When long-term interest rates are above short-term interest rates, this exposure of banks looks beneficial. Often, banks effectively lend at higher rates than the cost of their debts because of a positive spread between long-term rates and short-term rates. Unfortunately, the banks' interest income is at risk with the changes of shape and slope of the term structure. However, much less research efforts have been devoted to bank portfolio choice problem with stochastic interest rates. The optimal portfolio choice for banks with stochastic interest rates has been only discussed in the work of Witbooi et al. [30]. The main novel feature of their research is the combination of the interest rate model of Cox-Ingersoll and Ross and the Cox-Huang methodology to a banking fund portfolio. They obtained a closed form solution for the optimal equity allocation strategy that will optimize the terminal utility of the bank's shareholders under a power utility function. In fact, in some occasions they had to directly quote some results from Deelstra et al. [9]. For some other authors, a closed form solutions for some term structure models is determined. For instance, Korn and Kraft [17] investigated the case where interest rates follow a Vasicek and Ho-Lee model. Sørensen, [28], Wachter, [29], and Munk, Sørensen and Vinther, [25] suggested a similar problem under the Vasicek interest rates model of a CRRA agent. Likewise, Bajeux-Besnainou et al. [1] and Kim and Omberg, [16], considered the dynamic asset allocation problem for the Hyperbolic Absolute Risk Aversion (HARA) utility function where investors can invest in a bank account, stocks and bonds.

The closest literature to the present paper is the work of Witbooi et al. [30]. Having the same objectives, they solved for the exact solution of optimal portfolio strategies through the advent of loans as a new asset class. The main differences between this work and ours are that: they consider the Cox-Ingersoll and Ross interest rates model while we include the Vasicek type model. The martingale method is the means of finding the result against the present research paper applies the dynamic programming principle to investigate the optimal asset allocation strategy. Moreover, the portfolio includes treasuries, securities and loans while in our study the bank can invest in a riskless asset and two risky investment alternatives with risks depicted by the variance. This is a scope of banks with low risk aversion to foster capital by attracting risk-free deposits which they use to invest in risk-bearing loans. Therefore, there is more complications to find a closed form solution of our portfolio choice problem. For this reason, we choose the CRRA utility function because it is very tractable and the optimal asset allocation strategy is independent of wealth level. Another type of portfolio choice problem related to this paper is the optimal portfolio with defaultable assets established by Korn [18] in the framework of Merton's firm value model (see Merton [21]). In this regard, our setup differs from that on the use of options as investment classes and the presentation of the worst-case investment approach that takes the possibility of stock market crashes into account.

## 3. BANK PORTFOLIO MODEL

In this section, we show that the bank's assets may be modelled as random variables that are driven by an associated standard and independent Brownian motions and can be bought and sold without incurring any transaction costs or restriction on short sales. The uncertainty is modelled by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where, $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is the filtration generated by the Brownian motions $W \equiv \{W(t), t \geq 0\}$.

The dynamics of a security price $S(t)$ are presented by the following stochastic differential equation (SDE):

$$\frac{dS(t)}{S(t)} = (r(t) + \lambda_S)dt + \sigma_S dW_S(t), \tag{1}$$

where, $\sigma_S$ is the security volatility, $\lambda_S$ denotes the risk premium. Under the Capital Asset Pricing Model (CAPM) it could be quantified by the relation $\lambda_S = \beta[E(R_m) - R_f]$, with $E(R_m)$ is the market expected return, $R_f$ is the risk-free interest rate and $\beta$ is the sensitivity of the expected excess asset returns to the expected excess market returns.

The instantaneous interest rate dynamics $r(t)$ are described by an Ornstein-Uhlenbeck process:

$$dr(t) = \theta(\mu - r(t))dt + \sigma_r dW_r(t), \tag{2}$$

where the parameter $\theta$, $\mu$ and $\sigma_r$ are strictly positive constants and correspond, respectively, to the degree of mean-reversion, the long-run mean and the volatility of the interest rate.

The interest rate term structure has the same form as in Vasicek (1977). In particular, the price of a zero coupon bond with time to maturity $(T - t)$ is given by:

$$P(r, t, T) = e^{-a(T-t)-b(T-t)r}, \tag{3}$$

where;

$$a(\tau) = R(\infty)((T - t) - b(T - t)) + \frac{\sigma_r^2}{4\theta}(b(T - t))^2,$$
$$b(T - t) = \frac{(1 - e^{-\theta(T-t)})}{\theta},$$

and where $R(\infty) = \mu + \frac{\sigma_r \lambda_r}{\theta} - \frac{1}{2}\frac{\sigma_r^2}{\theta^2}$ represents the yield to maturity of a zero-coupon bond and $\lambda_r$ denotes the interest rate risk premium.

Any loan is essentially an interest rate contingent claim and by Itô lemma, the dynamics of the loan price $L(t)$ can be assumed as follows:

$$\frac{dL(t)}{L(t)} = (r(t) + \lambda_L)dt + \sigma_L dW_r(t), \tag{4}$$

where $\lambda_L = \lambda_r \sigma_L + \delta$. As in Merton [21] the default risk premium, $\delta$ is the bank charged credit spread, which is the function[2] of the probability of default $PD$ and the loss given default $LGD$. It is also assumed that the investor available loans have a constant duration $D$ similar to a zero-coupon bond expiring at the finite investment horizon. Hence, loans' volatility is also constant and given by $\sigma_L = \sigma_r D$.

[2]Spread = PD*LGD.

Let $X(t)$ denote the value of the bank asset portfolio at time $t \in [0, T]$ and $\pi_L(t)$, $\pi_S(t)$ are the proportions invested in the loans and securities, respectively. Then, $(1 - \pi_L(t) - \pi_S(t))$ is the proportion invested in the bank account. Owing to the independence of the Brownian motions and the self-financing assumptions, the asset portfolio value can be expressed as the following stochastic process:

$$\begin{aligned} \frac{dX(t)}{X(t)} &= (1 - \pi_L(t) - \pi_S(t))\frac{dB(t)}{B(t)} + \pi_L(t)\frac{dL(t)}{L(t)} \\ &\quad + \pi_S(t)\frac{dS(t)}{S(t)} \\ &= (r(t) + \pi_L(t)\lambda_L + \pi_S(t)\lambda_S)dt \\ &\quad + \pi_L(t)\sigma_L dW_r(t) + \pi_S(t)\sigma_S dW_S(t), \end{aligned} \tag{5}$$

where, $X(0) = X_0$ stands for an initial wealth.

## 4. BANK OPTIMAL CONTROL PROBLEM

The bank shareholders expect a good return on their capital investment while minimizing the risk. In fact, bank management needs to strategically allocate the shareholders' equity in order to maximize the terminal wealth. However, the changes in the bank's asset value are reflected in the shareholders' equity fluctuations prompting the bank to maximize asset portfolio return relative to risk. In this regard, the associated utility function is assumed to belong to the CRRA utility function class.

The control problem on a time interval [0,T] is defined by:

$$\sup_{\pi(.)\in\mathcal{A}} \mathbb{E}[U(X_T)],$$

where,

$$U(X) = \frac{X^{1-\gamma}}{1-\gamma}, \qquad 0 < \gamma < 1.$$

For a well posed control problem one needs additional assumptions about admissible controls set to the effect that the SDE (5) below admits a unique, strong and almost surely positive solution. The set of admissible controls is given by:

$$\begin{aligned} \mathcal{A} &= \Big\{\pi(.) = \pi(t)_{t\in[0,T]}, \ \mathcal{F}\text{- adapted}, \\ & \int_0^T ((\pi_L(t)\sigma_L)^2 + (\pi_S(t)\sigma_S)^2)\mathrm{d}t < +\infty, \mathbb{P} - a.s.\Big\}. \end{aligned} \tag{6}$$

In this case, the basic source of uncertainty are due to changes in the interest rates or the value of the asset portfolio. Moreover, the state variables in equation (5) can be identified as the interest rates $r(t)$, and the value of the asset portfolio $X(t)$, and the control variables is the optimal proportion $\pi(t)$.

We are going to solve this problem via stochastic control.

**Theorem.** *Suppose that $J \in C^{1,2}$ is a solution of the HJB equation*

$$J(t,r,x) = \sup_{\pi(t)\in\mathcal{A}} \mathbb{E}[U(X_T^{t,r,x})], \tag{7}$$

*and the optimal asset allocation strategy $(\pi_L^*(t), \pi_S^*(t))$, if it exists*

$$(\pi_L^*(t), \pi_S^*(t)) = \arg\max_{\pi(t)\in\mathcal{A}} J(t,r,X).$$

*Proof.* We provide a mere outline of the proof.

Let $\pi(t) \in \mathcal{A}$ and $X(t)$ the corresponding asset portfolio value process, hence, the HJB equation associated with the optimal control problem is:

$$J_t + J_r(\theta(\mu - r(t))) + \frac{1}{2}J_{rr}\sigma_r^2 + \sup_{\pi(t)\in\mathcal{A}}\Big[XJ_X(r(t) + \pi_L(t)\lambda_L + \pi_S(t)\lambda_S) + \frac{1}{2}J_{XX}X^2[(\pi_L(t)\sigma_L)^2 + (\pi_S(t)\sigma_S)^2] + J_{Xr}X[(\pi_L(t)\sigma_L + \pi_S(t)\sigma_S)\sigma_r\Big] = 0.$$

Here, $J_t$, $J_r$, $J_X$, $J_{rr}$,$J_{XX}$, and $J_{Xr}$ denote the first and second-order partial derivatives with respect to $t$, $r$ and $X$ in the normal way.

By applying the first-order conditions we get:

$$\pi_S^*(t) = -\frac{\lambda_S}{\sigma_S^2}\frac{J_X}{XJ_{XX}} - \frac{\sigma_r}{\sigma_S}\frac{J_{Xr}}{XJ_{XX}},$$
$$\pi_L^*(t) = -\frac{\lambda_L}{\sigma_L^2}\frac{J_X}{XJ_{XX}} - \frac{\sigma_r}{\sigma_L}\frac{J_{Xr}}{XJ_{XX}}.$$

□

The standard approach to solve this kind of PDE is to try for separability condition. In Merton [20], the separability condition in wealth by product represents a common assumption in the attempt to solve explicitly optimal portfolio problems. Specifically, in order to obtain smooth analytic solution to the maximization problem, we choose a power utility function. The value function $J$ can be rewritten as:

$$J(t,r,X) = X^\gamma(T)f(t,r), \tag{8}$$

with terminal condition $f(T,r) = 1$ for all $r$.

Substituting the partial derivatives of the value function (8) and the optimal proportions $\pi(t)^*$ into HJB equation, leads to a second-order PDE for $f$ of the form:

$$(\gamma-1)ff_t + (\gamma-1)ff_r(\theta(\mu - r(t))) + \frac{1}{2}(\gamma-1)ff_{rr}\sigma_r^2 + \gamma(\gamma-1)f^2r - \frac{1}{2}\gamma f^2\left[\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\right] - \gamma ff_r\left[\frac{\lambda_L}{\sigma_L} + \frac{\lambda_S}{\sigma_S}\right]\sigma_r - \gamma f_r^2\sigma_r^2 = 0,$$

with the terminal condition $f(T,r) = 1$ for all $r$.

Therefore, by conjecture, a solution of $J$ have the following form: $f(t,r) = g(t)\exp(A(t)r)$[3] with terminal conditions: $g(T) = 1$, $A(T) = 0$.

Simplifications yields:

$$(\gamma-1).g' + \underbrace{(\gamma-1)(A'(t) + \gamma - A(t)\theta)}_{=:\alpha(t)}.rg + \underbrace{\left\{\begin{array}{l}(\gamma-1)\theta\mu A(t) - \frac{1}{2}(\gamma+1)A^2(t)\sigma_r^2 \\ -\gamma A(t)\left[\frac{\lambda_L}{\sigma_L} + \frac{\lambda_S}{\sigma_S}\right]\sigma_r - \frac{1}{2}\gamma\left[\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\right]\end{array}\right\}}_{=:h(t)}.g = 0.$$

The conjecture for $f$ is only meaningful, if $A$ can be calculated so the factor $\alpha(t)$ becomes zero. As a result we have to solve the inhomogeneous ODE for $A$ which has the following form:

$$A'(t) = \theta A(t) - \gamma,$$

with $A(t) = 0$ leading to: $A(t) = \frac{\gamma}{\theta}[1 - \exp(\theta(T-t))]$.

Choosing $A$ as calculated, a first-order homogeneous ODE for $g$ is obtained again:

$$(\gamma-1)g' + h(t)g = 0,$$

with $g(T) = 1$. Hence,

$$g(t) = \exp\left[\frac{1}{\gamma-1}H(t) - H(T)\right],$$

with $H(t)$ is the primitive of $h(t)$ (see appendix A).

Therefore,

$$J(t,r,X) = X^\gamma\exp\left[\left(\frac{1}{\gamma-1}H(t) - H(T)\right) + \frac{\gamma}{\theta}(1 - \exp(\theta(T-t)))r\right].$$

and the optimal solution of the portfolio choice problem:

$$\begin{aligned}\pi_S^*(t) &= \frac{1}{1-\gamma}\frac{\lambda_S}{\sigma_S^2} - \frac{\gamma\sigma_r k(t)}{1-\gamma}, \\ \pi_L^*(t) &= \underbrace{\frac{1}{1-\gamma}\frac{\lambda_L}{\sigma_L^2}}_{\text{Merton result}} - \underbrace{\frac{\gamma\sigma_r k(t)}{1-\gamma}}_{\text{Correction term}},\end{aligned} \tag{9}$$

with, $k(t) = \frac{1-e^{-\theta(T-t)}}{\theta}$.

The optimal proportions of power utility function are continuous function of time and directly related to the interest rate. The solution confirms the conjecture of the separable value function $J$. The first term coincides with the classical

[3] Is the solution of PDE with $g$ and $A$ are regular functions.

optimal one in Merton [20] when the coefficients are deterministic. The second term can be interpreted as a correction term, positively and monotonously decreasing to zero up to the terminal date T.

## 5. NUMERICAL RESULTS

In this section we present the numerical results for optimal control program with power utility function. The investment horizon is set to 10 years and the degree of risk aversion is $\gamma = 0.5$.

The estimation of parameters related to the risk and return of each asset is a more difficult task since the confidential nature and the little data sample. The estimation is based on maximum likelihood method[4] using historical data collected from over the counter market and the Tunisian stock exchange for the period 2004–2012. The interest rate parameters are adopted from the estimation of the Vasicek term structure[5] using the 52 weeks Treasury bond yields. Most bank loans are divided into corporate and consumer categories, respectively. However, we focus on corporate loans only because there is no information for consumer loans. The loans have constant duration similar to a 10 year zero-coupon bond. Using the historical bank financial statement data (Balance sheet and Cash flow statement) and by calibrating the Merton's [21] model we get the parameter of the loan's volatility and the default risk premium[6]. Regarding the securities, they are represented by a portfolio of three SICAV[7](SICAV Prosperity, SICAV Opportunity and SICAV Tresor). The estimation results are displayed in Table 1. Fig. 1 highlights how the evolution of the optimal asset allocation strategy is actually affected by the realization of the stochastic variables characterizing the economy. The optimal asset allocation strategy, shows that the optimal proportion invested in the bank account (represented by the downward sloping curve) decreases from 75.68% to 23.57%. On the other hand, the optimal proportion invested in securities and loans increase with respect to time. In particular, the loans' proportion rises from an initial value of about 19.94% to just above 46%, while the proportion invested in the securities increases from an initial value close to 4.38% to a proportion of about 30.43%. However, the bank account plays a residual role in the optimal portfolio composition. At the beginning of the investment period, the need of a conservative strategy for creating a higher wealth level and a lower risk leads to a high proportion of bank account in the optimal portfolio, while the investment in the loans and securities asset is very low. Consistently, as time approaches to maturity T, a shift in wealth from bank account investment to the risky assets will be noticed. The riskiness of strategy increases both the investment in securities and loans increases and the proportion of wealth invested in the bank account decreases. Then, the bank manager maintains a diversified portfolio until maturity with a high percentage of wealth is allocated to the loans. These results are very intuitive and reasonable since it indicates that the bank optimal strategy is to borrow money to invest in securities.

*Table 1. Parameter values*

| Definition | Symbol | value | Standard Error |
|---|---|---|---|
| Interest rate premium | $\lambda_r$ | 0.0002 | $5.26 \times 10^{-5}$ |
| Mean rate | $\mu$ | 0.044 | $6.96 \times 10^{-5}$ |
| Volatility | $\sigma_r$ | 0.0003 | $4.47 \times 10^{-5}$ |
| Mean-reversion | $\theta$ | 0.0037 | $2.53 \times 10^{-4}$ |
| Securities risk premium | $\lambda_S$ | 0.007 | $2.27 \times 10^{-4}$ |
| Securities volatility | $\sigma_S$ | 0.046 | $1.5 \times 10^{-4}$ |
| Default risk premium | $\delta$ | 0.023 | $3 \times 10^{-4}$ |
| Loans volatility | $\sigma_L$ | 0.1 | 0.0062 |
| Investment horizon | T | 10 | |

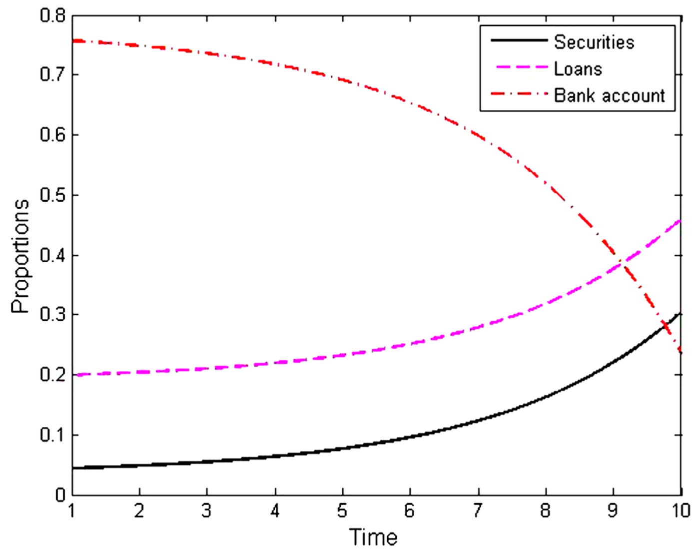


*Figure 1. Optimal proportions invested in bank account, loans and securities.*

[4]See Appendix B, C and D for more detail.
[5]See Chakroun and Abid, [7].
[6]

$$\text{Default risk premium} = R(\tau) - r = -\frac{1}{\tau}\ln\left\{\phi(h_2) + \frac{1}{d}\phi(-h_1)\right\},$$

where;

$$d = \frac{De^{-R(\tau)}}{L},$$
$$h_1 = [\ln(d) - (r + \frac{1}{2}\sigma_L^2)\tau]/\sigma_L\sqrt{\tau},$$
$$h_2 = h_1 - \sigma_L\sqrt{\tau}.$$

We denote by: $R(\tau)$ is the yield to maturity on the loans, $\tau = T - t$ is length of time until maturity, $\sigma_L$ is the loan's volatility, $D$ is the deposit reimbursed at time T and $\phi(.)$ is the cumulative Normal distribution function.
[7]Investment company with variable capital held by the bank.

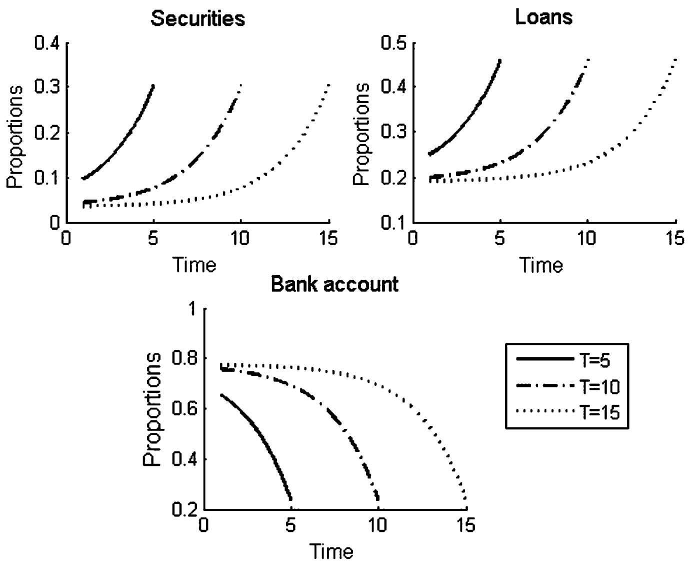


*Figure 2. Effect of time.*

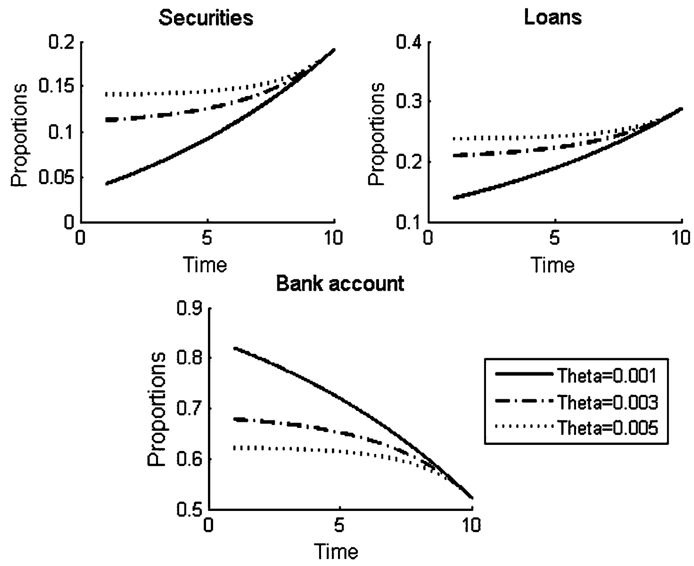


*Figure 4. Effect of the mean-reversion parameter.*

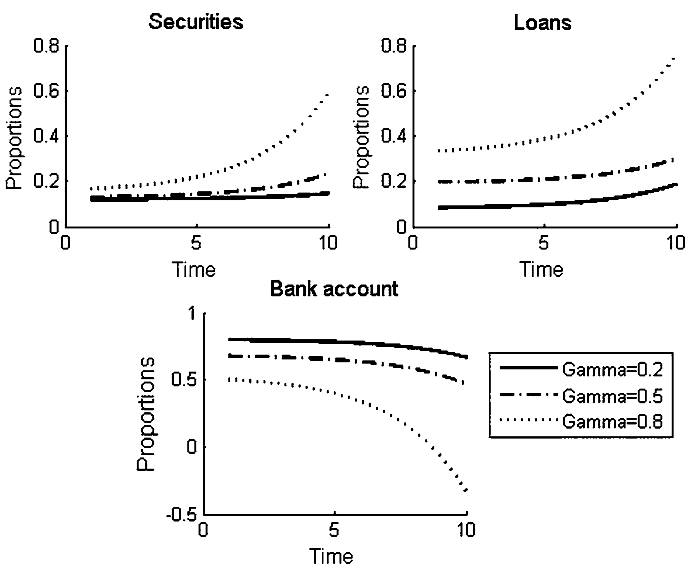


*Figure 3. Effect of the risk aversion degree.*

To test how sensitive the optimal strategy is to changes in the different underlying variables, we have performed a sensitivity analysis, keeping the parameters in Table 1 and changing each time the value of one parameter.

Fig. 2 shows for a given value of gamma ($\gamma = 0.5$), how the proportions are modified as time passes. With horizons ranging from 5 years to 15 years the proportion in bank account increases and remains positive. However, the allocation to loans and securities decrease as the investment horizon increases. As a result, it seems that a long horizon bank manager behaves more conservatively. Fig. 3 presents the effect of varying the degree of risk aversion. The optimal asset allocation strategy is quite sensitive to the risk aversion. For a given time horizon ($t \in [5, 10]$) the proportion invested in securities and loans increases with risk aversion. For shorter horizon and higher risk aversion, the proportion in a bank account remains positive. However, the allocation to the asset are constant across time for a risk aversion lower than 0.8 and the investment behavior seems to be stabilized until maturity. Fig. 4 shows the effect of changes the mean-reversion parameter on the proportions of the optimal asset allocation strategy. Obviously, increasing the interest rate mean-reversion parameter has the same impact on the allocation of the loans and securities. As time approaches to maturity, the asset allocation is relatively insensitive to the interest rate mean-reversion parameter. As a consequence, a strong correlation has been established between assets and interest rate. In order to monitor the fluctuations in the interest rate, in practice, the securities may partially be used to hedge real interest rate uncertainty. Increasing interest rate volatility causes the bank manager to shift money from securities and loans into a bank account (see Figure 5). Therefore, this result can be explained by the fact that the loans and securities becomes more risky for the same risk premium. Varying the interest rate long-run mean has no effect. This is linked to the assumption of CRRA utility function, which yields proportions that are mainly a function of the risk premiums and independent of the interest rates level.

Our analysis has great potential implications on the bank management. Indeed, it is often said that the portfolio choice is an important decision making and a key source of competitive advantage. Actually most bankers implicitly promote an ALM approach to bank asset management. In particular, ALM cannot be separated from the decision about how much equity the bank owner should invest. This means that banking decisions and equity policies have to be simultaneously addressed by bank managers. In our research, we

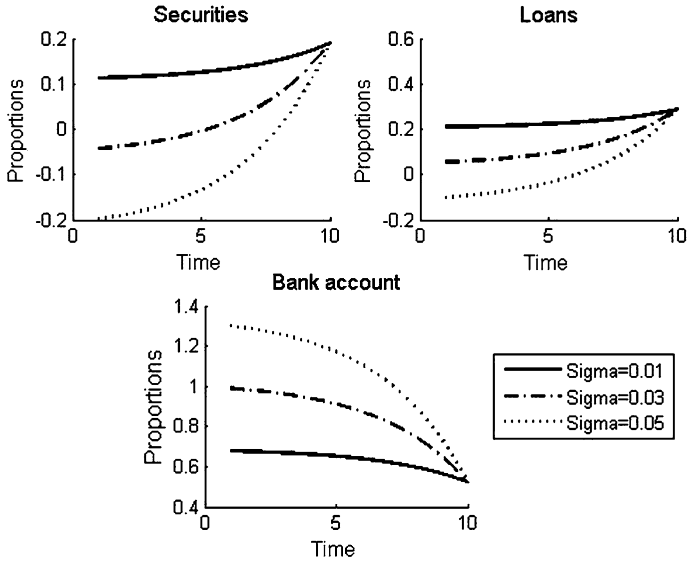


*Figure 5. Effect of the interest rate volatility.*

provide a simplified framework suggesting that ALM can ensure that bank managers are able to adequately manage their asset portfolios. Our analysis shows that taking an ALM approach generates two main benefits. First, it has a direct impact on the selection of asset classes, in particular, it leads to a focus on the liability hedging properties of various asset classes. Second, it leads to define the risk and returns in relative rather than in absolute terms, with the liability portfolio used as a benchmark or numeraire. This is a critical improvement on the portfolio choice models which fail to recognise that changes in the asset values must be analysed in comparison to the changes in the liability values.

This work can be extended in a number of directions. First, it would be desirable to incorporate the impact of inflation risk in the analysis of the optimal asset allocation strategy. Other elements that are left for further research are the introduction of capital adequacy decisions as well as the extension of the portfolio choice model to more general forms of state dependent optimal allocation strategies.

## 6. CONCLUSION

This paper addresses the problem of bank optimal portfolio choice. The bank shareholders have a power utility function and can invest in the bank account, securities and loans in a complete market setting where the Vasicek term structure model applies. The solution approach is based on the dynamic programming principle. Indeed, a verification theorem claims that the related HJB equation has a closed form solution under the separation condition. The estimation of parameters is based on the maximum likelihood method. With this parameterization a case study confirms the practical potential of the results and shows that this model can adequately account for the essential aspects of the bank. The sensitivity analysis highlights the importance of dynamic considerations in optimal asset allocation depending on the stochastic characteristics of the investment opportunity set.

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## APPENDICES

### Appendix A

$H(t)$ is the primitive of $h(t)$, by replacing A(t) we get:

$$\begin{aligned} H(t) = \frac{\gamma}{\theta}(t - \frac{e^{\theta\tau}}{\theta})\Big\{ & \theta\mu(\gamma-1) \\ & - \frac{1}{2}(\gamma+1)\sigma_r^2\frac{\gamma}{\theta}\left(t - \frac{2e^{\theta\tau}}{\theta} + \frac{e^{2\theta\tau}}{2\theta}\right) \\ & - \gamma\big[\frac{\lambda_L}{\sigma_L} + \frac{\lambda_S}{\sigma_S}\big]\sigma_r\Big\} - \frac{1}{2}\gamma\left[\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\right]. \end{aligned}$$

### Appendix B

The estimation of the Vasicek parameters which maximise the likelihood function. Given $N$ observations of 52-week Treasury bond yields $\{r_{t_i}, i = 1, ..., N\}$. The likelihood function is as follow:

$$L(\psi) = \prod_{i=1}^{N-1} p(r_{t_{i+1}}|r_{t_i};\psi;\Delta), \tag{10}$$

with $\Delta t$ time step, $\psi \equiv (\theta, \mu, \sigma)$ a parameter vector to be estimated and $p(r_i|\psi)$ defined as the transition function of the Vasicek and CIR process respectively. Then, the log-likelihood function is,

$$\ln L(\psi) = \sum_{i=1}^{N-1} \ln p(r_{t_{i+1}}|r_{t_i};\psi;\Delta t). \tag{11}$$

Therefore, the maximum likelihood estimator $\hat{\psi}$ of parameter vector $\psi$ is:

$$\psi \equiv (\hat{\theta}, \hat{\mu}, \hat{\sigma}) = \arg\max_{\psi} \ln L(\psi). \tag{12}$$

Moreover, the application of the maximum likelihood requires the specification of the transition function of each process. Hence, the conditional density function for Vasicek model is given by:

$p(r_{t+\Delta t}|r_t;\psi,\Delta t)$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{\left(r_{t_{i+1}} - r_{t_i} e^{-\theta\Delta t} - \mu(1 - e^{-\theta\Delta t})\right)^2}{2\sigma^2}\right],$$

with, $\sigma^2 = \hat{\sigma}^2 \frac{1-e^{-2\theta\Delta t}}{2\theta}$. The corresponding log-likelihood function is:

$$\ln L(\psi) = -\frac{N}{2}\ln(2\pi) - N\ln(\sigma)$$
$$-\frac{1}{2\sigma^2}\sum_{i=1}^{N}\left[r_{t_{i+1}} - r_{t_i}e^{-\theta\Delta t} - \mu\left(1 - e^{-\theta\Delta t}\right)\right]^2.$$

## Appendix C

Let $X, Y$ and $Z$ be three random variables that define the SICAV Opportunity, SICAV Prosperity and SICAV Tresor, respectively. In order to determine the maximum likelihood function, we only consider random variables with multivariate normal transition density,

$$f(x;\mu,\Sigma) = (2\pi)^{-n/2}\frac{1}{(\det\Sigma)^{1/2}}e^{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)},$$

where, $\mu \in \mathbb{R}^N$ is the mean vector and $\Sigma \in \mathbb{R}^{N\times N}$ is symmetric and positive definite covariance matrix. Where,

$$\Sigma^{-1} = \begin{bmatrix} K_{1,1} & \cdots & K_{1,n} \\ \vdots & \ddots & \vdots \\ K_{n,1} & \cdots & K_{n,n} \end{bmatrix}.$$

The maximum likelihood estimation of the parameters $(\mu, \Sigma)$ of a multivariate distribution $X \sim N(\mu, \Sigma)$ can be solved efficiently as a convex optimization problem.

The log-likelihood function of the observations is:

$$L(x;\mu,\Sigma) = \ln\prod_i f(x_i) = -\frac{N}{2}\ln\det(\Sigma)$$
$$-\frac{1}{2}\Sigma_i(x_i-\mu)^T\Sigma^{-1}(x_i-\mu).$$

Define the sample estimates $\bar{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i$ and $\bar{\Sigma} = \frac{1}{N}\Sigma_{i=1}^{N}(x_i - \bar{\mu})(x_i - \bar{\mu})^T$. Then, the likelihood function can be written as:

$$L(x;\mu,\Sigma) = \frac{N}{2}(-\ln(\det(\Sigma) - \mathrm{Tr}(\Sigma^{-1}\bar{\Sigma})$$
$$-(\mu-\bar{\mu})^T\Sigma^{-1}(\mu-\bar{\mu})).$$

Thus, the maximum likelihood problem is as follows:

$$\max \ln\det(K) - \mathrm{Tr}(K\bar{\Sigma}).$$

In our implementation we used the Tunisia Stock Market index (TUNINDEX) to estimate the risk premium.

## Appendix D

The estimation of Merton's [21] parameters are proposed by Duan et al. [10, 11] based on the transformed-data maximum likelihood estimation method.

The loan value process follows the geometric Brownian motion, we can derive its discrete-time form with time step $\tau_i - \tau_{i-1} = h$ as:

$$\ln L_{\tau_{i+1}} = \ln L_{\tau_i} + (\mu - \frac{\sigma_L^2}{2})h + \sigma\sqrt{h\epsilon_{i+1}},$$

where, $\epsilon_i, i = 1, N$ are i.i.d standard normal random variables. We denote the log-likelihood function of observed data set under a specific model as $L(\theta;\text{data})$ where $\theta$ is the set of unknown parameters under the model. The maximum likelihood estimation is to find the value of $\theta$ at which the data set has the highest likelihood of occurrence under Merton [21] model and assuming that one could directly observe the firm's loan values $\{L_0, L_k, ..., L_{nk}\}$ the log-likelihood function could be written as:

$$L^L(\mu_L, \sigma_L; L_0, L_k, ..., L_{nk})$$
$$= -\frac{n}{2}\ln(2\pi\frac{\sigma_L^2}{2}h) - \frac{1}{2}\sum_{k=1}^{n}\frac{(R_k - (\mu - \sigma_L^2)h)^2}{\sigma_L^2 h} - \sum_{k=1}^{n}\ln L_{kh},$$

where, $R_k = \ln(\frac{L_{kh}}{L_{(k-1)h}})$.

Recognizing that Merton's [21] model implicitly provides a one-to-one smooth relationship between the equity $E(t)$ and loan values, one can invoke the standard transformations to derive the log-likelihood function solely based on the bank observed equity data. If we denote the density of the loan value as $f(L)$, the density associated with the equity will be given by $f(L)|\frac{\delta g(L;\sigma)}{\delta L}|$. Applying this knowledge yields the following log-likelihood function on the bank observed equity data:

(13) $$L^E(\mu_L, \sigma_L; E_0, E_h, ..., E_{nh})$$
$$= L^L\left(\mu_L, \sigma_L; \hat{L}_0(\sigma_L), \hat{L}_h(\sigma_L), ..., \hat{L}_{nh}(\sigma_L)\right)$$
$$-\sum_{k=1}^{n}\ln(\Phi(\hat{d}_{kh}(\sigma_L))),$$

where, $\hat{L}_{kh}(\sigma_L) = g^{-1}(E_{kh};\sigma_L)$ and $\hat{d}_{kh} = \frac{\ln(\hat{L}_{kh}(\sigma_L/F) + (r + \frac{\sigma_L^2}{2}(T-kh)}{\sigma_L\sqrt{T-kh}}$. One can easily find the maximum-likelihood estimates by numerically maximizing the function (13) based on the function fmincon in MATLAB. Then, the estimation procedure is as follows:

**Step 1**: Estimate the Merton [21] model using the log-likelihood function in equation (13).

**Step 2**: Compute $\delta$ which represent the point estimates for default risk premium.

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