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Uniform asymptotics for ruin probabilities in a nonstandard compound renewal risk model*

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In this paper, we consider a nonstandard compound renewal risk model with or without a constant interest rate, in which claims at each accident moment are aggregated from a number of widely orthant dependent individual claims, and inter-arrival times are widely lower orthant dependent. We establish some asymptotic formulae for the finite-time and infinite-time ruin probabilities, when the individual claims are heavy-tailed. The obtained asymptotics hold uniformly on a finite or infinite time interval.

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1. COMPOUND RENEWAL RISK MODEL

In this paper, we investigate the uniformly asymptotic behavior of the finite-time and infinite-time ruin probabilities in a nonstandard compound renewal risk model with or without a constant interest rate. The compound renewal risk model is a natural modification of the classical one. In such a model, the claims at each accident moment are aggregated from a number of individual claims, meanwhile, in a classical renewal risk model one claim at each accident time appears. More specifically, the compound renewal risk model satisfies the following three requirements:

Assumption H$_1$. The individual claim sizes $\{X_k,\ k \ge 1\}$ form a sequence of identically distributed but not necessarily independent nonnegative random variables (r.v.s) with common distribution F and finite mean $\mu > 0$.

Assumption H$_2$. The inter-arrival times $\{\theta_k,\ k \ge 1\}$ are also identically distributed but not necessarily independent nonnegative r.v.s, which are independent of $\{X_k,\ k \ge 1\}$ and not degenerate at zero.

Assumption H$_3$. The individual claim sizes and the claim number caused by the nth accident at the accident time $\tau_n = \sum_{k=1}^{n} \theta_k$ are $\{X_k^{(n)},\ k \ge 1\}$ and N_n, respectively. Here, $\{X_k^{(n)},\ k \ge 1\}$ are independent copies of $\{X_k,\ k \ge 1\}$ and $\{N_k,\ k \ge 1\}$ are independent and identically distributed (i.i.d.) nonnegative integer-valued r.v.s with common distribution G and finite mean $\nu > 0$. In addition, the random sequences $\{N_k,\ k \ge 1\}$, $\{\theta_k,\ k \ge 1\}$ and $\{\{X_k^{(n)},\ k \ge 1\},\ n \ge 1\}$ are mutually independent.

If the individual claim sizes $\{X_k,\ k \ge 1\}$ and the inter-arrival times $\{\theta_k,\ k \ge 1\}$ are both independent r.v.s, respectively, the model is the so-called independent compound renewal risk model, which was introduced by Tang et al. (2001). If each claim number N_k is degenerate at 1, the model above reduces to the classical one.

In such a compound renewal risk model, the times of successive accidents $\{\tau_n,\ n \ge 1\}$ constitute a quasi-renewal counting process

$$\tau(t) = \sup\{n \ge 0 :\ \tau_n \le t\},\ t \ge 0, \tag{1}$$

which represents the number of accidents in the interval $[0, t]$ with mean function $\lambda(t) = \mathrm{E}\tau(t)$. The total claim amount at time τ_n and the total claim amount up to time $t \ge 0$ are, respectively,

$$S_{N_n}^{(n)} = \sum_{k=1}^{N_n} X_k^{(n)} \quad \text{and} \quad \sum_{n=1}^{\tau(t)} S_{N_n}^{(n)} = \sum_{n=1}^{\tau(t)} \sum_{k=1}^{N_n} X_k^{(n)}. \tag{2}$$

The total amount of premiums accumulated up to time $t \ge 0$, denoted by $C(t)$ with $C(0) = 0$ and $C(t) < \infty$ almost surely (a.s.) for every $t > 0$, is a nonnegative and nondecreasing stochastic process, which is independent of $\{N_k,\ k \ge 1\}$, $\{\theta_k,\ k \ge 1\}$ and $\{\{X_k^{(n)},\ k \ge 1\},\ n \ge 1\}$. Let $r \ge 0$ be the constant interest rate (that is, after time t, the capital x becomes xe^{rt}). We remark that if $r = 0$, then the model has no interest rate. The discounted surplus process of an insurance company, which plays an important role in the study of ruin probabilities, is expressed by

$$U_r(x, t) = x + \tilde{C}(t) - D_r(t),$$

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where x is the initial capital reserve, $\tilde{C}(t)=\int_{0-}^{t}e^{-rs}C(\mathrm{d}s)$ and $D_r(t)=\sum_{n=1}^{\tau(t)}S_{N_n}^{(n)}e^{-r\tau_n}$ represent the total discounted premium amount and the total discounted claim amount up to time $t\geq 0$, respectively. Assume that $\tilde{C}(t)<\infty$ a.s. for all $0\leq t\leq\infty$. In this described model the ruin probability within a finite time $t\geq 0$ and the infinite-time ruin probability are defined, respectively, by

$$\Psi(x,t)=\mathrm{P}\Big(\inf_{0\leq u\leq t}U_r(x,u)<0|U_r(x,0)=x\Big)$$

and

$$\Psi(x)=\mathrm{P}\Big(\inf_{u\geq 0}U_r(x,u)<0|U_r(x,0)=x\Big).$$

This paper aims to investigate the asymptotics for such ruin probabilities in some dependent compound renewal risk model, holding uniformly for all t such that the quasi-renewal function $\lambda(t)$ is positive, which implies that the set $\Lambda=\{t:\ \lambda(t)>0\}$ is needed. Define $\underline{t}=\inf\{t:\ \lambda(t)>0\}=\inf\{t:\ \mathrm{P}(\theta_1\leq t)>0\}$, then, clearly,

$$\Lambda=\begin{cases}[\underline{t},\infty], & \text{if } \mathrm{P}(\theta_1=\underline{t})>0;\\ (\underline{t},\infty], & \text{if } \mathrm{P}(\theta_1=\underline{t})=0.\end{cases}$$

Then, for all $t\in\Lambda$ and $x\geq 0$,

$$\begin{aligned}\Psi(x,t)&=\mathrm{P}\Big(\sup_{0\leq u\leq t}\big(D_r(u)-\tilde{C}(u)\big)>x\Big)\\ \Psi(x)&=\mathrm{P}\Big(\sup_{u\geq 0}\big(D_r(u)-\tilde{C}(u)\big)>x\Big).\end{aligned}\tag{3}$$

In the noncompound renewal risk model, where $N_1=N_2=\cdots=1$, the renewal counting process $\tau(t)$ in (1) can be explained as the number of claims within $[0,t]$, then the total claim amount up to time $t\geq 0$ in (2) is simplified by $\sum_{n=1}^{\tau(t)}X_n$. Therefore, the finite-time and infinite-time ruin probabilities in (3) reduce to

$$\mathrm{P}\left(\sup_{0\leq u\leq t}\left(\sum_{n=1}^{\tau(u)}X_ne^{-r\tau_n}-\tilde{C}(u)\right)>x\right)$$

and

$$\mathrm{P}\left(\sup_{u\geq 0}\left(\sum_{n=1}^{\tau(u)}X_ne^{-r\tau_n}-\tilde{C}(u)\right)>x\right).$$

To better illuminate our results, we denote the finite-time and infinite-time ruin probabilities in the noncompound renewal risk model by $\psi(x,t)$ and $\psi(x)$, respectively.

Throughout this paper, all limit relationships hold for x tending to ∞ unless stated otherwise. For two positive functions $a(x)$ and $b(x)$, we write $a(x)\sim b(x)$ if $\lim a(x)/b(x)=1$; write $a(x)\prec b(x)$ if $\limsup a(x)/b(x)\leq 1$; write $a(x)\succ b(x)$ if $\liminf a(x)/b(x)\geq 1$; and $a(x)=o(b(x))$ if $\lim a(x)/b(x)=0$. Furthermore, for two positive bivariate functions $a(x,t)$ and $b(x,t)$, we write $a(x,t)\sim b(x,t)$ uniformly for all t in a nonempty set A, if

$$\lim_{x\to\infty}\sup_{t\in A}\left|\frac{a(x,t)}{b(x,t)}-1\right|=0;$$

write $a(x,t)\prec b(x,t)$ or $b(x,t)\succ a(x,t)$ uniformly for all $t\in A$, if

$$\limsup_{x\to\infty}\sup_{t\in A}\frac{a(x,t)}{b(x,t)}\leq 1.$$

For real y, denote by $\lceil y\rceil$ the greatest integer smaller than or equal to y.

2. PRELIMINARIES AND MAIN RESULTS

We shall restrict the claim-size distribution F to some classes of heavy-tailed distributions. An important class of heavy-tailed distributions is $\mathcal{D}$, which consists of all distributions with dominated variation. A distribution $V=1-\overline{V}$ on $(-\infty,\infty)$ belongs to the class $\mathcal{D}$, if $\limsup\overline{V}(xy)/\overline{V}(x)<\infty$ for any $0<y<1$. A slightly smaller class is $\mathcal{C}$ of consistently varying distributions. A distribution V on $(-\infty,\infty)$ belongs to the class $\mathcal{C}$, if $\lim_{y\searrow 1}\liminf_{x\to\infty}\overline{V}(xy)/\overline{V}(x)=1$. Closely related distribution class is the class $\mathcal{L}$ of long-tailed distributions. A distribution V on $(-\infty,\infty)$ belongs to the class $\mathcal{L}$, if $\overline{V}(x+y)\sim\overline{V}(x)$ for any $y>0$. It is well known that the following inclusion relationships hold:

$$\mathcal{C}\subset\mathcal{L}\cap\mathcal{D}\subset\mathcal{L}$$

Furthermore, for a distribution V on $(-\infty,\infty)$, denote its upper and lower Matuszewska indices, respectively, by

$$J_V^+=-\lim_{y\to\infty}\frac{\log\overline{V}_*(y)}{\log y}$$

with $\overline{V}_*(y):=\liminf\overline{V}(xy)/\overline{V}(x)$ for $y>1$,

$$J_V^-=-\lim_{y\to\infty}\frac{\log\overline{V}^*(y)}{\log y}$$

with $\overline{V}^*(y):=\limsup\overline{V}(xy)/\overline{V}(x)$ for $y>1$. Define another important parameter $L_V=\lim_{y\searrow 1}\overline{V}_*(y)$. The following assertions are equivalent: (i) $V\in\mathcal{D}$; (ii) $L_V>0$; (iii) $J_V^+<\infty$. It also holds that $V\in\mathcal{C}$ if and only if $L_V=1$. For more details, see Bingham et al. (1987, Chapter 2.1) and Foss et al. (2011).

Many works have been devoted to ruin probabilities with constant interest rate and heavy-tailed claim sizes. After some earlier works in some noncompound renewal risk models, Tang et al. (2001) and Aleškevičienė et al. (2008) started to consider compound renewal risk models. Most of these works were conducted for i.i.d. claims and inter-arrival times with the premium process $C(t)$ a deterministic linear function. Recently, Wang et al. (2013) introduced some new and wider dependence structures (widely upper/lower orthant dependence, see the definitions below), under which they considered the noncompound renewal risk model and derived a uniformly asymptotic estimate for the finite-time ruin probability on the interval $\Lambda \cap [0, T]$ and some $T \in \Lambda$, when F belonging to the class $\mathcal{L} \cap \mathcal{D}$. We would like to remark that their result is important and interesting in view of the fact that it is in the form of exact equivalence, allowing the interest rate $r = 0$ and dropping the restriction $J_F^- > 0$, even though the uniformity is over a bounded time interval. Later, Liu et al. (2012) extended Wang et al.'s result from widely upper orthant dependent claims to upper tail asymptotically independent claims; and Yang and Wang (2012) investigated the uniform asymptotics for the finite-time and infinite-time ruin probabilities, and extended the uniformity region to the whole interval Λ. Some related results in some dependent noncompound renewal risks can be found in Yang and Wang (2010) among others. Under some dependence structures, the asymptotics for finite-time ruin probability in compound risk models have been studied by using the investigation of random sums, see, e.g., Yang et al. (2009), Zong (2010) and Yang et al. (2012) among others. A recent paper Yang et al. (2013) investigated the uniformly asymptotic behavior of finite-time and infinite-time ruin probabilities in a dependent compound Poisson risk model.

Motivated by the above results, this paper considers a dependent compound renewal risk model, and investigates the uniformly asymptotic behavior of ruin probabilities on any finite time interval or the whole interval Λ.

We firstly introduce some dependence structures, which will be commonly used in our risk model. Maulik and Resnick (2004) proposed a pairwise dependence structure. R.v.s $\{\xi_n,\ n \geq 1\}$ are said to be upper tail asymptotically independent (UTAI), if $\mathrm{P}(\xi_n > x) > 0$ for all $x \in (-\infty, \infty)$ and $n \geq 1$, and for each $1 \leq i \neq j < \infty$

$$\lim_{\min\{x,y\}\to\infty} \mathrm{P}(\xi_i > x | \xi_j > y) = 0.$$

Such a dependence structure is wider than the following widely upper orthant dependence, which was recently proposed by Wang et al. (2013). R.v.s $\{\xi_n,\ n \geq 1\}$ are said to be widely upper orthant dependent (WUOD), if there exists a finite real sequence $\{g_U^\xi(n),\ n \geq 1\}$ such that, for each $n \geq 1$ and all $x_1, \ldots, x_n$,

$$\mathrm{P}\left(\bigcap_{k=1}^{n} \{\xi_k > x_k\}\right) \leq g_U^\xi(n) \prod_{k=1}^{n} \mathrm{P}(\xi_k > x_k); \tag{4}$$

they are said to be widely lower orthant dependent (WLOD), if there exists a finite real sequence $\{g_L^\xi(n),\ n \geq 1\}$ such that, for each $n \geq 1$ and all $x_1, \ldots, x_n$,

$$\mathrm{P}\left(\bigcap_{k=1}^{n} \{\xi_k \leq x_k\}\right) \leq g_L^\xi(n) \prod_{k=1}^{n} \mathrm{P}(\xi_k \leq x_k); \tag{5}$$

and they are said to be widely orthant dependent (WOD), if they are both WUOD and WLOD.

Clearly, if r.v.s $\{\xi_n,\ n \geq 1\}$ are WUOD, then they are also UTAI. Recall that if $g_U^\xi(n) = g_L^\xi(n) = M$ for some constant $M > 0$ and all $n \geq 1$ in (4) and (5), the r.v.s $\{\xi_n,\ n \geq 1\}$ are said to be upper extended negatively dependent (UEND) and lower extended negatively dependent (LEND), respectively (see Chen et al. (2010)); they are extended negatively dependent (END), if they are both UEND and LEND (see Liu (2009)). In particular, if $g_U^\xi(n) = g_L^\xi(n) = 1$ for all $n \geq 1$ in (4) and (5), the r.v.s $\{\xi_n,\ n \geq 1\}$ are said to be upper negatively dependent (UND) and lower negatively dependent (LND), respectively (see Ebrahimi and Ghosh (1981) or Block et al. (1982)); they are negatively dependent (ND), if they are both UND and LND (see Lehmann (1966)). So, the WUOD and WLOD structures allow many common negatively dependent r.v.s, as well as some positively dependent r.v.s. See the examples in Section 3 of Wang et al. (2013).

The following lemma can be obtained directly from the definitions, which is due to Proposition 1.1 of Wang et al. (2013).

Lemma 2.1. *(1) If r.v.s $\{\xi_n,\ n \geq 1\}$ are nonnegative and WUOD with the dominating sequence $\{g_U^\xi(n),\ n \geq 1\}$, then for each $n \geq 1$*

$$\mathrm{E}\left(\prod_{k=1}^{n} \xi_k\right) \leq g_U^\xi(n) \prod_{k=1}^{n} \mathrm{E}\xi_k.$$

(2) Assume that r.v.s $\{\xi_n,\ n \geq 1\}$ are WUOD (WLOD) with the dominating sequence $\{g_U^\xi(n),\ n \geq 1\}$ ($\{g_L^\xi(n),\ n \geq 1\}$). If functions $\{f_n(\cdot),\ n \geq 1\}$ are all nondecreasing then $\{f_n(\xi_n),\ n \geq 1\}$ are still WUOD (WLOD); while if $\{f_n(\cdot),\ n \geq 1\}$ are all nonincreasing then $\{f_n(\xi_n),\ n \geq 1\}$ are WLOD (WUOD). For each case, the dominating sequence remains unchanged.

Some related asymptotic independence structures can be referred to Chen and Yuen (2009), Yang and Wang (2013), Yang and Hashorva (2013), Li (2013) among others.

Now we state the main results of the paper.

Theorem 2.1. *Consider a dependent compound renewal risk model with constant interest rate $r \geq 0$ described in Section 1. Under Assumptions H_1–H_3, assume that the individual claim sizes $\{X_n,\ n \geq 1\}$ are WOD r.v.s with common distribution $F \in \mathcal{L} \cap \mathcal{D}$ and the dominating sequences $\{g_U^X(n),\ n \geq 1\}$, $\{g_L^X(n),\ n \geq 1\}$ satisfying for any $0 < \epsilon < 1$,*

$$\lim_{n\to\infty} g_U^X(n) \big(n\overline{F}(n)\big)^\epsilon = 0; \tag{6}$$

the inter-arrival times $\{\theta_n,\ n \geq 1\}$ are WLOD r.v.s with the dominating sequence $\{g_L^\theta(n),\ n \geq 1\}$ satisfying for some $\epsilon_0 > 0$,

$$\lim_{n\to\infty} g_L^\theta(n) e^{-\epsilon_0 n} = 0; \tag{7}$$

and the claim numbers $\{N_n,\ n \geq 1\}$ *are i.i.d. r.v.s with common distribution* $G \in \mathcal{C}$. *Then, for any fixed* $T \in \Lambda$,

$$\Psi(x,t) \sim \int_{0-}^{t} \big(\nu \overline{F}(xe^{ru}) + \overline{G}(\mu^{-1}xe^{ru})\big)\lambda(\mathrm{d}u) \tag{8}$$

holds uniformly for all $t \in \Lambda \cap [0,T]$, *if either of the following two conditions is satisfied:*

Condition A. $\mathrm{E}X_1^2 < \infty$ *and for any* $0 < \epsilon < 1$,

$$\lim_{n\to\infty} g_L^X(n)e^{-\epsilon n} = 0. \tag{9}$$

Condition B. *There exists a nonnegative and nondecreasing function* $g(x)$, *such that* $\max\{g_U^X(n),\ g_L^X(n)\} \leq g(n)$ *for all* $n \geq 1$, *where* $g(x)$ *satisfies* $g(x) \to \infty$ *and* $x^{-\alpha}g(x)$ *is semi-decreasing for some* $0 < \alpha < 1$. *Here, the semi-decreasing function* $x^{-\alpha}g(x)$ *means that there exists a constant* $C > 0$ *such that* $x_1^{-\alpha}g(x_1) \geq Cx_2^{-\alpha}g(x_2)$ *for all* $x_2 > x_1 \geq 0$.

We remark that if the individual claim sizes $\{X_n,\ n \geq 1\}$ are END r.v.s and the inter-arrival times $\{\theta_n,\ n \geq 1\}$ are LEND r.v.s, then all conditions on dominating sequences in Theorem 2.1 are, clearly, satisfied, and the strong law of large numbers of END r.v.s also holds without Condition B. The obtained Theorem 2.1 significantly extends and improves the result in Zong (2010) from the following aspects. We extend the dependence structures among individual claim size and inter-arrival times to the more general case; enlarge the scope of the claim-size distribution to the class $\mathcal{L}\cap\mathcal{D}$; obtain the equivalent formula holding uniformly for t in a finite interval; drop the condition $J_F^- > 0$; and allow the constant interest rate $r \geq 0$.

Our second result investigates the uniform asymptotics for ruin probability on the whole interval Λ. We remark that it includes the estimate for infinite-time ruin probability.

Theorem 2.2. *Consider a dependent compound renewal risk model with constant interest rate* $r > 0$ *described in Section 1. Under the conditions in Theorem 2.1, if* $F \in \mathcal{C}$, $J_F^- > 0$, $J_G^- > 0$ *and (7) holds for some* $0 < \epsilon_0 < -\log \mathrm{E}e^{-rJ_F^-\theta_1}$, *then (8) holds uniformly for all* $t \in \Lambda$.

3. PROOFS OF MAIN RESULTS

In the sequel, C always represents a finite and positive constant whose value may vary from place to place. We start this section by a lemma before proving our main results. The following lemma deals with the asymptotics for the tail probability of the random sum, which plays a crucial role in the proof of our main results, and is of independent interest in its own right.

Lemma 3.1. *Let* $\{X_n,\ n \geq 1\}$ *be WOD nonnegative r.v.s with common distribution* $F \in \mathcal{L}\cap\mathcal{D}$, *finite mean* $\mu > 0$ *and the dominating sequences* $\{g_U^X(n),\ n \geq 1\}$, $\{g_L^X(n),\ n \geq 1\}$, *and let* N, *independent of* $\{X_n,\ n \geq 1\}$, *be a nonnegative integer-valued r.v. with distribution* $G \in \mathcal{C}$ *and finite mean* $\nu > 0$. *If (6) and either of Conditions A and B in Theorem 2.1 are satisfied, then*

$$\mathrm{P}\left(\sum_{k=1}^{N} X_k > x\right) \sim \nu\overline{F}(x) + \overline{G}(\mu^{-1}x). \tag{10}$$

Proof. We follow the line of Zong (2010). Denote by $S_N = \sum_{k=1}^N X_k$. We firstly estimate the upper bound of $\mathrm{P}(S_N > x)$. For any $0 < \varepsilon < 1$, any fixed positive integer m and all $x > 0$,

$$\begin{aligned}(11)\quad & \\ &\mathrm{P}(S_N > x) \\ &= \left(\sum_{n=1}^{m} + \sum_{m<n\leq(1-\varepsilon)\mu^{-1}x} + \sum_{n>(1-\varepsilon)\mu^{-1}x}\right)\mathrm{P}(S_n > x)\mathrm{P}(N = n) \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

By Lemma 2.1 of Liu et al. (2012), we have that

$$\lim_{m\nearrow\infty}\lim_{x\to\infty}\frac{I_1}{\overline{F}(x)} = \lim_{m\nearrow\infty}\sum_{n=1}^{m} n\mathrm{P}(N = n) = \nu. \tag{12}$$

As for I_2, by (6) and $F \in \mathcal{D}$, according to the large-deviation upper bound for WUOD r.v.s (see Theorem 1 of Wang et al. (2012)), there exists a positive constant C, depending only on F, such that for sufficiently large m and all $m < n \leq (1-\varepsilon)\mu^{-1}x$,

$$\begin{aligned}\mathrm{P}(S_n > x) &\leq Cn\overline{F}\big(x-(n-1)\mu\big) \\ &\leq Cn\overline{F}(\varepsilon x) \\ &\leq Cn\overline{F}(x),\end{aligned}$$

which, combined with $\nu = \mathrm{E}N < \infty$, implies that

$$\lim_{m\nearrow\infty}\limsup_{x\to\infty}\frac{I_2}{\overline{F}(x)} \leq C\lim_{m\nearrow\infty}\sum_{n=m+1}^{\infty} n\mathrm{P}(N = n) = 0. \tag{13}$$

By $G \in \mathcal{C}$, we have that

$$\begin{aligned}(14)\quad &\lim_{\varepsilon\searrow 0}\limsup_{x\to\infty}\frac{I_3}{\overline{G}(\mu^{-1}x)} \\ &\leq \lim_{\varepsilon\searrow 0}\limsup_{x\to\infty}\frac{\overline{G}((1-\varepsilon)\mu^{-1}x)}{\overline{G}(\mu^{-1}x)} = 1.\end{aligned}$$

From (11)–(14), we obtain that

$$\mathrm{P}(S_N > x) \prec \nu\overline{F}(x) + \overline{G}(\mu^{-1}x). \tag{15}$$

We next deal with the lower bound of $\mathrm{P}(S_N > x)$. For any $0 < \varepsilon < 1$, any fixed positive integer m and all $x > 0$,

$$\mathrm{P}(S_N > x) \geq \left(\sum_{n=1}^{m} + \sum_{n > (1+\varepsilon)\mu^{-1}x}\right)\mathrm{P}(S_n > x)\mathrm{P}(N = n) =: I_1 + I_4. \tag{16}$$

The first term I_1 has been treated in (12). We mainly estimate I_4 under Condition A or B in Theorem 2.1. Since $\{X_n,\ n \geq 1\}$ are nonnegative, it holds that

$$I_4 \geq \mathrm{P}(S_{\lceil (1+\varepsilon)\mu^{-1}x\rceil} > x)\overline{G}\big((1+\varepsilon)\mu^{-1}x\big). \tag{17}$$

Under Condition A, for any $v > 0$, since $\{X_n,\ n \geq 1\}$ are WLOD r.v.s, by Markov's inequality and Lemma 2.1 we have that

$$\begin{aligned} &\mathrm{P}(S_{\lceil (1+\varepsilon)\mu^{-1}x\rceil} \leq x) \\ &\leq g_L^X\big(\lceil (1+\varepsilon)\mu^{-1}x\rceil\big)e^{vx}\big(\mathrm{E}e^{-vX_1}\big)^{\lceil (1+\varepsilon)\mu^{-1}x\rceil}. \end{aligned} \tag{18}$$

According to the inequalities $1+y \leq e^y,\ y \in (-\infty, \infty)$, and $e^{-y} \leq 1 - y + y^2,\ y \geq 0$,

$$\begin{aligned} \mathrm{E}e^{-vX_1} &\leq \exp\big\{\mathrm{E}\big(e^{-vX_1} - 1\big)\big\} \\ &\leq \exp\big\{-v\mu + v^2\mathrm{E}X_1^2\big\}. \end{aligned}$$

Plugging this into (18), we obtain that for any $v > 0$ and sufficiently large x,

$$\begin{aligned} &\mathrm{P}(S_{\lceil (1+\varepsilon)\mu^{-1}x\rceil} \leq x) \\ &\leq g_L^X\big(\lceil (1+\varepsilon)\mu^{-1}x\rceil\big) \\ &\quad \times \exp\big\{-v(\varepsilon x - \mu) + v^2\mathrm{E}X_1^2\mu^{-1}(1+\varepsilon)x\big\} \\ &\leq g_L^X\big(\lceil (1+\varepsilon)\mu^{-1}x\rceil\big)\exp\left\{-\frac{v\varepsilon x}{2} + \frac{v^2\mathrm{E}X_1^2(1+\varepsilon)x}{\mu}\right\}. \end{aligned}$$

Choose $v = \mu\varepsilon/(4\mathrm{E}X_1^2(1+\varepsilon)) > 0$, then

$$\begin{aligned} &\mathrm{P}(S_{\lceil (1+\varepsilon)\mu^{-1}x\rceil} \leq x) \\ &\leq g_L^X\big(\lceil (1+\varepsilon)\mu^{-1}x\rceil\big)\exp\left\{-\frac{\mu\varepsilon^2}{16\mathrm{E}X_1^2(1+\varepsilon)}x\right\} \\ &\leq g_L^X\big(\lceil (1+\varepsilon)\mu^{-1}x\rceil\big) \\ &\quad \times \exp\left\{-\frac{\mu^2\varepsilon^2}{16\mathrm{E}X_1^2(1+\varepsilon)^2}\lceil (1+\varepsilon)\mu^{-1}x\rceil\right\} \to 0, \end{aligned} \tag{19}$$

where the last step follows from (9). Condition B implies that the strong law of large numbers for WOD r.v.s holds, see Theorem 1.4 of Wang and Cheng (2011). Thus,

$$\begin{aligned} &\mathrm{P}(S_{\lceil (1+\varepsilon)\mu^{-1}x\rceil} > x) \\ &= \mathrm{P}\left(\frac{S_{\lceil (1+\varepsilon)\mu^{-1}x\rceil}}{\lceil (1+\varepsilon)\mu^{-1}x\rceil} - \mu > \frac{x}{\lceil (1+\varepsilon)\mu^{-1}x\rceil} - \mu\right) \\ &\geq \mathrm{P}\left(\frac{S_{\lceil (1+\varepsilon)\mu^{-1}x\rceil}}{\lceil (1+\varepsilon)\mu^{-1}x\rceil} - \mu > \frac{-\varepsilon\mu + \mu^2 x^{-1}}{1+\varepsilon - \mu x^{-1}}\right) \to 1. \end{aligned} \tag{20}$$

Hence, plugging (19) or (20) into (17) and by $G \in \mathcal{C}$, we obtain that

$$\begin{aligned} &\lim_{\varepsilon \searrow 0}\liminf_{x\to\infty}\frac{I_4}{\overline{G}(\mu^{-1}x)} \\ &\geq \lim_{\varepsilon \searrow 0}\liminf_{x\to\infty}\frac{\overline{G}((1+\varepsilon)\mu^{-1}x)}{\overline{G}(\mu^{-1}x)} = 1. \end{aligned} \tag{21}$$

It follows from (16), (21) and (12) that

$$\mathrm{P}(S_N > x) \succsim \nu\overline{F}(x) + \overline{G}\big(\mu^{-1}x\big). \tag{22}$$

Therefore, the desired (10) follows from (15) and (22). This ends the proof of the lemma.

Now we start the proofs of the main results.

Proof of Theorem 2.1. Clearly, Assumption H_3 shows that $\{S_{N_n}^{(n)},\ n \geq 1\}$ are i.i.d. r.v.s with common distribution, denoted by H. By Lemma 3.1, we have that

$$\overline{H}(x) \sim \nu\overline{F}(x) + \overline{G}\big(\mu^{-1}x\big), \tag{23}$$

which, by $F \in \mathcal{L}\cap\mathcal{D}$, $G \in \mathcal{C}$ and combined with the inequality $\frac{a_1+\cdots+a_n}{b_1+\cdots+b_n} \leq \max\{\frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n}\}$, implies that $H \in \mathcal{L}\cap\mathcal{D}$. By (3) and applying Theorem 1.1 of Liu et al. (2012), which estimates the finite-time ruin probability in the noncompound risk model, we obtain that for any fixed $T \in \Lambda$,

$$\Psi(x,t) \sim \int_{0-}^{t}\overline{H}\big(xe^{ru}\big)\lambda(\mathrm{d}u) \tag{24}$$

holds uniformly for all $t \in \Lambda \cap [0, T]$. Plugging (23) into (24), we derive the desired (8). This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. The proof of the theorem is similar to that of Theorem 2.1 by using Theorem 3 of Yang and Wang (2012), and noting $J_H^- > 0$. Indeed, (23), together with $F \in \mathcal{C}$ and $G \in \mathcal{C}$, yields $H \in \mathcal{C}$; and again by (23) and the inequality $\frac{a_1+\cdots+a_n}{b_1+\cdots+b_n} \leq \max\{\frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n}\}$, for any $y > 1$,

$$\begin{aligned} \overline{H}^*(y) &= \limsup\frac{\nu\overline{F}(xy) + \overline{G}(\mu^{-1}xy)}{\nu\overline{F}(x) + \overline{G}(\mu^{-1}x)} \\ &\leq \limsup\max\left\{\frac{\overline{F}(xy)}{\overline{F}(x)}, \frac{\overline{G}(\mu^{-1}xy)}{\overline{G}(\mu^{-1}x)}\right\} \\ &\leq \max\{\overline{F}^*(y), \overline{G}^*(y)\}, \end{aligned}$$

which, by $J_F^- > 0$ and $J_G^- > 0$, leads to $J_H^- \geq \min\{J_F^-, J_G^-\} > 0$. □

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