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Functional regular variations, Pareto processes and peaks over threshold

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History: The latest developments of extreme value theory focus on the functional framework and much effort has been put in the theory of max-stable processes and functional regular variations. Paralleling the univariate extreme value theory, this work focuses on the exceedances of a stochastic process above a high threshold and their connections with generalized Pareto processes. More precisely we define an exceedance through a homogeneous cost functional ℓ and show that the limiting (rescaled) distribution is a ℓ–Pareto process whose spectral measure can be characterized. Three equivalent characterizations of the ℓ–Pareto process are given using either a constructive approach, either a homogeneity property or a peak over threshold stability property. We also provide non parametric estimators of the spectral measure and give some examples.

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1. INTRODUCTION

Balkema and de Haan [4] and Pickands [17] have made a major contribution to the extreme value theory with the introduction of the generalized Pareto distribution and its connection with exceedances above a large threshold. They established that the linearly normalized maximum of independent random variables converges to an extreme value distribution if and only if the normalized exceedance above a threshold converges to a generalized Pareto distribution. For statistical purposes, the use of peaks over threshold rather than block maxima is often more convenient since it usually wastes less observations. Extensions to the multivariate case have been proposed by Rootzen and Tajvidi [19] and Falk et al. [11].

More recently, the infinite dimensional setting, i.e., the functional framework and continuous random processes, enjoyed renewed interests. The generalized Pareto processes, also known as GPD processes or functional generalized Pareto distributions, have been introduced by Buishand et al. [5], Aulbach et al. [3] and de Haan and Ferreira [12]. Similarly to the finite dimensional case, the domain of attraction of a generalized Pareto process and that of the associated max-stable process coincide. Several equivalent characterizations of Pareto processes are given, including the peak over threshold stability and a homogeneity property. Statistical issues such as local asymptotic normality or tests for the class of generalized Pareto processes are addressed in Aulbach and Falk [1, 2].

Often exceedances above a high threshold can be defined through a uniform supremum. More precisely, a peak over threshold of a stochastic process $\{X(t)\}_{t\in T}$ can be defined by

$$\sup_{t\in T}\frac{X(t)-b_n(t)}{a_n(t)}>0$$

where $\{a_n>0\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are normalizing functions. Since we restrict our attention to the tails of the process, it is sensible to have

$$\mathbb{P}\left\{\sup_{t\in T}\frac{X(t)-b_n(t)}{a_n(t)}>0\right\}\longrightarrow 0,\qquad n\to\infty.$$

Theorem 3.2 in de Haan and Ferreira [12] states that if there exists continuous normalizing functions $\{a_n>0\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ such that

$$\frac{X-b_n}{a_n}\;\Big|\;\left\{\sup_{t\in T}\frac{X(t)-b_n(t)}{a_n(t)}>0\right\}$$

converges weakly in the space of continuous functions as $n\to\infty$, then the limit must be a generalized Pareto process. In particular when X is nonnegative and $a_n=b_n>0$ are constants, we have

$$\sup_{t\in T}\frac{X(t)-b_n}{a_n}>0\quad\text{if and only if}\quad\|X\|>a_n$$

where $\|\cdot\|$ denotes the uniform norm.

Although from a theoretical point of view the use of a uniform supremum seems sensible when working with the space of continuous functions, for practical purposes other kinds of thresholds might be relevant. More generally, an exceedance over a threshold can be defined as an event $\{\ell(X)>a_n\}$, for some functional ℓ and where the threshold a_n is such that

$$\mathbb{P}\{\ell(X)>a_n\}\longrightarrow 0,\qquad n\to\infty.$$

For example, Buishand et al. [5] were interested in the total amount of rain over a catchment T, i.e., $\int_T X(t)\mathrm{d}t$ where

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$X(t)$ represents the amount of rain at $t \in T$. In this context, it seems appropriate to let $\ell(x) = \int_T x(t)\mathrm{d}t$ and to derive the limiting distribution of

$$a_n^{-1} X \mid \{\ell(X) > a_n\}, \qquad n \to \infty.$$

Other possibilities are $\ell_1(x) = \int_T x(t)^2 \mathrm{d}t$, $\ell_2(x) = \inf_{t\in T} x(t)$ or $\ell_3(x) = x(t_0)$. The choice ℓ_1 is natural in the context of an energy functional, for example if X stands for the strength of the wind in space. A high threshold with respect to ℓ_2 occurs when the random field takes large values at any point $t \in T$ and might be relevant for modeling sea levels along a dike. The use of ℓ_3 puts the emphasis on a specific point $t_0 \in T$ and might be of interest for modeling extreme flows at the confluence of two rivers. We may see these functionals as *cost functionals* and for our purposes we will restrict our attention to the class of measurable nonnegative homogeneous functional $\ell : \mathcal{C} \to [0, +\infty)$, with $\mathcal{C} = \mathcal{C}\{T, [0, +\infty)\}$ the space of nonnegative continuous functions over a compact parameter set T. All the previous examples belong to this class. A functional ℓ is said to be homogeneous of order $\beta > 0$ if

$$\ell(ux) = u^\beta \ell(x), \qquad \text{for all } u > 0,\ x \in \mathcal{C}.$$

Without loss of generality, we can assume that $\beta = 1$. Indeed since the functional ℓ is nonnegative, the functional $\tilde{\ell} = \ell^{1/\beta}$ is clearly homogeneous of order 1 and satisfies

$$\{\ell(X) > a_n\} = \{\tilde{\ell}(X) > a_n^{1/\beta}\}.$$

The paper is organized as follows. The background on functional extreme value theory is introduced in Section 2. Section 3 is devoted to the ℓ-Pareto processes and their properties in connection to exceedances over high thresholds. Non parametric estimators of the spectral measure are introduced in Section 4 and their consistency is shown. Section 4 deals with some examples related to Brown-Resnick processes and the simulation of Pareto processes is discussed.

2. FUNCTIONAL EXTREME VALUE THEORY

The theory of continuous max-stable processes was initiated by de Haan [7], de Haan and Pickands [10] and de Haan and Lin [9]. Connections with functional regular variations are well known, see e.g. Hult and Lindskog [13, 14] or Davis and Mikosch [6]. For a background on functional extreme value theory, we refer to the monograph de Haan and Ferreira [8]. Note that for the sake of simplicity, we will only consider in this paper standardized (or simple) processes, i.e. processes whose marginal distributions are in the domain of attraction of a Fréchet distribution with index $\alpha > 0$.

For the convenience of the reader, we start with some standard results on univariate extreme value theory. Let $\alpha > 0$ and X, $\{X_i\}_{i\geq 1}$ be i.i.d. positive random variables with common distribution function F. For $u > 1$, we note $a(u) = F^{\leftarrow}(1 - 1/u)$, where $F^{\leftarrow}$ denotes the quantile function. It is well known that the following statements are equivalent:

1. the tail function $1 - F$ is regularly varying at infinity with index $-\alpha$;
2. $u\mathbb{P}\{X/a(u) \in \cdot\} \xrightarrow{v} \lambda_\alpha(\cdot)$ as $u \to +\infty$, where $\lambda_\alpha(dr) = \alpha r^{-\alpha-1}\mathrm{d}r$ and $\xrightarrow{v}$ stands for vague convergence in the space $M\{(0, +\infty]\}$ of Radon measures on $(0, +\infty]$;
3. the normalized sample point process $\sum_{i=1}^n \delta_{X_i/a(n)}$ converges weakly in $M\{(0, +\infty]\}$ as $n \to +\infty$ to a Poisson point process on $(0, +\infty]$ with intensity λ_α;
4. the normalized maximum $\max(X_1, \dots, X_n)/a(n)$ converges in distribution as $n \to +\infty$ to an α-Fréchet distribution;
5. the distribution $\mathbb{P}\{X/u \in \cdot \mid X > u\}$ of normalized exceedances over high threshold converges in distribution as $u \to +\infty$ to a Pareto distribution with index α.

These equivalence are well known but rarely presented all at the same time. The reader shall refer to Resnick [18] Proposition 1.11 and 3.21 for the equivalence betwen 1, 2, 3 and 4. The equivalence between 4 and 5 follows from Balkema and de Haan [4].

We explain how this can be generalized to the functional setting and introduce first the notion of functional regular variation. Let T be a compact metric space and $\mathcal{C} = \mathcal{C}\{T, [0, +\infty)\}$ the Banach space of nonnegative continuous functions $x\colon T \to [0, +\infty)$ endowed with the uniform norm $\|x\| = \sup_{t\in T} |x(t)|$. Let $\mathcal{C}_0 = \mathcal{C} \setminus \{0\}$ and $\mathcal{S} = \{x \in \mathcal{C}; \|x\| = 1\}$ be the unit sphere. Given any metric space $\mathcal{X}$, we denote by $\mathcal{B}(\mathcal{X})$ its Borel σ-algebra.

Definition 1. *A random process X with sample path in $\mathcal{C}_0$ is said to be regularly varying with exponent $\alpha > 0$ and spectral probability measure σ on $\mathcal{S}$, noted shortly $X \in RV_{\alpha,\sigma}(\mathcal{C}_0)$, if there exists a positive function $a(\cdot)$ such that $a(u) \to +\infty$ as $u \to +\infty$ and*

$$u\mathbb{P}\{X/\|X\| \in B,\ \|X\| > ra(u)\} \longrightarrow \sigma(B) r^{-\alpha} \tag{1}$$

for all $r > 0$ and all $B \in \mathcal{B}(\mathcal{S})$ such that $\sigma(\partial B) = 0$, where ∂B denotes the boundary of B.

The exponent α and the spectral measure σ are uniquely determined while the function $a(\cdot)$ is unique up to asymptotic equivalence and regularly varying at infinity with exponent $1/\alpha$. Similarly to the univariate case, a convenient choice is

$$a(u) = \inf\{x \geq 0\colon \mathbb{P}(\|X\| \leq x) \leq 1 - 1/u\}, \tag{2}$$

and in the remainder of this paper we will always assume this choice.

We now introduce some technical backgrounds on function and measure spaces that are useful when using point processes. A first step is to introduce a suitable modification of the space $\mathcal{C}$ in order to deal with points at infinity. In the univariate case, this is done by working with the space $(0,+\infty]$ instead of $(0,+\infty)$. In the functional framework, the polar decomposition $\mathcal{C}_0 \to (0,+\infty)\times\mathcal{S}$ given by $x \mapsto (\|x\|, x/\|x\|)$ is bijective and bi-continuous and allows to identify $\mathcal{C}_0$ and $(0,+\infty)\times\mathcal{S}$. We consider the complete separable metric space $\overline{\mathcal{C}}_0 = (0,+\infty]\times\mathcal{S}$ equipped with the metric

$$d\{(r_1,s_1),(r_2,s_2)\} = |1/r_1 - 1/r_2| + \|s_1 - s_2\|.$$

A set B is bounded in $\overline{\mathcal{C}}_0$ if and only if there exists some $\varepsilon > 0$ such that $B \subset [\varepsilon,+\infty]\times\mathcal{S}$ (see e.g. Davis and Mikosch [6] for more details).

Definition 2. *Let $M(\overline{\mathcal{C}}_0)$ be the set of Borel measures m on $\overline{\mathcal{C}}_0$ that are boundedly finite, i.e., such that $m(B) < \infty$, for all bounded sets $B \in \mathcal{B}(\overline{\mathcal{C}}_0)$.*

A sequence $\{m_n\}_{n\geq 1}$ in $M(\overline{\mathcal{C}}_0)$ is said to converge to m in the $\hat{w}$-topology if

$$\int f \mathrm{d}m_n \longrightarrow \int f \mathrm{d}m, \qquad n \to \infty,$$

for all bounded and continuous functions $f : \overline{\mathcal{C}}_0 \to \mathbb{R}$ with bounded support.

The notion of $\hat{w}$–convergence generalizes the notion of vague convergence and takes into account the fact that $\overline{\mathcal{C}}_0$ is not locally compact. The $\hat{w}$-topology defined by this notion of convergence ensures that $M(\overline{\mathcal{C}}_0)$ is a Polish space. The subspace $M_p(\overline{\mathcal{C}}_0)$ consisting of all boundedly finite point measures is a closed subset of $M(\overline{\mathcal{C}}_0)$ and is endowed with the induced $\hat{w}$-topology. It is the suitable space when working with point processes in functional extreme value theory.

In the following, we emphasize on the connections between regular variations, sample point measures, sample maxima and exceedances above high thresholds in a functional framework. Before generalizing the analogous of statements i)–v) in the univariate case to the functional framework, we first need some notations and to introduce the limiting objects that will appear in the functional extreme value theory. For a general background on functional extreme value theory, the reader shall refer to de Haan and Ferreira [8], chapter 9.

Definition 3. *For $\alpha > 0$ and σ a probability measure on $\mathcal{S}$, we define*

- *$m_{\alpha,\sigma}$ the unique measure on $\overline{\mathcal{C}}_0$ such that*

  $$m_{\alpha,\sigma}\{[r,+\infty)\times B\} = r^{-\alpha}\sigma(B), \qquad r>0,\ B\in\mathcal{B}(\mathcal{S});$$

- *$\Pi_{\alpha,\sigma}$ a Poisson point measure on $\overline{\mathcal{C}}_0$ with intensity $m_{\alpha,\sigma}$;*
- *$M_{\alpha,\sigma}$ a continuous max-stable process on T with exponent measure $m_{\alpha,\sigma}$;*
- *$P_{\alpha,\sigma}$ a Pareto process with index $\alpha > 0$ and spectral measure σ, i.e.,*

  $$P_{\alpha,\sigma}(t) = P_\alpha Y(t), \qquad t \in T$$

  where P_α has an α–Pareto distribution, i.e.,

  $$\mathbb{P}(P_\alpha > r) = r^{-\alpha}, \quad r \geq 1,$$

  and is independent of the continuous process Y defined on T and whose distribution is σ.

Note that $m_{\alpha,\sigma}$ is boundedly finite and homogeneous of order $-\alpha$, i.e.,

$$m_{\alpha,\sigma}(uA) = u^{-\alpha} m_{\alpha,\sigma}(A),$$

for all $u > 0$ and $A \in \mathcal{B}(\overline{\mathcal{C}}_0)$ bounded.

The Poisson point measure $\Pi_{\alpha,\sigma}$ can be seen as a random element of $M_p(\overline{\mathcal{C}}_0)$ and might be defined as follows. Let $\{\Gamma_i\}_{i\geq 1}$ be a Poisson point process on $(0,+\infty)$ with Lebesgue intensity and, independently, let $\{Y_i\}_{i\geq 1}$ be a sequence of independent processes with common distribution σ, then

$$\Pi_{\alpha,\sigma} = \sum_{i\geq 1} \delta_{\Gamma_i^{-1/\alpha} Y_i}$$

is a Poisson point process with intensity $m_{\alpha,\sigma}$. Similarly, the max-stable process

$$M_{\alpha,\sigma}(t) = \max_{i\geq 1} \Gamma_i^{-1/\alpha} Y_i(t), \qquad t \in T,$$

is a max-stable process with exponent measure $m_{\alpha,\sigma}$.

Theorem 1. *Let $X_1, X_2, \dots$ be independent copies of a random process X with sample path in $\mathcal{C}_0$. The following statements are equivalent:*

1. *$X \in RV_{\alpha,\sigma}(\mathcal{C}_0)$;*
2. *the following $\hat{w}$-convergence holds in $M(\overline{\mathcal{C}}_0)$*

   $$t\,\mathbb{P}\{a(t)^{-1}X \in \cdot\} \xrightarrow{\hat{w}} m_{\alpha,\sigma}, \qquad \text{as } t\to\infty;$$

3. *the normalized point measure*

   $$N_n = \sum_{i=1}^n \delta_{X_i/a(n)}$$

   converges weakly in $M_p(\overline{\mathcal{C}}_0)$ to the Poisson point measure $\Pi_{\alpha,\beta}$ as $n\to\infty$;
4. *the normalized sample maximum*

   $$M_n(t) = a(n)^{-1}\max\{X_1(t),\dots,X_n(t)\}, \qquad t\in T,$$

   converges weakly in $\mathcal{C}$ to the max-stable random process $M_{\alpha,\sigma}$ as $n\to\infty$;

5. *the conditional distribution of normalized exceedances*

$$\mathbb{P}\left(t^{-1}X \in A \mid \|X\| > t\right), \qquad A \in \mathcal{B}(\mathcal{C}),$$

converges weakly in $\mathcal{C}$ *to the generalized Pareto process* $P_{\alpha,\sigma}$ *as* $n \to \infty$.

Proof of Theorem 1. The equivalence 1 ⇔ 2 is due to Hult and Lindskog [13]. The equivalence 1 ⇔ 3 ⇔ 4 is proved in Davis and Mikosch [6, Lemma 2.2], note that an essential part of the proof is due to de Haan and Lin [9, Theorem 2.4]. The equivalence 4 ⇔ 5 is essentially a consequence of Theorem 3.1 in Ferreira and de Haan [12]. □

3. PEAKS OVER THRESHOLDS AND PARETO PROCESSES

As explained in the introduction, the peaks over threshold approach amounts to consider the conditional distribution of a random field X given that $\ell(X) > u$, where $u > 0$ is the threshold level and $\ell\colon \mathcal{C} \to [0, +\infty)$ is a homogeneous measurable cost functional associated to the threshold method.

The notion of ℓ-Pareto process relies on the following theorem providing several equivalent characterizations. Setting $\ell(\cdot) = \|\cdot\|$, we retrieve essentially Theorem 2.1 of Ferreira and de Haan [12]. The main difference is that some extra positivity condition is made in [12] that prevent W from being identically equal to zero at some point $t \in T$ (this is useful when considering finite dimensional distributions).

Theorem 2. *Let W be a continuous stochastic process with sample path in $\mathcal{C}_0$. The following three statements are equivalent:*

1. *Constructive approach:*
   - 1a. $\ell(W)$ *has a Pareto distribution with index* $\alpha > 0$, *i.e.,* $\mathbb{P}(\ell(W) > u) = u^{-\alpha}$, $u > 1$;
   - 1b. $\ell(W)$ *and* $W/\ell(W)$ *are independent.*
2. *Homogeneity property:*
   - 2a. $\mathbb{P}\{\ell(W) > 1\} = 1$;
   - 2b. *For all* $u \geq 1$ *and measurable* $A \subset \{f \in \mathcal{C}\colon \ell(f) \geq 1\}$,
   $$\mathbb{P}(W \in uA) = u^{-\alpha}\mathbb{P}(W \in A).$$
3. *Peaks over threshold stability:*
   - 3a. $\mathbb{P}\{\ell(W) > 1\} > 0$;
   - 3b. *For all* $A \in \mathcal{B}(\mathcal{C})$ *and all* $u \geq 1$ *such that* $\mathbb{P}\{\ell(W) > u\} > 0$,
   $$\mathbb{P}\{u^{-1}W \in A \mid \ell(W) > u\} = \mathbb{P}(W \in A).$$

Note that the tail index α is the same in 1a. and 2b. Characterization 3. is more implicit and does not involve the tail index α.

Definition 4. *The distribution σ_ℓ of $W/\ell(W)$ is called the spectral measure on $\{x \in \mathcal{C}; \ell(x) = 1\}$. The process W is called a simple ℓ–Pareto process with tail index α and spectral measure σ_ℓ and is denoted by $W \sim P^{\ell}_{\alpha,\sigma_\ell}$.*

Proof of Theorem 2. We first prove that 1. ⇒ 2. Condition 2a. follows trivially from 1a. Consider the set

$$A_{v,B} = \{f \in \mathcal{C}\colon \ell(f) \geq v,\ f/\ell(f) \in B\}. \tag{3}$$

with $v \geq 1$ and $B \subset \{f \in \mathcal{C}\colon \ell(f) = 1\}$ measurable. Clearly, 1a. and 1b. entail

$$\mathbb{P}(W \in A_{v,B}) = \mathbb{P}(\ell(W) \geq v, W/\ell(W) \in B) = v^{-\alpha}\sigma_\ell(B).$$

Using the relation $uA_{v,B} = A_{uv,B}$, we obtain

$$\mathbb{P}(W \in uA_{v,B}) = u^{-\alpha}\mathbb{P}(W \in A_{v,B}).$$

The sets of the form $A_{v,B}$ form a π-system and generate the σ-algebra of Borel sets $A \subset \{f \in \mathcal{C}\colon \ell(f) \geq 1\}$. Hence condition 2b. holds for all Borel set A.

We prove that 2. ⇒ 3. Let $A \subset \mathcal{C}$ be a Borel set. Using conditions 2a. and 2b., we obtain

$$\begin{aligned}\mathbb{P}\{u^{-1}W \in A,\ \ell(W) > u\} &= u^{-\alpha}\mathbb{P}\{W \in A,\ \ell(W) > 1\}\\ &= u^{-\alpha}\mathbb{P}(W \in A).\end{aligned}$$

When $A = \mathcal{C}$ we have $\mathbb{P}\{\ell(W) > u\} = u^{-\alpha} > 0$ and hence

$$\begin{aligned}\mathbb{P}\{u^{-1}W \in A \mid \ell(W) > u\} &= \frac{\mathbb{P}\{u^{-1}W \in A,\ \ell(W) > u\}}{\mathbb{P}\{\ell(W) > u\}}\\ &= \mathbb{P}(W \in A).\end{aligned}$$

It remains to check that 3. ⇒ 1. Condition 3b. with $A = \{f \in \mathcal{C}\colon \ell(f) > v\}$ gives for $u, v \geq 1$

$$\mathbb{P}\{\ell(W) > uv\} = \mathbb{P}\{\ell(W) > u\}\mathbb{P}\{\ell(W) > v\}$$

and hence the tail function $u \mapsto \bar{F}(u) = \mathbb{P}\{\ell(W) > u\}$ satisfies the functional equation

$$\bar{F}(uv) = \bar{F}(u)\bar{F}(v), \qquad u, v \geq 1. \tag{4}$$

Condition 3a. gives the initial condition $\bar{F}(1) > 0$. Clearly (4) implies $\bar{F}(1) = \bar{F}(1)^2$ and the initial condition ensures that $\bar{F}(1) = 1$. We then prove that $\bar{F}$ is positive on $[1, \infty)$. Since F is right continuous and $\bar{F}(\infty) = 0$, there exists some $u_0 > 1$ such that $\bar{F}(u_0) \in (0, 1)$. Using (4), we have for all $n \geq 1$, $\bar{F}(u_0^n) = \bar{F}(u_0)^n > 0$ and letting $u_0^n \to \infty$, $\bar{F}$ must be positive since it must be non-increasing.

Any non-increasing positive solution of the functional equation (4) must be of the form $F(u) = u^{-\alpha}$ for some $\alpha > 0$. This proves that $\ell(W)$ satisfies 1a. It remains to prove that $\ell(W)$ and $W/\ell(W)$ are independent. To this aim, we consider $B \in \mathcal{B}(\mathcal{C})$ and we set $A = \{f \in \mathcal{C} \colon f/\ell(f) \in B\}$. Condition 3b. ensures that for all $u \geq 1$,

$$\begin{aligned} & \mathbb{P}\{W/\ell(W) \in B,\ \ell(W) > u\} \\ = & \mathbb{P}\{W \in A,\ \ell(W) > u\} \\ = & \mathbb{P}\{u^{-1}W \in A,\ \ell(W) > u\} \\ = & \mathbb{P}\{W \in A\}\mathbb{P}\{\ell(W) > u\} \\ = & \mathbb{P}\{W/\ell(W) \in B\}\mathbb{P}\{\ell(W) > u\}, \end{aligned}$$

and this proves condition 1b. □

The following result shows that Pareto processes naturally appear as the possible limiting distributions of (normalized) exceedances over a high threshold. The proof relies on the characterization of Pareto processes by peak over threshold stability.

Proposition 1. *Assume ℓ is continuous and let X be a stochastic process such that*

$$\mathbb{P}\{u^{-1}X \in \cdot \mid \ell(X) > u\} \longrightarrow \mathbb{P}(W \in \cdot), \qquad u \to \infty,$$

weakly in $\mathcal{C}$. Then either W is a simple ℓ–Pareto process or $\mathbb{P}\{\ell(W) = 1\} = 1$.

Proof of Proposition 1. Let $A = \{f \in \mathcal{C} \colon \ell(f) \geq 1\}$. Clearly, $\mathbb{P}\{u^{-1}X \in A \mid \ell(X) > u\} = 1$ for all $u > 0$. Furthermore, A is closed by the continuity of ℓ so that the Portmanteau theorem implies

$$\mathbb{P}(W \in A) \geq \limsup_{u \to +\infty} \mathbb{P}\{u^{-1}X \in A \mid \ell(X) > u\} = 1,$$

and hence $\ell(W) \geq 1$ almost surely.

We suppose that $\mathbb{P}\{\ell(W) = 1\} < 1$ and prove that W is a simple ℓ–Pareto process. Clearly, in this case, $\mathbb{P}\{\ell(W) > 1\} > 0$ and condition 3a. of Theorem 2 is satisfied. We prove that the limit W satisfies also the peak over threshold stability condition 3b. so that it must be a simple ℓ–Pareto process. Clearly, for all $u_1, u_2 \geq 1$ and all set $A_{v,B}$ of the form (3), we have

$$\begin{aligned} & \mathbb{P}\{u_1^{-1}u_2^{-1}X \in A_{v,B},\ \ell(u_1^{-1}X) > u_2 \mid \ell(X) > u_1\} \\ = & \mathbb{P}\{u_1^{-1}u_2^{-1}X \in A_{v,B} \mid \ell(X) > u_1u_2\} \\ & \times \mathbb{P}\{\ell(u_1^{-1}X) > u_2 \mid \ell(X) > u_1\}. \end{aligned}$$

As $u_1 \to \infty$ the weak convergence entails

$$\begin{aligned} & \mathbb{P}\{u_2^{-1}W \in A_{v,B},\ \ell(W) > u_2\} \\ = & \mathbb{P}(W \in A_{v,B})\,\mathbb{P}\{\ell(W) > u_2\}, \end{aligned} \tag{5}$$

provided $\mathbb{P}\{\ell(W) = v\} = 0$ and $\mathbb{P}\{W/\ell(W) \in \partial B\} = 0$. Indeed since ℓ is continuous, the boundary set of $\{f \in \mathcal{C} \colon \ell(f) > v\}$ is $\{f \in \mathcal{C} \colon \ell(f) = v\}$. Finally, equation (5) is extended to all $v \geq 1$ and all $A \in \mathcal{B}(\mathcal{C})$ thanks to the fact that the sets of the form $A_{v,B}$ with $\mathbb{P}\{\ell(W) = v\} = 0$ and $\mathbb{P}\{W/\ell(W) \in \partial B\} = 0$ form a π-system generating the σ-algebra of Borel sets $A \subset \{f \in \mathcal{C} \colon \ell(f) \geq 1\}$. Hence the two probability measures $\mathbb{P}(W \in \cdot)$ and $\mathbb{P}\{u_2^{-1}W \in \cdot \mid \ell(W) < u_2\}$ are equal and condition 3b. is satisfied. □

Theorem 3. *Suppose that $X \in RV_{\alpha,\sigma}(\mathcal{C}_0)$. If ℓ is continuous at the origin and does not vanish σ–a.e., then*

$$\mathbb{P}\{u^{-1}X \in \cdot \mid \ell(X) > u\} \longrightarrow P^{\ell}_{\alpha,\sigma_\ell}, \qquad u \to \infty,$$

weakly in $\mathcal{C}$ and the spectral measure σ_ℓ is given by

$$\sigma_\ell(B) = \frac{1}{c_\ell}\int_{\mathcal{S}} \ell(f)^\alpha 1_{\{f/\ell(f) \in B\}}\,\sigma(df), \qquad B \in \mathcal{B}(\mathcal{C}), \tag{6}$$

with $c_\ell = \int_{\mathcal{S}} \ell(f)^\alpha \sigma(df)$.

Proof of Theorem 3. In order to use regular variations, we introduce the normalizing function $a(\cdot)$ from Definition 1 and prove

$$\mathbb{P}\{a(u)^{-1}X \in \cdot \mid \ell(X) > a(u)\} \longrightarrow P^{\ell}_{\alpha,\sigma_\ell}, \qquad u \to \infty.$$

To this aim, it is enough to prove that for all bounded continuous functional $F \colon \mathcal{C} \to \mathbb{R}$ we have

$$u\mathbb{E}\left[F\{a(u)^{-1}X\}1_{\{\ell(X) > a(u)\}}\right] \longrightarrow c_\ell \mathbb{E}\{F(W)\}, \qquad u \to \infty, \tag{7}$$

with $W \sim P^{\ell}_{\alpha,\sigma_\ell}$. Indeed, taking $F \equiv 1$ implies

$$u\mathbb{P}\{\ell(X) > a(u)\} \longrightarrow c_\ell, \qquad u \to \infty.$$

Since ℓ is nonnegative and does not vanish σ–a.e., $c_\ell > 0$ and we have

$$\begin{aligned} & \mathbb{E}[F(a(u)^{-1}X) \mid \ell(X) > a(u)] \\ = & \frac{\mathbb{E}\left[F(a(u)^{-1}X)1_{\{\ell(X) > a(u)\}}\right]}{\mathbb{P}\{\ell(X) > a(u)\}} \longrightarrow \mathbb{E}\{F(W)\}, \quad u \to \infty, \end{aligned}$$

proving the required weak convergence. We now prove (7). By the homogeneity of ℓ,

$$u\mathbb{E}\left[F\{a(u)^{-1}X\}1_{\{\ell(X) > a(u)\}}\right] = u\mathbb{E}\left[\tilde{F}\{a(u)^{-1}X\}\right], \tag{8}$$

with $\tilde{F}(f) = F(f)1_{\{\ell(f) > 1\}}$. According to Theorem 1, $X \in \mathrm{RV}_{\alpha,\sigma}(\mathcal{C}_0)$ implies the $\hat{w}$-convergence in $\overline{\mathcal{C}}_0$

$$u\,\mathbb{P}\{a(u)^{-1}X \in \cdot\} \to m_{\alpha,\beta}(\cdot)$$

and therefore we have

$$u\,\mathbb{E}\left[\tilde{F}\{a(u)^{-1}X\}\right] \longrightarrow \int_{\overline{\mathcal{C}}_0} \tilde{F}(f) m_{\alpha,\sigma}(\mathrm{d}f), \tag{9}$$

provided that $\tilde{F}$ has a bounded support in $\overline{\mathcal{C}}_0$ and is continuous $m_{\alpha,\sigma}$–a.e. We will check these conditions later. The right-hand side of (9) is equal to

$$\begin{aligned} & \int_0^\infty \int_{\mathcal{S}} \tilde{F}(rf)\, \alpha r^{-\alpha-1} \mathrm{d}r \sigma(\mathrm{d}f) \\ (10) \quad = & \int_0^\infty \int_{\mathcal{S}} F(rf) 1_{\{r\ell(f)>1\}}\, \alpha r^{-\alpha-1} \mathrm{d}r \sigma(\mathrm{d}f). \end{aligned}$$

On the other hand, the right-hand side of (7) can be computed using a simple change of variable

$$\begin{aligned} & \int_1^\infty \int_{\mathcal{C}} F(rf)\, \alpha r^{-\alpha-1} \mathrm{d}r c_\ell \sigma_\ell(\mathrm{d}f) \\ = & \int_0^\infty \int_{\mathcal{S}} 1_{\{r>1\}} F(rf/\ell(f)) \ell(f)^\alpha\, \alpha r^{-\alpha-1} \mathrm{d}r \sigma(\mathrm{d}f) \\ (11) \quad = & \int_0^\infty \int_{\mathcal{S}} 1_{\{r\ell(f)>1\}} F(rf)\, \alpha r^{-\alpha-1} \mathrm{d}r \sigma(\mathrm{d}f). \end{aligned}$$

Equations (8)–(11) imply (7). It remains to prove that $\tilde{F}$ has a bounded support in $\overline{\mathcal{C}}_0$ and is continuous $m_{\alpha,\sigma}$–a.e. The continuity and homogeneity of ℓ implies that there exists some $M>0$ such that $\ell(f) \leq M\|f\|$ for all $f \in \mathcal{C}$. Hence $\ell(f)>1$ implies $\|f\| > M^{-1}$ and the support of $\tilde{F}$ is included in $[M^{-1}, +\infty] \times \mathcal{S}$ and is bounded in $\overline{\mathcal{C}}_0$. Furthermore since F is continuous, $f \mapsto \tilde{F}(f) = F(f) 1_{\{\ell(f)>1\}}$ is continuous at every point f such that $\ell(f) \neq 1$. Finally, it holds

$$m_{\alpha,\sigma}(\{\ell(f)=1\}) = \int_0^\infty \int_{\mathcal{S}} 1_{\{r\ell(f)=1\}}\, \alpha r^{-\alpha-1} \mathrm{d}r \sigma(\mathrm{d}f) = 0,$$

and $\tilde{F}$ is continuous $m_{\alpha,\sigma}$–a.e. □

When different functionals ℓ and ℓ' are involved, the corresponding spectral measures σ_ℓ and $\sigma_{\ell'}$ defined by (6) are linked by a simple relation.

Proposition 2. *Let ℓ and ℓ' be homogeneous measurable functionals $\mathcal{C} \to [0,+\infty)$ and suppose that $\ell'(f)>0$ σ_ℓ–a.e. Then,*

$$(12) \quad \sigma_\ell(B) = \frac{\int_{\mathcal{C}} \ell(f)^\alpha 1_{\{f/\ell(f) \in B\}}\, \sigma_{\ell'}(\mathrm{d}f)}{\int_{\mathcal{C}} \ell(f)^\alpha\, \sigma_{\ell'}(\mathrm{d}f)}, \qquad B \in \mathcal{B}(\mathcal{C}).$$

As a direct consequence, if $\ell(f)>0$ σ–a.e., then (6) can be inverted and we have

$$(13) \quad \sigma(B) = c_\ell \int_{\mathcal{C}} \|f\|^\alpha 1_{\{f/\|f\| \in B\}} \sigma_\ell(\mathrm{d}f), \quad B \in \mathcal{B}(\mathcal{S}).$$

Proof of Proposition 2. Using the definition of σ_ℓ and $\sigma_{\ell'}$, we have for $B \in \mathcal{B}(\mathcal{C})$,

$$\begin{aligned} & \int_{\mathcal{C}} \ell(f)^\alpha 1_{\{f/\ell(f)\in B\}}\, \sigma_{\ell'}(\mathrm{d}f) \\ = & \frac{1}{c_{\ell'}} \int_{\mathcal{C}} \ell'(f)^\alpha \{\ell(f)/\ell'(f)\}^\alpha 1_{\{f/\ell(f)\in B\}}\, \sigma(\mathrm{d}f) \\ = & \frac{c_\ell}{c_{\ell'}} \sigma_\ell(B), \end{aligned}$$

where we used in the last equality the fact that $\ell'(f)>0$ σ_ℓ–a.e. Taking $B=\mathcal{C}$, we get

$$c_\ell = c_{\ell'} \int_{\mathcal{C}} \ell(f)^\alpha \sigma_{\ell'}(\mathrm{d}f)$$

and (12) follows easily. □

4. ESTIMATION OF SPECTRAL MEASURES

Under the assumptions of Theorem 3, we consider a natural non-parametric estimator of the spectral measure σ_ℓ in (6) associated to the regularly varying random field $X \in \mathrm{RV}_{\alpha,\sigma}(\mathcal{C}_0)$. It is based on independent copies $X_1, X_2, \ldots$ of X, and especially on exceedances over large thresholds, i.e., such that $\ell(X_i) > u_n$ for some large threshold level u_n.

Proposition 3. *Suppose that $X \in \mathrm{RV}_{\alpha,\sigma}(\mathcal{C}_0)$ and that ℓ is continuous at the origin and does not vanish σ–a.e. Consider a sequence $u_n>0$ such that $u_n \to \infty$ and $u_n/a(n) \to 0$, with $a(\cdot)$ given by (2). Then*

$$\hat{\sigma}_{\ell,n} = \frac{\sum_{i=1}^n 1_{\{\ell(X_i)>u_n\}} \delta_{X_i/\ell(X_i)}}{\sum_{i=1}^n 1_{\{\ell(X_i)>u_n\}}}, \qquad n \geq 1,$$

is a consistent estimator of the spectral measure σ_ℓ in the sense that $\hat{\sigma}_{\ell,n}(B)$ converges in probability as $n \to +\infty$ to $\sigma_\ell(B)$ for all $B \in \mathcal{B}(\mathcal{C})$ such that $\sigma_\ell(\partial B)=0$.

In some applications, it may happen that the observations are obtained by thresholding with respect to a functional ℓ' different from the functional ℓ of interest. Propositions 2 and 3 suggest the following generalized estimator.

Proposition 4. *Suppose that $X \in \mathrm{RV}_{\alpha,\sigma}(\mathcal{C}_0)$. Let ℓ and ℓ' be homogeneous functionals $\mathcal{C} \to [0,+\infty)$ such that*

- *ℓ' is continuous at the origin and does not vanish σ–a.e.*
- *ℓ is continuous and satisfies $\ell \leq M\ell'$ for some $M>0$.*

Consider a sequence $u_n>0$ such that $u_n \to \infty$ and $u_n/a_n \to 0$ with $a(\cdot)$ given by (2). Then

$$\tilde{\sigma}_{\ell,n} = \frac{\sum_{i=1}^n \{\ell(X_i)/\ell'(X_i)\}^\alpha 1_{\{\ell'(X_i)>u_n\}} \delta_{\{X_i/\ell(X_i)\}}}{\sum_{i=1}^n \{\ell(X_i)/\ell'(X_i)\}^\alpha 1_{\{\ell'(X_i)>u_n\}}}.$$

is a consistent estimator of σ_ℓ.

Proof of Proposition 3. Let $N_n = \sum_{i=1}^n 1_{\{\ell(X_i)>u_n\}}$ be the number of observations above threshold u_n in the sample $X_1, \ldots, X_n$. The estimator $\hat{\sigma}_{\ell,n}$ is well defined as soon as $N_n>0$. If $N_n=0$, define $\hat{\sigma}_{\ell,n} = \sigma_0$ with σ_0 an arbitrary probability measure. The choice of σ_0 is irrelevant from an asymptotic point of view since we will see that $\mathbb{P}(N_n=0) \to 0$ as $n \to \infty$.

Clearly N_n has a binomial distribution with parameters n and $p_n = \mathbb{P}\{\ell(X) > u_n\}$. In particular, N_n has mean np_n and variance $np_n(1-p_n)$. The conditions $u_n \to \infty$ and $u_n/a_n \to 0$ imply $p_n \to 0$ and $np_n \to \infty$. Since $N_n/(np_n)$

has mean 1 and its variance goes to 0 as $n \to \infty$, $N_n/(np_n)$ converges in probability to 1.

Let $B \in \mathcal{B}(\mathcal{C})$ such that $\sigma_\ell(\partial B) = 0$ and define $p_{n,B} = \mathbb{P}\{\ell(X) > u_n,\ X/\ell(X) \in B\}$. The normalized sum

$$\frac{1}{np_n} \sum_{i=1}^{n} 1_{\{\ell(X_i) > u_n,\ X_i/\ell(X_i) \in B\}} \tag{14}$$

has expectation $p_{n,B}/p_n$ and variance $p_{n,B}(1-p_{n,B})/(np_n^2)$. Theorem 3 combined with the condition $\sigma_\ell(\partial B) = 0$ yields

$$\begin{aligned} \frac{p_{n,B}}{p_n} &= \mathbb{P}\{X/\ell(X) \in B \mid \ell(X) > u_n\} \\ &\longrightarrow \sigma_\ell(B), \qquad n \to \infty. \end{aligned}$$

Since

$$\frac{p_{n,B}(1-p_{n,B})}{np_n^2} \sim \frac{\sigma_\ell(B)}{np_n} \longrightarrow 0,$$

(14) converges in probability to $\sigma_\ell(B)$ as $n \to \infty$. Finally, the expression

$$\begin{aligned} \hat{\sigma}_{\ell,n}(B) &= \sigma_0 1_{\{N_n = 0\}} \\ &+ \frac{np_n}{N_n} \frac{1}{np_n} \left[\sum_{i=1}^{n} 1_{\{\ell(X_i) > v_n,\ X_i/\ell(X_i) \in B\}} \right] 1_{\{N_n > 0\}} \end{aligned}$$

and the convergences in probability mentioned above combined with Slutsky's lemma imply $\hat{\sigma}_{\ell,n}(B) \to \sigma_\ell(B)$ in probability and proves Proposition 3. ◻

Proof of Proposition 4. According to Proposition 2,

$$\sigma_\ell(B) = \frac{\int_{\mathcal{C}} \ell(f)^\alpha 1_{\{f/\ell(f) \in B\}}\, \sigma_{\ell'}(\mathrm{d}f)}{\int_{\mathcal{C}} \ell(f)^\alpha\, \sigma_{\ell'}(\mathrm{d}f)}, \qquad B \in \mathcal{B}(\mathcal{C}). \tag{15}$$

The estimator $\tilde{\sigma}_{\ell,n}$ is obtained by replacing $\sigma_{\ell'}$ in this expression by the non-parametric estimator $\hat{\sigma}_{\ell',n}$ from Proposition 3:

$$\tilde{\sigma}_{\ell,n}(B) = \frac{\int_{\mathcal{C}} \ell(f)^\alpha 1_{\{f/\ell(f) \in B\}}\, \hat{\sigma}_{\ell',n}(\mathrm{d}f)}{\int_{\mathcal{C}} \ell(f)^\alpha\, \hat{\sigma}_{\ell',n}(\mathrm{d}f)}, \qquad B \in \mathcal{B}(\mathcal{C}). \tag{16}$$

As a consequence of Proposition 3, the probability measures $\hat{\sigma}_{\ell',n}$ converge in probability as $n \to \infty$ to $\sigma_{\ell'}$ in the space of probability measures on $\mathcal{C}$ endowed with the metric of weak convergence. This entails the convergence in probability $\int F(f)\hat{\sigma}_{\ell',n}(\mathrm{d}f) \to \int F(f)\sigma_{\ell'}(\mathrm{d}f)$ for all functional F that are bounded and continuous $\sigma_{\ell'}$–a.e.

Let $B \in \mathcal{B}$ be such that $\sigma_\ell(\partial B) = 0$ and consider the particular choice $F_B(f) = \ell(f)^\alpha 1_{\{f/\ell(f) \in B\}}$. The condition $\ell \le M\ell'$ entails $F_B(f) \le M^\alpha$ for all f such that $\ell'(f) = 1$. Since ℓ is continuous, F_B is continuous except at points f such that $f/\ell(f) \in \partial B$. It is easily checked that the condition $\sigma_\ell(\partial B) = 0$ implies $\sigma_{\ell'}(\{f/\ell(f) \in \partial B\}) = 0$ so that F_B is continuous $\sigma_{\ell'}$–a.e. Hence, we get as $n \to \infty$

$$\begin{aligned} &\int_{\mathcal{C}} \ell(f)^\alpha 1_{\{f/\ell(f) \in B\}}\, \hat{\sigma}_{\ell',n}(\mathrm{d}f) \\ &\xrightarrow{\mathbb{P}} \int_{\mathcal{C}} \ell(f)^\alpha 1_{\{f/\ell(f) \in B\}}\, \hat{\sigma}_{\ell'}(\mathrm{d}f), \end{aligned}$$

and similarly

$$\int_{\mathcal{C}} \ell(f)^\alpha \hat{\sigma}_{\ell',n}(\mathrm{d}f) \xrightarrow{\mathbb{P}} \int_{\mathcal{C}} \ell(f)^\alpha 1_{\{f/\ell(f) \in B\}}\, \hat{\sigma}_{\ell'}(\mathrm{d}f).$$

Then (15)–(16) and Slutsky's Lemma imply that $\tilde{\sigma}_{\ell,n}(B) \to \sigma_\ell(B)$ in probability. ◻

5. AN EXAMPLE

For standard examples of regularly varying random fields, the reader shall refer to section 4.1 in Davis and Mikosh [6]. These examples include simple multiplicative processes, symmetric α-stable processes, max-stable processes. We develop here an example in connection with Brown-Resnick processes: we describe the associated Pareto processes and provide some simulations.

The Brown-Resnick processes introduced in Kabluchko et al. [16] form a flexible class of stationary max-stable random fields. The construction is as follows: consider $B = (B(t))_{t \in \mathbb{R}^d}$ a centered continuous stationary increments Gaussian random field on $\mathbb{R}^d$ with variance function $\sigma^2(t) = \mathbb{E}[B(t)^2]$; let $(B_i)_{i \ge 1}$ be independent copies of B and, independently, let $\{\Gamma_i\}_{i \ge 1}$ be a Poisson point process on $(0, +\infty)$ with Lebesgue intensity; then for $\alpha > 0$, the random field

$$M(t) = \max_{i \ge 1} \Gamma_i^{-1/\alpha} \exp\Big(\frac{1}{\alpha}(B_i(t) - \sigma^2(t)/2) \Big), \quad t \in \mathbb{R}^d$$

is an α-Fréchet max-stable random field which is stationary on $\mathbb{R}^d$. Furthermore, the distribution of M depends only on α and on the variogram $\gamma(t) = \mathbb{E}[(B(t) - B(0))^2]$.

For any compact $T \subset \mathbb{R}^d$, M is regularly varying on $\mathcal{C}_0$ with index α and spectral measure given by

$$\sigma(A) = \frac{\mathbb{E}[\|Z\|^\alpha 1_{\{Z/\|Z\| \in A\}}]}{\mathbb{E}[\|Z\|^\alpha]}, \qquad A \in \mathcal{B}(\mathcal{S}),$$

where Z denotes the log-normal process

$$Z(t) = \exp\big(\frac{1}{\alpha}(B(t) - \sigma^2(t)/2) \big), \quad t \in \mathbb{R}^d.$$

We consider a random field X in the max-domain of attraction of M, or equivalently $X \in RV_{\alpha,\sigma}(\mathcal{C}_0)$ and a cost functional ℓ and we are interested in the behavior of X given that $\ell(X)$ is large. Theorem 3 states that the (normalized) exceedances of X over high threshold converge to a ℓ-Pareto process $P^\ell_{\alpha,\sigma_\ell}$ with spectral measure given by

$$\sigma_\ell(A) = \frac{\mathbb{E}[\ell(Z)^\alpha 1_{\{Z/\ell(Z) \in A\}}]}{\mathbb{E}[\ell(Z)^\alpha]}, \qquad A \in \mathcal{B}(\mathcal{C}).$$

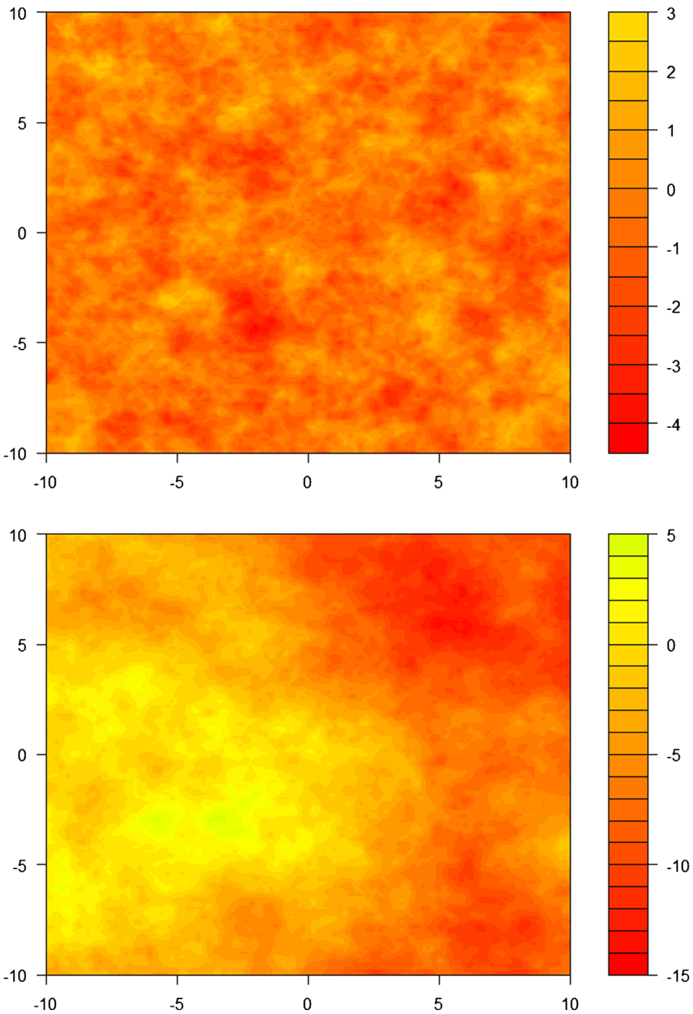

Figure 1. Simulations in logarithmic scale from the spectral measure σ_ℓ in the Brown-Resnick framework with exponential (top) and Brownian (bottom) variogram respectively. The cost functional ℓ is the mean value on the domain $T = [-10, 10]$.

It is worth noting that for the simple cost function $\ell(x) = x(t_0)$ giving the value of x at a given location $t_0 \in T$, the spectral measure is simply the distribution of the log-normal process $\tilde{Z}(t) = \exp(\frac{1}{\alpha}(\tilde{B}(t) - \tilde{\sigma}^2(t)/2))$ where $\tilde{B}(t) = B(t) - B(t_0)$ and $\tilde{\sigma}^2(t) = \mathbb{E}[\tilde{B}(t)^2]$. This can be checked by standard computations on Gaussian random vectors since in this case

$$\sigma_\ell(A) = \mathbb{E}\Big[\exp(B(t_0) - \sigma^2(t_0)/2) 1_{\{Z/Z(t_0) \in A\}}\Big], \quad A \in \mathcal{B}(\mathcal{C}).$$

and we can use a change of measure with exponential density (see Kabluchko [15, Proposition 2] for more details).

More generally, one can simulate (approximatively) the ℓ-Pareto process thanks to a Metropolis-Hastings procedure. The construction uses independent copies $(Z_n)_{n\geq 1}$ of the log-normal process Z as proposals and is as follows:

i) set $Z_1' = Z_1$;
ii) for $n \geq 2$, compute the acceptance probability $p_n = \min((\ell(Z_n)/\ell(Z_{n-1}))^\alpha, 1)$ and set then $Z_n' = Z_n$ with probability p_n and $Z_n' = Z_{n-1}'$ with probability $1 - p_n$.

The sequence of random fields (Z_n') is a reversible Markov chain with stationary distribution

$$\mathbb{P}[Z' \in A] = \frac{\mathbb{E}[\ell(Z)^\alpha 1_{\{Z \in A\}}]}{\mathbb{E}[\ell(Z)^\alpha]}, \qquad A \in \mathcal{B}(\mathcal{C}).$$

Hence for large n, the distribution of $Z_n'/\ell(Z_n')$ converges to σ_ℓ.

We apply this procedure to sample from the spectral measure σ_ℓ when $\ell(x) = |T|^{-1}\int_T x(t)dt$ is the mean value of x on the domain T. Simulation of the stationary increments Gaussian random fields is performed with the function GaussRF from the R package RandomFields. We use the domain $T = [-10, 10]$ and a discretization of the integral with a grid of size 100×100. The tail index is $\alpha = 1$. Two different variograms are considered: the exponential model $\gamma(t) = 2(1 - e^{-\|t\|})$ and the Brownian model $\gamma(t) = \|t\|$. Sample path from the spectral measure are shown in Figure 1. Both random fields have mean value equal to 1 but the behaviors are quite different: in the exponential model, many moderate peaks ($\approx$15) appear at various places in the domain; in the Brownian model, there is one large peak ($\approx$80) in a localized place. This is related to behavior of the log-normal process Z which is stationary in the exponential model while it vanishes at infinity in the Brownian model. Hence an extremal event has an impact in the whole domain in the first case, while the impact remains more localized in the second case.

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