Statistics and Its Interface Volume 8 (2015) 63–69

Tail dependence for two skew slash distributions

Chengxiu Ling*,† and Zuoxiang Peng‡

Coefficients of tail dependence measure the dependencies between extreme values. In this paper, the upper tail dependence coefficients of two classes of skew slash distributions are derived. The difference of tail dependence coefficients between the two types skew slash distributions sheds light on the model choice for random variables with asymptotic dependence.

AMS 2000 subject classifications: Primary 60G15; secondary 60G70.
Keywords and phrases: Tail dependence coefficient, Skew slash distribution, Skew-normal distribution, Variance-mean mixture.

1. INTRODUCTION

Let $\boldsymbol{X} = (X_1, X_2)^\top$ be a bivariate random vector with marginal distribution functions (dfs) F_1 and F_2, respectively ($^\top$ stands for transpose sign). The upper tail dependence coefficient of $\boldsymbol{X}$ is defined by

$$\lambda_U = \lim_{u\uparrow 1} \mathbb{P}\left(F_1(X_1) \geq u | F_2(X_2) \geq u\right) \tag{1}$$

provided that the limit λ_U exists; see [23, 12]. This quantity provides insight into the tendency for the distribution to describe joint extreme events since it measures the strength of dependence (or association) in the tail of a bivariate distribution. Generally, $\boldsymbol{X}$ is said to have asymptotic upper tail dependence if λ_U is positive. In particular, trivial values $\lambda_U = 1$ and $\lambda_U = 0$ correspond to full dependence and independence, respectively.

Tail independence of bivariate normal distributions was first addressed by Sibuya [27] (see also [12]) while the tail dependence of symmetry t-distributions was established by [10]. Their skew-versions were further considered by [5, 13]. The skew t-distributions are more popular and useful since they provide tail dependence of some extent as well as skewness and heavy tails compared with (skew) normal distributions. For more related studies see, e.g., [17, 14, 24], and references therein.

*Corresponding author.
†Partial support by SNSF Grant No. 200021-140633/1 and the project RARE-318984 (an FP7 Marie Curie IRSES Fellowship).
‡Supported by the National Natural Science Foundation of China grant no. 11171275 and the Natural Science Foundation Project of CQ no. cstc2012jjA00029.

In recent years, the multivariate skew slash distributions alternatively (see (2) and (3) below for the precise definitions) have received considerable attention in theoretical studies for their numerous stochastic properties and in applied studies for robust statistical modeling of datasets involving distributions with skewness and heavy tails; see, e.g., conditional distributions, moments and applying skew slash distributions to fit AIS and glass-fiber data ([28]) and characteristics functions ([19]) for skew slash distributions in (3), and parameters estimation procedure such as the EM based on MLE in [3, 7, 8], MLE in [22], and empirical Bayes estimations in [30] for the skew slash distributions in (2). For more details see, e.g., [15, 25], and references therein.

Recently, tail dependence has been discussed in financial applications related to market or credit risk; see, e.g., [26, 11]. A generalized tail dependence measure, namely tail quotient correlation coefficient, was proposed by [31] where new test statistics of tail independence were developed; see [29] for more related studies. In this paper, we shall investigate the tail dependence coefficient for two classes of skew slash distributions. The first class is defined by the normal variance-mean method. Specifically, a random vector $\boldsymbol{X} = (X_1, X_2)^\top$ is called skew slash distributed random vector with parameters $(\lambda, \boldsymbol{\theta}, \mathbf{R})$, denoted by $\boldsymbol{X} \sim SS(\lambda, \boldsymbol{\theta}, \mathbf{R})$, if $\boldsymbol{X}$ has the following stochastic representation (see [2, 3])

$$\boldsymbol{X} = \frac{\boldsymbol{\theta}}{V} + \frac{\boldsymbol{Z}}{\sqrt{V}}, \tag{2}$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2)^\top \in \mathbb{R}^2$ and $V \sim Beta(\lambda, 1), \lambda > 0$ with probability density function (pdf) $f(x) = \lambda x^{\lambda-1}, x \in (0, 1)$, independent of $\boldsymbol{Z} \sim N_2(\mathbf{0}, \mathbf{R})$, a bivariate normal distribution with mean $\mathbf{0}$ and correlation matrix $\mathbf{R}$ with correlation entry $\rho \in (-1, 1)$. This skew slash distribution introduces randomness into the variance and mean of a normal distribution via a beta random variable so that it is more flexible and can provide useful asymmetric and heavy-tailed extensions of their symmetric counterparts ($\boldsymbol{\theta} = \mathbf{0}$) for robust statistical modeling of datasets. For more related studies on model (2) see, e.g., generalized hyperbolic skew t-distributions in [20], skew grouped t-distributions in [5] and skew t-distributions in [14].

The second class of skew slash distributions is defined as the scale-mixed skew-normal distribution (see [4]). A random vector $\boldsymbol{X}$ is called the second type skew slash distribution, denoted by $\boldsymbol{X} \sim ASS(\lambda, \boldsymbol{\theta}, \mathbf{R})$, if $\boldsymbol{X}$ is given by

$$\boldsymbol{X} = \frac{\boldsymbol{\mathcal{Z}}}{\sqrt{V}}, \tag{3}$$

where $V \sim Beta(\lambda, 1), \lambda > 0$, independent of $\boldsymbol{\mathcal{Z}} = (Z_1, Z_2)^\top \sim SN_2(\boldsymbol{\theta}, \mathbf{R})$, a bivariate skew normal distribution with pdf

$$2\phi_2(\boldsymbol{z}, \mathbf{R})\Phi(\boldsymbol{\theta}^\top \boldsymbol{z}),$$

where $\phi_2(\cdot, \mathbf{R})$ is the bivariate normal density function with mean $\mathbf{0}$ and correlation matrix $\mathbf{R}$, and $\Phi(\cdot)$ is the standard normal distribution function. For more related studies on model (3) see, e.g., [19, 21] for other scaled positive variable V.

The goal of this paper is to establish the limit of the conditional distributions and to derive the upper tail dependence coefficient of $\boldsymbol{X}$ given by (2) and (3), respectively. Comparison with the findings of tail independence of bivariate normal ([27, 12]), skew-bivariate normal ([6]); tail dependence of two skew t-distributions ([13, 6]), the tail dependence of the first class of skew slash distributions exists with trivial values 0 or 1 for some special cases (Theorem 3.1), while the second class has wider region of tail dependence (Theorem 3.2).

The rest of the paper is organized as follows. The main results are provided in Section 3. All proofs are postponed to Section 4.

2. PRELIMINARIES AND NOTATION

In this section, we first introduce some important functions with their asymptotic properties established in Lemma 2.1, and then give Lemma 2.2 for the distribution properties of the skew slash random vector $\boldsymbol{X}$ given by (2) via the normal variance-mean mixture.

Let $K_\tau(x; \omega), x \geq 0, \omega > 0$ be the incomplete modified Bessel function of the third kind with index $\tau \in \mathbb{R}$ defined by

$$K_\tau(x; \omega) = \frac{1}{2}\int_x^\infty t^{\tau-1} \exp\left(-\frac{\omega}{2}\left(t + t^{-1}\right)\right)\, dt. \tag{4}$$

It follows from (7.5) in [18] that for $\tau \in \mathbb{R}$ and sufficiently large ω

$$K_\tau(0; \omega) = \sqrt{\frac{\pi}{2\omega}} e^{-\omega}\left(1 + \frac{4\tau^2 - 1}{8\omega} + o\left(\frac{1}{\omega}\right)\right). \tag{5}$$

Define further $P_\tau(a; b)$ and $Q_\upsilon(x; a)$, respectively by

$$P_\tau(a; b) = \int_0^1 t^{\tau-1} \exp\left(-\frac{1}{2}\left(a^2 t + \frac{b^2}{t}\right)\right)\, dt \tag{6}$$

and

$$Q_\upsilon(x; a) = \int_{-\infty}^x \int_0^a t^{\upsilon-1} e^{-(1+u^2)t}\, dt du, \quad x \in \mathbb{R}, \tag{7}$$

where $\tau > 0, a, b \geq 0$ and $\upsilon \geq 1$. For simplicity, we write $\Gamma(\cdot)$ for the Euler gamma function.

The following result is about the asymptotic behaviors of both $P_\tau(a; b)$ and $Q_\upsilon(x; a)$.

Lemma 2.1. *Let $P_\tau(a; b)$ and $Q_\upsilon(x; a)$ be those defined as in (6) and (7). Then for $\tau > 0$ and some $\omega > 0$*

$$P_\tau(a; b) = \begin{cases} \left(\dfrac{2}{a^2}\right)^\tau \Gamma(\tau)(1 + o(1)), & b = 0, a \to \infty; \\ \dfrac{b^{\tau-1/2}}{a^{\tau+1/2}}\sqrt{2\pi} e^{-ab}(1 + o(1)), & b > 0, a \to \infty; \\ 2\left(\dfrac{b}{a}\right)^\tau K_\tau(0; \omega)(1 + o(1)), & b \to 0, ab \to \omega, \end{cases}$$

and for $\upsilon \geq 1, x \in \mathbb{R}$ and $a \to \infty$

$$Q_\upsilon(x; a) \to \Gamma(\upsilon)\int_{-\infty}^x (1 + u^2)^{-\upsilon}\, du =: Q_\upsilon(x; \infty). \tag{8}$$

Recall that λ_U is equivalent to

$$\begin{aligned} \lambda_U = & \lim_{x_1 \to \infty} \mathbb{P}\left(X_2 \geq F_2^{-1}(F_1(x_1)) | X_1 = x_1\right) \\ & + \lim_{x_2 \to \infty} \mathbb{P}\left(X_1 \geq F_1^{-1}(F_2(x_2)) | X_2 = x_2\right) \end{aligned} \tag{9}$$

provided that the marginal distributions are continuous (cf. [23], p. 11, 36). In the following we derive the marginal distribution and the conditional distribution of $\boldsymbol{X}$ given by (2).

Lemma 2.2. *For $\boldsymbol{X} \sim SS(\lambda, \boldsymbol{\theta}, \mathbf{R})$ given by (2), denote by $f_2(\cdot)$ and $f_{1.2}(\cdot|x_2)$ the pdfs of X_2 and $X_{1.2} := (X_1|X_2 = x_2)$, respectively. Then, with $P_\tau(a; b)$ given by (6)*

$$f_2(x_2) = \frac{\lambda e^{\theta_2 x_2}}{\sqrt{2\pi}} P_{\lambda+1/2}(|x_2|; |\theta_2|)$$

$$f_{1.2}(x_1|x_2) = \frac{e^{\beta(x_1 - \rho x_2)}}{\sqrt{2\pi(1-\rho^2)}} \frac{P_{\lambda+1}\left(\sqrt{x_1'^2 + x_2^2}; \sqrt{\theta_1'^2 + \theta_2^2}\right)}{P_{\lambda+1/2}(|x_2|; |\theta_2|)},$$

where $\beta(1-\rho^2) = \theta_1 - \rho\theta_2, x_1'\sqrt{1-\rho^2} = x_1 - \rho x_2$ and $\theta_1'\sqrt{1-\rho^2} = \theta_1 - \rho\theta_2$. Furthermore, for $\boldsymbol{\theta} \neq \mathbf{0}$

$$\mathbb{E}e^{-sX_{1.2}} = \frac{P_{\lambda+1/2}\left(|x_2|; \sqrt{\theta_2^2 + 2\beta(1-\rho^2)s - (1-\rho^2)s^2}\right)}{P_{\lambda+1/2}(|x_2|; |\theta_2|) e^{\rho x_2 s}},$$

where $s \in \beta \pm \sqrt{\frac{\boldsymbol{\theta}^\top \mathbf{R}^{-1}\boldsymbol{\theta}}{1-\rho^2}}$.

Remark 2.1. *Let $F_2(\cdot)$ be the df of X_2 for $\boldsymbol{X} = (X_1, X_2)^\top$ defined as in (2). Then, using Lemma 2.1 and Lemma 2.2, we have as $x_2 \to \infty$*

$$1 - F_2(x_2) = \begin{cases} \left(\theta_2/x_2\right)^\lambda (1 + o(1)), & \theta_2 > 0; \\ \left(\widetilde{\lambda}/x_2\right)^{2\lambda}(1 + o(1)), & \theta_2 = 0; \\ \dfrac{\lambda}{2}\dfrac{|\theta_2|^{\lambda-1}}{x_2^{\lambda+1}} e^{-2|\theta_2|x_2}(1 + o(1)), & \theta_2 < 0, \end{cases} \tag{10}$$

with

$$\widetilde{\lambda} = \left(\frac{2^{\lambda-1}\Gamma(\lambda+1/2)}{\sqrt{\pi}}\right)^{1/(2\lambda)}. \tag{11}$$

3. MAIN RESULTS

In this section, we provide the main results on the upper tail dependence coefficient λ_U of two classes of skew slash distributions given by (2) and (3), respectively.

Theorem 3.1. *Let $\boldsymbol{X} \sim SS(\lambda, \boldsymbol{\theta}, \mathbf{R})$ be defined as in (2), and let $T_{2\lambda+1}(\cdot)$ be the student's t distribution function (df) with $2\lambda+1$ degrees of freedom. Then, with $\widetilde{\lambda}$ given by (11)*

(1). for $\theta_1 = \theta_2 = 0$,

$$\lambda_U = 2\left(1 - T_{2\lambda+1}\left(\sqrt{\frac{(2\lambda+1)(1-\rho)}{1+\rho}}\right)\right);$$

(2). for $\theta_1 > 0, \theta_2 > 0,\ \lambda_U = 1$;
(3). for $\theta_1 > 0, \theta_2 = 0$ or $\theta_1 = 0, \theta_2 > 0$,

$$\lambda_U = \int_0^1 \left(1 - \Phi\left(\widetilde{\lambda} u^{1/(2\lambda)}\right)\right) du - \frac{1}{2\lambda+1}\int_0^1 u \ d\left(1 - \Phi\left(\widetilde{\lambda} u^{1/(2\lambda)}\right)\right);$$

(4). for the remaining cases, $\lambda_U = 0$.

From (10) and Theorem 3.1, we see that the skew slash random vector $\boldsymbol{X}$ has asymptotic upper tail dependence provided that both of their marginals posses power laws, i.e., $\theta_1, \theta_2 \geq 0$; see also [13] for the two skew t-distributions. Therefore, regular varying tails play an important role in the presence of tail dependence. Theorem 3.1 shows that tail dependence of the first class of skew slash distributions exists with trivial values 0 or 1, which implies that it has extremal tail behavior (independence and full dependence), contrary to the second class of skew slash distributions showing that the tail dependence has nontrivial values.

In order to state our next theorem, we need to define

$$g_{\theta_1,\theta_2}(z) = f_{2\lambda+1}(z) Q_{\lambda+3/2}\left(\frac{\theta_2\sqrt{\frac{1-\rho^2}{2\lambda+1}}z + \theta_1 + \rho\theta_2}{\sqrt{1+\frac{z^2}{2\lambda+1}}}; \infty\right), \tag{12}$$

with $z \in \mathbb{R}$ and $(\theta_1, \theta_2) \in \mathbb{R}^2$, and

$$h(\mu) = \left(\int_{-\infty}^{\mu} (1+u^2)^{-(\lambda+1)} du\right)^{\frac{1}{2\lambda}}, \quad \mu \in \mathbb{R}, \tag{13}$$

where $f_{2\lambda+1}(\cdot)$ is the probability density function (pdf) of student's t distribution with $2\lambda+1$ degrees of freedom and $Q_{\lambda+3/2}(\cdot; \infty)$ is given by (8).

Theorem 3.2. *Let $\boldsymbol{X} \sim ASS(\lambda, \boldsymbol{\theta}, \mathbf{R})$ be defined as in (3). Then*

$$\lambda_U = \frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)}\left(\frac{1}{Q_{\lambda+1}(\mu_1; \infty)}\int_{z_0'}^{\infty} g_{\theta_1,\theta_2}(z)\ dz + \frac{1}{Q_{\lambda+1}(\mu_2; \infty)}\int_{z_0}^{\infty} g_{\theta_2,\theta_1}(z)\ dz\right),$$

where $Q_{\lambda+3/2}(\cdot; \infty), g_{\theta_1,\theta_2}(\cdot)$ are given by (8) and (12), respectively. And

$$\mu_1 = \frac{\theta_1 + \rho\theta_2}{\sqrt{1+\theta_2^2(1-\rho^2)}}, \quad z_0 = \left(\frac{h(\mu_1)}{h(\mu_2)} - \rho\right)\sqrt{\frac{2\lambda+1}{1-\rho^2}};$$

$$\mu_2 = \frac{\theta_2 + \rho\theta_1}{\sqrt{1+\theta_1^2(1-\rho^2)}}, \quad z_0' = \left(\frac{h(\mu_2)}{h(\mu_1)} - \rho\right)\sqrt{\frac{2\lambda+1}{1-\rho^2}}.$$

4. PROOFS

PROOF OF LEMMA 2.1. First we consider $P_\tau(a; b)$. We will consider the following three cases in turn: (1) $b = 0, a \to \infty$; (2) $b > 0, a \to \infty$ and (3) $b \to 0, ab \to \omega > 0$.

Case (1) as $b = 0$ and $a \to \infty$. Using integration by substitution we have as $a \to \infty$

$$P_\tau(a; 0) = \left(\frac{2}{a^2}\right)^\tau \int_0^{\frac{a^2}{2}} t^{\tau-1}e^{-t}\ dt = \left(\frac{2}{a^2}\right)^\tau \Gamma(\tau)(1 + o(1))$$

since $\int_x^\infty t^{\tau-1}e^{-t}\,dt = x^{\tau-1}e^{-x}(1+o(1))$ as $x \to \infty$, the claim for $P_\tau(a; 0)$ follows.

Case (2) as $b > 0$ and $a \to \infty$. Using $K_\tau(\cdot;\cdot)$ given by (4) we rewrite $P_\tau(a; b)$ as

$$P_\tau(a; b) = 2\left(\frac{b}{a}\right)^\tau \left(K_\tau(0; ab) - K_\tau\left(\frac{a}{b}; ab\right)\right). \tag{14}$$

Noting that

$$K_\tau\left(\frac{a}{b}; ab\right) = \frac{1}{2}\left(\frac{2}{ab}\right)^\tau \int_{\frac{a^2}{2}}^{\infty} t^{\tau-1}\exp\left(-\left(t + \frac{a^2b^2}{4t}\right)\right) dt$$

and

$$\exp\left(-\frac{a^2b^2}{4t}\right) = e^{-\frac{b^2}{2}}\sum_{n=0}^{\infty}\left(\frac{u}{u+1}\right)^n \left(\frac{b^2}{2}\right)^n, \ u = \frac{2t}{a^2} - 1,$$

we have

$$\int_{a^2/2}^{\infty} t^{\tau-1}\exp\left(-\left(t + \frac{a^2b^2}{4t}\right)\right) dt = \left(\frac{a^2}{2}\right)^\tau \exp\left(-\frac{a^2+b^2}{2}\right)\sum_{n=0}^{\infty}\left(\frac{b^2}{2}\right)^n d_n,$$

with

$$d_n = \frac{1}{n!}\int_0^\infty u^n(u+1)^{\tau-n-1}\exp\left(-\frac{a^2}{2}u\right) du =: U(n+1; \tau+1; a^2/2),$$

where U is the confluent hypergeometric function and $U(n+1;\tau+1;a^2/2) = (a^2/2)^{-n-1}(1+o(1))$ as $a \to \infty$ (cf. [9]). Hence,

$$K_\tau\left(\frac{a}{b};ab\right) = \frac{a^{\tau-2}}{b^\tau}\exp\left(-\frac{a^2+b^2}{2}\right)(1+o(1)).$$

This together with (5) yields that

$$\begin{aligned}&\frac{K_\tau\left(\frac{a}{b};ab\right)}{K_\tau(0;ab)}\\ &= \sqrt{\frac{2}{\pi}}\frac{a^{\tau-3/2}}{b^{\tau-1/2}}\exp\left(ab-\frac{a^2+b^2}{2}\right)(1+o(1)),\end{aligned} \tag{15}$$

which tends to zero as $a \to \infty$. Consequently, the claim for $P_\tau(a;b)$ as $b>0, a\to\infty$ follows.

Case (3) as $b \to 0, a \to \infty$ and $ab \to w > 0$. The proof is similar to that of Case (2), and thus the details are omitted here.

Next, we consider $Q_\upsilon(x;a)$. Note that for all $x \in \mathbb{R}$

$$\begin{aligned}Q_\upsilon(x;a) &= \Gamma(\upsilon)\int_{-\infty}^{x}(1+u^2)^{-\upsilon}\,du\\ &\quad - \int_{-\infty}^{x}\int_{a}^{\infty}t^{\upsilon-1}e^{-(1+u^2)t}\,dtdu.\end{aligned} \tag{16}$$

Further, recalling that $\upsilon \ge 1$ we have

$$\begin{aligned}0 &\le \int_{-\infty}^{x}\int_{a}^{\infty}t^{\upsilon-1}e^{-(1+u^2)t}\,dtdu\\ &\le \left(\int_{-\infty}^{x}(1+u^2)^{-\upsilon}\,du\right)\left(\int_{a}^{\infty}t^{\upsilon-1}e^{-t}\,dt\right),\end{aligned}$$

which tends to 0 as $a \to \infty$. It follows thus that

$$\int_{-\infty}^{x}\int_{a}^{\infty}t^{\upsilon-1}e^{-(1+u^2)t}\,dtdu \to 0, \quad a\to\infty.$$

Therefore, for all $x \in \mathbb{R}$ and $\upsilon \ge 1$

$$Q_\upsilon(x;a) \to \Gamma(\upsilon)\int_{-\infty}^{x}(1+u^2)^{-\upsilon}\,du \quad \text{as } a\to\infty.$$

The proof is complete. □

PROOF OF LEMMA 2.2. Recall that $\boldsymbol{X}|V = t \sim N_2(\boldsymbol{\theta}/t, \mathbf{R}/t)$ with $t \in (0,1)$ given. It follows from the total probability formula that, the pdf of $\boldsymbol{X}$ defined as in (2), denoted by $f_{\boldsymbol{X}}(\cdot)$, is

$$\begin{aligned}f_{\boldsymbol{X}}(\boldsymbol{x}) &= \frac{\lambda e^{\boldsymbol{\theta}^\top\mathbf{R}^{-1}\boldsymbol{x}}}{2\pi\sqrt{1-\rho^2}}\int_0^1 t^{\lambda+1-1}\\ &\quad\times\exp\left(-\frac{1}{2}\left(\boldsymbol{x}^\top\mathbf{R}^{-1}\boldsymbol{x}t+\frac{\boldsymbol{\theta}^\top\mathbf{R}^{-1}\boldsymbol{\theta}}{t}\right)\right)dt,\end{aligned}$$

with $\boldsymbol{x} = (x_1,x_2)^\top \in \mathbb{R}^2$. Hence, the pdf of X_2, denoted by $f_2(\cdot)$, satisfies

$$f_2(x_2) = \frac{\lambda e^{\theta_2 x_2}}{\sqrt{2\pi}}\int_0^1 t^{\lambda+1/2-1}\exp\left(-\frac{1}{2}\left(x_2^2 t+\frac{\theta_2^2}{t}\right)\right)dt.$$

Consequently, the conditional density of $X_{1.2} := X_1|X_2 = x_2$, denoted by $f_{1.2}(\cdot|x_2)$, is

$$\begin{aligned}f_{1.2}(x_1|x_2) &= \frac{f_{\boldsymbol{X}}(\boldsymbol{x})}{f_2(x_2)}\\ &= \frac{e^{\beta(x_1-\rho x_2)}}{\sqrt{2\pi(1-\rho^2)}}\frac{P_{\lambda+1}\left(\sqrt{x_1'^2+x_2^2};\sqrt{\theta_1'^2+\theta_2^2}\right)}{P_{\lambda+1/2}(|x_2|;|\theta_2|)},\end{aligned}$$

with $\beta(1-\rho^2) = \theta_1-\rho\theta_2, x_1'\sqrt{1-\rho^2} = x_1-\rho x_2, \theta_1'\sqrt{1-\rho^2} = \theta_1-\rho\theta_2$. Therefore, we have with $s' = \sqrt{1-\rho^2}s$

$$\mathbb{E}e^{-sX_{1.2}} = e^{-\rho x_2 s}\mathbb{E}e^{-s'(X_{1.2}-\rho x_2)/\sqrt{1-\rho^2}}$$

and

$$\mathbb{E}e^{-s'(X_{1.2}-\rho x_2)/\sqrt{1-\rho^2}} = \frac{P_{\lambda+1/2}(|x_2|;|\theta_2'|)}{P_{\lambda+1/2}(|x_2|;|\theta_2|)},$$

with $\theta_2'^2 = \theta_1'^2+\theta_2^2-(\theta_1'-s')^2$ and s' satisfying $\theta_1'^2+\theta_2^2-(\theta_1'-s')^2>0$, i.e.,

$$\theta_2'^2 = \theta_2^2+2\beta(1-\rho^2)s-(1-\rho^2)s^2, \quad s\in\beta\pm\sqrt{\frac{\theta_1'^2+\theta_2^2}{1-\rho^2}}.$$

The proof is complete. □

PROOF OF THEOREM 3.1. For $\boldsymbol{\theta} = \mathbf{0}$, the skew slash random variable $\boldsymbol{X}$ is symmetry and has the same marginal distributions with regular varying tail index 2λ (see (10)), and thus the claim follows by Theorem 1 (i) in [1] (see also [16] for the multivariate copula extensions). Next, we derive the remaining cases, i.e., $\boldsymbol{\theta} \ne \mathbf{0}$.

To this end, we need to derive the asymptotic distribution of $W(x_2)$ where

$$W(x_2) = x_2^{-1/2}\left(X_{1.2}-\left(\rho x_2+\frac{\beta(1-\rho^2)}{|\theta_2|}\sqrt{x_2^2+2\lambda}\right)\right)$$

for $\theta_2 \ne 0$; otherwise $x_2^{-2}X_{1.2}$. It follows from Lemma 2.2 that $\mathbb{E}e^{-sW(x_2)}$ as $\theta_2 \ne 0$ equals

$$\begin{aligned}&\exp\left(\frac{\rho x_2+\frac{\beta(1-\rho^2)}{|\theta_2|}\sqrt{x_2^2+2\lambda}}{\sqrt{x_2}}s\right)\mathbb{E}\exp\left(-\frac{s}{\sqrt{x_2}}X_{1.2}\right)\\ &= \exp\left(\frac{\beta(1-\rho^2)}{|\theta_2|}\sqrt{x_2+\frac{2\lambda}{x_2}}s\right)\frac{P_{\lambda+1/2}\left(|x_2|;\sqrt{\tilde{g}(x_2)}\right)}{P_{\lambda+1/2}(|x_2|;|\theta_2|)},\end{aligned}$$

which, in view of Lemma 2.1, is asymptotically equal to

$$\exp\left(\frac{\beta(1-\rho^2)}{|\theta_2|}\sqrt{x_2+\frac{2\lambda}{x_2}s}\right)\left(\frac{\tilde g(x_2)}{\theta_2^2}\right)^{\frac{\lambda}{2}}$$
$$\times\exp\left(|\theta_2|x_2-\sqrt{\tilde g(x_2)}x_2\right)$$
$$=\exp\left(\frac{\beta(1-\rho^2)}{|\theta_2|}\sqrt{x_2+\frac{2\lambda}{x_2}s}\right)$$
$$\times\exp\left(|\theta_2|x_2\left[1-\left(1+\frac{2\beta(1-\rho^2)s}{\theta_2^2\sqrt{x_2}}-\frac{(1-\rho^2)s^2}{\theta_2^2x_2}x_2\right)^{\frac{1}{2}}\right]\right)$$
$$\sim\exp\left(-\frac{(1-\rho^2)(\theta_1'^2+\theta_2^2)}{2|\theta_2|^3}s^2+O\left(\frac{1}{\sqrt{x_2}}\right)\right)$$
$$\to\exp\left(-\frac{(1-\rho^2)(\theta_1'^2+\theta_2^2)}{2|\theta_2|^3}s^2\right),\quad x_2\to\infty,$$

where $\tilde g(x_2)=\theta_2^2+\frac{2\beta(1-\rho^2)s}{\sqrt{x_2}}-\frac{(1-\rho^2)s^2}{x_2}$ and $\theta_1'=(\theta_1-\rho\theta_2)/\sqrt{1-\rho^2}$. Therefore, by the Laplace inverse transform, we have the following convergence in distribution (denoted by $\stackrel{d}{\to}$)

$$W(x_2)\stackrel{d}{\to}Z_1\sim N\left(0,\frac{(1-\rho^2)(\theta_1'^2+\theta_2^2)}{|\theta_2|^3}\right)\tag{17}$$

as $x_2\to\infty$. For $\theta_2=0$, and thus $\theta_1\neq 0$. It follows from Lemma 2.1 and Lemma 2.2 that as $x_2\to\infty$

$$\mathbb{E}e^{-sW(x_2)}\to\frac{2(\sqrt{2\theta_1 s})^{\lambda+1/2}K_{\lambda+1/2}(0;\sqrt{2\theta_1 s})}{2^{\lambda+1/2}\Gamma(\lambda+1/2)},$$

which is the Laplace transform of θ_1/Y where $Y\sim\Gamma(1/2+\lambda,1/2)$, a Gamma distributed random variable with parameters $1/2+\lambda,1/2$. Therefore

$$W(x_2)\stackrel{d}{\to}\frac{\theta_1}{Y},\quad x_2\to\infty.\tag{18}$$

Further, we need the asymptotic expression of the function $c(x_2)=F_1^{-1}(F_2(x_2))$. In view of Lemma 3.1 in [5] we have for $\theta_1>0$ with $t(\theta_1,\theta_2)=((2|\theta_2|)/(\lambda+1))^{1/\lambda}\theta_1/|\theta_2|$

$$c(x_2)=\begin{cases}\frac{\theta_1}{\theta_2}x_2(1+o(1)), & \theta_2>0;\\ \frac{\theta_1}{\widetilde\lambda^2}x_2^2(1+o(1)), & \theta_2=0;\\ t(\theta_1,\theta_2)x_2^{1+\frac{1}{\lambda}}\exp\left(\frac{2|\theta_2|x_2}{\lambda}\right)(1+o(1)), & \theta_2<0\end{cases}\tag{19}$$

as $x_2\to\infty$, where $\widetilde\lambda$ is given by (11).

Next, we give the proofs of assertions (2)–(4).

Assertion (2) as $\theta_1>0,\theta_2>0$. Using (17), (19) and $\beta(1-\rho^2)=\theta_1-\rho\theta_2$, we have

$$\lim_{x_2\to\infty}\mathbb{P}\left(X_1\ge F_1^{-1}(F_2(x_2))|X_2=x_2\right)$$
$$=\lim_{x_2\to\infty}\mathbb{P}\left(W(x_2)\ge\frac{c(x_2)-\left(\rho x_2+\frac{\beta(1-\rho^2)}{|\theta_2|}\sqrt{x_2^2+2\lambda}\right)}{\sqrt{x_2}}\right)$$
$$=\mathbb{P}(Z_1\ge 0)=1/2.$$

Similarly, $\lim_{x_1\to\infty}\mathbb{P}\left(X_2\ge F_2^{-1}(F_1(x_1))|X_1=x_1\right)=1/2$. Therefore, in view of (9), we have

$$\lambda_U=\lim_{x_2\to\infty}\mathbb{P}\left(X_1\ge F_1^{-1}(F_2(x_2))|X_2=x_2\right)+\lim_{x_1\to\infty}\mathbb{P}\left(X_2\ge F_2^{-1}(F_1(x_1))|X_1=x_1\right)=1.$$

Assertion (3) as $\theta_1>0,\theta_2=0$ and $\theta_1=0,\theta_2>0$. For this, we only present the proof of $\theta_1>0,\theta_2=0$ since another case follows by the similar arguments. Using (18) and (19) we have

$$\lim_{x_2\to\infty}\mathbb{P}\left(X_1\ge F_1^{-1}(F_2(x_2))|X_2=x_2\right)$$
$$=\lim_{x_2\to\infty}\mathbb{P}\left(W(x_2)\ge\frac{c(x_2)}{x_2^2}\right)=\mathbb{P}\left(Y\le\widetilde\lambda^2\right),$$

where $Y\sim\Gamma(1/2+\lambda,1/2)$ and $\widetilde\lambda$ is defined by (11). Similarly

$$\lim_{x_1\to\infty}\mathbb{P}\left(X_2\ge F_2^{-1}(F_1(x_1))|X_1=x_1\right)$$
$$=\lim_{x_1\to\infty}\mathbb{P}\left(\frac{X_{2.1}-\left(\rho x_1+\frac{\beta'(1-\rho^2)}{\theta_1}\sqrt{x_1^2+2\lambda}\right)}{\sqrt{x_1}}\ge\frac{\widetilde\lambda\sqrt{\frac{x_1}{\theta_1}}-\left(\rho x_1+\frac{\beta'(1-\rho^2)}{\theta_1}\sqrt{x_1^2+2\lambda}\right)}{\sqrt{x_1}}\right)$$
$$=\mathbb{P}\left(Z_1'\ge\frac{\widetilde\lambda}{\sqrt{\theta_1}}\right),$$

where

$$\beta'(1-\rho^2)=\theta_2-\rho\theta_1,\quad Z_1'\sim N\left(0,\frac{(1-\rho^2)\boldsymbol{\theta}^\top\mathbf{R^{-1}}\boldsymbol{\theta}}{\theta_1^3}\right).\tag{20}$$

Therefore, using integration by parts, we have

$$\lambda_U=1-\Phi(\widetilde\lambda)+\mathbb{P}\left(Y\le\tilde\lambda^2\right)=\int_0^1\left(1-\Phi(\widetilde\lambda u^{1/(2\lambda)})\right)du-\frac{1}{2\lambda+1}\int_0^1 u\,d\left(1-\Phi(\widetilde\lambda u^{1/(2\lambda)})\right).$$

Assertion (4) as $\theta_1\theta_2<0$ and $\theta_1<0,\theta_2<0$. Here, we only present the proof of $\theta_1>0,\theta_2<0$. The other cases follow by similar arguments and thus are omitted here. Using (17), (19) and $x_2^{-1}c(x_2)\to\infty$, we have

$$\lim_{x_2\to\infty}\mathbb{P}\left(X_1\ge F_1^{-1}(F_2(x_2))|X_2=x_2\right)$$
$$=\lim_{x_1\to\infty}\mathbb{P}\left(X_2\ge F_2^{-1}(F_1(x_1))|X_1=x_1\right)=0.$$

Consequently, $\lambda_U=0$ for $\theta_1>0,\theta_2<0$. The proof is complete. □

Proof of Theorem 3.2. Note that $\boldsymbol{X}|V=t$ is skew normal distributed with pdf $2\phi_2(\boldsymbol{x};\mathbf{R}/t)\Phi(\sqrt{t}\boldsymbol{\theta}^\top\boldsymbol{x})$ with $t\in(0,1)$ given. It follows from the total probability formula that the pdf of $\boldsymbol{X}$, denoted by $f_{\boldsymbol{X}}(\boldsymbol{x}),\boldsymbol{x}\ne 0$, is

(21)

$$f_{\boldsymbol{X}}(\boldsymbol{x})=\frac{2\lambda}{(2\pi)^{3/2}|\mathbf{R}|^{1/2}}\int_0^1\int_{-\infty}^{\boldsymbol{\theta}^\top\boldsymbol{x}}t^{\lambda+1/2}$$
$$\times\exp\left(-\frac{\boldsymbol{x}^\top\mathbf{R}^{-1}\boldsymbol{x}+u^2}{2}t\right)dudt$$
$$=\frac{2\lambda}{(2\pi)^{3/2}|\mathbf{R}|^{1/2}}\frac{2^{\lambda+3/2}}{(\boldsymbol{x}^\top\mathbf{R}^{-1}\boldsymbol{x})^{\lambda+1}}$$
$$\times\int_{-\infty}^{\frac{\boldsymbol{\theta}^\top\boldsymbol{x}}{\sqrt{\boldsymbol{x}^\top\mathbf{R}^{-1}\boldsymbol{x}}}}\int_0^{\frac{\boldsymbol{x}^\top\mathbf{R}^{-1}\boldsymbol{x}}{2}}t^{\lambda+1/2}\exp\left(-(1+u^2)t\right)dtdu$$
$$=\frac{2^{\lambda+1}\lambda}{(\pi^3|\mathbf{R}|)^{\frac12}(\boldsymbol{x}^\top\mathbf{R}^{-1}\boldsymbol{x})^{\lambda+1}}Q_{\lambda+\frac32}\left(\frac{\boldsymbol{\theta}^\top\boldsymbol{x}}{\sqrt{\boldsymbol{x}^\top\mathbf{R}^{-1}\boldsymbol{x}}};\frac{\boldsymbol{x}^\top\mathbf{R}^{-1}\boldsymbol{x}}{2}\right).$$

Consequently, the pdf of X_i, denoted by $f_i(\cdot)$, is given by

$$f_i(x)=\frac{\lambda}{\pi}\frac{2^{\lambda+1}}{|x|^{2\lambda+1}}Q_{\lambda+1}\left(\mu_i\mathrm{sign}(x);x^2/2\right),\quad i=1,2,\tag{22}$$

with

$$\mu_1=\frac{\theta_1+\rho\theta_2}{\sqrt{1+\theta_2^2(1-\rho^2)}},\quad \mu_2=\frac{\theta_2+\rho\theta_1}{\sqrt{1+\theta_1^2(1-\rho^2)}}.$$

Hence we have by Lemma 2.1 (recall $h(\cdot)$ given by (13))

$$1-F_2(x_2)\sim\frac{x_2}{2\lambda}f_2(x_2)\sim\frac{\Gamma(\lambda+1)}{\pi}\frac{2^\lambda}{x_2^{2\lambda}}h(\mu_2)$$

as $x_2\to\infty$. Consequently, as $x_2\to\infty$

$$c(x_2)=F_1^{-1}(F_2(x_2))=\frac{h(\mu_1)}{h(\mu_2)}x_2(1+o(1))\tag{23}$$

and the pdf of $X_1|X_2=x_2$, denoted by $f_{1.2}(\cdot|x_2)$, satisfies

$$f_{1.2}(x_1|x_2)=\frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)}\frac{f_{2\lambda+1}\left((x_1-\rho x_2)/s(x_2)\right)}{s(x_2)}$$
$$\times\frac{Q_{\lambda+3/2}\left(\frac{\theta_1x_1+\theta_2x_2}{\sqrt{x_1'^2+x_2^2}};\frac{x_1'^2+x_2^2}{2}\right)}{Q_{\lambda+1}\left(\mu_2\mathrm{sign}(x_2);\frac{x_2^2}{2}\right)},$$

where $f_{2\lambda+1}(\cdot)$ is the pdf of student's t with $2\lambda+1$ degrees of freedom and

$$x_1'\sqrt{1-\rho^2}=x_1-\rho x_2,\quad s(x_2)=\sqrt{\frac{(1-\rho^2)x_2^2}{2\lambda+1}}.$$

It thus follows by the dominated convergence theorem and Lemma 2.1 that

$$\lim_{x_2\to\infty}\mathbb{P}\left(X_1\ge F_1^{-1}(F_2(x_2))|X_2=x_2\right)$$
$$=\frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)}\frac{1}{Q_{\lambda+1}(\mu_2;\infty)}\int_{z_0}^\infty g_{\theta_2,\theta_1}(z)\,dz,\tag{24}$$

where $g_{\theta_2,\theta_1}(\cdot)$ is given by (12) and

$$z_0=\lim_{x_2\to\infty}\frac{c(x_2)-\rho x_2}{s(x_2)}=\left(\frac{h(\mu_1)}{h(\mu_2)}-\rho\right)\sqrt{\frac{2\lambda+1}{1-\rho^2}}.$$

Similarly

$$\lim_{x_1\to\infty}\mathbb{P}\left(X_2\ge F_2^{-1}(F_1(x_1))|X_1=x_1\right)$$
$$=\frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)}\frac{1}{Q_{\lambda+1}(\mu_1;\infty)}\int_{z_0'}^\infty g_{\theta_1,\theta_2}(z)\,dz,\tag{25}$$

with

$$z_0'=\left(\frac{h(\mu_2)}{h(\mu_1)}-\rho\right)\sqrt{\frac{2\lambda+1}{1-\rho^2}}.$$

The desired result follows by (24) and (25). □

ACKNOWLEDGEMENTS

We are grateful to the Editor-in-Chief (Professor. Dr. Yazhen, Wang) and the referees for their careful reading and helpful suggestions which greatly improved the paper.

Received 19 September 2013

REFERENCES

[1] Abdous, B., Anne-Laure Fougeres, A. L. and Ghoudi, K. (2005). Extreme behaviour for bivariate elliptical distributions. *Canad. J. Statist.* **33** 317–334. MR2193978

[2] Arslan, O. (2008). An alternative multivariate skew-slash distribution. *Statist. Probab. Lett.* **78** 2756–2761. MR2465119

[3] Arslan, O. (2009). Maximum likelihood parameter estimation for the multivariate skew-slash distribution. *Statist. Probab. Lett.* **79** 2158–2165. MR2572047

[4] Azzalini, A. and Dalla Valle, A. (1996). The multivariate skew-normal distribution. *Biometrika.* **83** 715–726. MR1440039

[5] Banachewicz, K. and van der Vaart, A. (2008). Tail dependence of skewed grouped t-distributions. *Statist. Probab. Lett.* **78** 2388–2399. MR2462674

[6] Bortot, P. (2010). Tail dependence in bivariate skew-normal and skew-t distributions. (http://www2.stat.unibo.it/bortot/ricerca/paper-sn-2.pdf).

[7] Cabral, C. R. B., Lachos, V. H. and Prates, M. O. (2012). Multivariate mixture modeling using skew-normal independent distributions. *Comput. Statist. Data Anal.* **56** 126–142. MR2833042

[8] Cabral, C. R. B., Lachos, V. H. and Zeller, C. (2014) Multivariate measurement error models using finite mixtures of skew-Student t distributions. *J. Multivariate Anal.* **124** 179–198. MR3147319

[9] Chaudhry, M. A., Temme, N. M. and Veling, E. J. M. (1996). Asymptotics and closed form of a generalized incomplete gamma function. *J. Comput. Appl. Math.* **67** 371–379. MR1390191

[10] Demarta, S. and McNeil, A. J. (2005). The t copula and related copulas. *Internat. Statist. Rev.* **73** 111–129.

[11] Durante, F. (2013). *Copulas, tail dependence and applications to the analysis of financial time series.* In Bustince, H., Fernandez, J., Mesiar, R. and Calvo, T. (Eds.), Aggregation Functions in Theory and in Practice, pages 17–22. *Springer, New York.*

[12] Embrechts, P., McNeil, A. and Straumann, D. (2002). *Correlation and dependence in risk management: properties and pitfalls.* In Dempster, M. H. A. (editor), Risk management: Value at Risk and Beyond, pages 176–233. *Cambridge University Press, Cambridge.* MR1892190

[13] Fung, T. and Seneta, E. (2010). Tail dependence for two skew t distributions. *Stat. Probab. Lett.* **80** 784–791. MR2608817

[14] Fung, T. and Seneta, E. (2011). Tail dependence and skew distributions. *Quant. Finance.* **11** 327–333. MR2777077

[15] Genç, A. I. (2013). A skew extension of the slash distribution via beta-normal distribution. *Statist. Papers.* **54** 427–442. MR3043298

[16] Hashorva, E. and Jaworski, P. (2012). Gaussian approximation of conditional elliptical copulas. *J. Multivariate Anal.* **111** 397–407. MR2944429

[17] Heffernan, J. E. (2000). A directory of coefficients of tail dependence. *Extremes.* **3** 279–390. MR1856201

[18] Jones, D. S. (2007). Incomplete Bessel function I. *Proc. Edinb. Math. Soc. (2).* **50** 173–183. MR2294011

[19] Kim, H. M. and Genton, M. G. (2011). Characteristic functions of scale mixtures of multivariate skew-normal distributions. *J. Multivariate Anal.* **102** 1105–1117. MR2805651

[20] Kjersti, A. and Ingrid, H. H. (2006). The generalized hyperbolic skew Student's t-distribution. *J. Financ. Econom.* **4** 275–309.

[21] Lachos, V. H., Ghosh, P. and Arellano-Valle, R. B. (2010). Likelihood based inference for skew-normal independent linear mixed models. *Statist. Sinica.* **20** 303–322. MR2640696

[22] Lachos, V. H., Labra, F. V., Bolfarine, H. and Ghosh, P. (2010). Multivariate measurement error models based on scale mixtures of the skew-normal distribution. *Statistics.* **44** 541–556. MR2739411

[23] Nelsen, R. B. (1999). *An Introduction to Copulas. Lecture Notes in Statistics.* **139** Springer Verlag. MR1653203

[24] Padoan, S. A. (2011). Multivariate extreme models based on underlying skew t and skew-normal distributions. *J. Multivariate Anal.* **102** 977–991. MR2776583

[25] Punathumparambath, B. and Gorge, S. (2013). A new family of skewed slash logistic distributions. *J. Appl. Statist. Sci.* **19** 253–260. MR3077501

[26] Schmidt, R. (2005). Tail dependence. In Čížek, P., Hädle, W. and Weron, R. (Eds.), Statistical Tools for Finance and Insurance, pages 65–91. *Springer, New York.* MR2156760

[27] Sibuya, M. (1960). Bivariate extreme statistics, I. *Ann. Inst. Statist. Math. Tokyo.* **11** 195–210. MR0115241

[28] Wang, J. and Genton, M. (2006). The multivariate skew-slash distribution. *J. Statist. Plann. Inference.* **136** 209–220. MR2207182

[29] Wu, J., Zhang, Z. and Zhao, Y. (2012). Study of the tail dependence structure in global financial markets using extreme value theory. *J. Reviews on Global Economics.* **1** 62–81.

[30] Zareifard, H. and Khaledi, M. J. (2013). Empirical Bayes estimation in regression models with generalized skew-slash errors. *Comm. Statist. Theory Methods.* **42** 1105–1123. MR3028997

[31] Zhang, Z. (2008). Quotient correlation: a sample based alternative to Pearson's correlation. *Ann. Statist.* **36** 1007–1030. MR2396823

Chengxiu Ling
Department of Actuarial Science
University of Lausanne
UNIL-Dorigny, 1015 Lausanne
Switzerland
E-mail address: chengxiu.ling@unil.ch

Zuoxiang Peng
School of Mathematics and Statistics
Southwest University
Beibei, 400715 Chongqing
China
E-mail address: pzx@swu.edu.cn