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Convergence rate of maxima of bivariate Gaussian arrays to the Hüsler-Reiss distribution

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The limit distribution of maxima formed by a triangular array of independent and identically distributed bivariate Gaussian random vectors is the Hüsler-Reiss max-stable distribution if and only if the correlation of each vector approaches one with a certain rate. In this paper, we introduce a second-order condition on the convergence rate of this correlation. Under this condition we derive the uniform convergence rate of the distribution of normalized bivariate maxima to its ultimate limit distribution.

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1. INTRODUCTION

Let $\{(\xi_{ni}, \eta_{ni}), 1 \le i \le n, n \ge 1\}$ be a triangular array of bivariate Gaussian random vectors, which are independent for each fixed n. For a given $n \ge 1$, let $F(x, y)$ denote the bivariate Gaussian distribution function of (ξ_{ni}, η_{ni}), and the correlation coefficient of unit Gaussian distributed ξ_{ni} and η_{ni} is represented by $\rho_n, 1 \le i \le n$. The bivariate maxima $\mathbf{M}_n$ is defined componentwise by

$$\mathbf{M}_n = (M_{n1}, M_{n2}) = \left(\max_{1\le i\le n} \xi_{ni}, \max_{1\le i\le n} \eta_{ni}\right).$$

For fixed $x, y \in \mathbb{R}$, [16] showed that

$$\begin{aligned}(1.1)\qquad &\lim_{n\to\infty} \mathbb{P}\left(M_{n1} \le b_n + \frac{x}{b_n}, M_{n2} \le b_n + \frac{y}{b_n}\right)\\ &= \lim_{n\to\infty} F^n\left(b_n + \frac{x}{b_n}, b_n + \frac{y}{b_n}\right)\\ &= H_\lambda(x, y)\end{aligned}$$

if ρ_n satisfies the following Hüsler-Reiss condition (which is also the necessary condition, see Lemma 21 in [17])

$$(1.2)\qquad \lim_{n\to\infty} b_n^2(1-\rho_n) = 2\lambda^2 \quad \text{with } \lambda \in [0, \infty],$$

where the norming constant b_n satisfies

$$(1.3)\qquad \sqrt{2\pi} n^{-1} b_n \exp\left(\frac{b_n^2}{2}\right) = 1$$

and $H_\lambda(x, y)$, the Hüsler-Reiss max-stable distribution, is given by

$$\begin{aligned}(1.4)\qquad H_\lambda(x, y) = \exp\bigg(&-\Phi\left(\lambda + \frac{x-y}{2\lambda}\right)e^{-y}\\ &- \Phi\left(\lambda + \frac{y-x}{2\lambda}\right)e^{-x}\bigg),\end{aligned}$$

with $\Phi(x)$ denoting the standard Gaussian distribution. Note that from the discussion in [16],

$$H_0(x, y) = \lim_{\lambda\downarrow 0} H_\lambda(x, y) = \Lambda(\min(x, y))$$

and

$$H_\infty(x, y) = \lim_{\lambda\uparrow\infty} H_\lambda(x, y) = \Lambda(x)\Lambda(y),$$

where $\Lambda(x) = \exp(-e^{-x}), x \in \mathbb{R}$, the standard Gumbel distribution function. We say that $\{\xi_{ni}, 1 \le i \le n, n \ge 1\}$ and $\{\eta_{ni}, 1 \le i \le n, n \ge 1\}$ are asymptotic complete dependent and independent if (1.1) holds with $H_0(x, y)$ and $H_\infty(x, y)$, respectively.

Motivated by the seminal work of [16], numerous contributions on limiting distributions of extremes of bivariate triangular arrays have appeared in the literature. [15] derived general results for asymptotic dependence structures of bivariate maxima in a triangular array of independent random vectors. [14] considered the maxima of independent and identically distributed bivariate Gaussian random vectors with respect to two arbitrary directions. [9, 10] extended the results to the case of triangular arrays of independent elliptical random vectors. Related results can be found in [4, 11, 12]. For statistical applications of Hüsler-Reiss distributions, see [5].

In this paper, we are interested in the uniform convergence rate of bivariate maxima $\mathbf{M}_n$ to its ultimate Hüsler-Reiss max-stable distribution. For the univariate case, [3] considered the uniform convergence rate of maxima to its extreme value distribution by imposing some second order regular variation conditions. For the extreme value distributions of given distributions and their associated uniform

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convergence rates, we refer to [7, 8, 19, 21] and references therein. There are relatively few studies on the convergence rates of extremes under multivariate settings. [2] considered the convergence rates of bivariate extreme order statistics under second-order regular varying conditions. For bivariate Hüsler-Reiss Gaussian sequences, recently [6] considered the penultimate and ultimate convergence rate of $(n(\max_{1\le i\le n}\Phi(\xi_{ni})-1), n(\max_{1\le i\le n}\Phi(\eta_{ni})-1))$, and [13] derived the second order expansions of the distribution of normalized $\mathbf{M}_n$ under the following second order Hüsler-Reiss condition

$$\lim_{n\to\infty} b_n^2(\lambda_n-\lambda)=\alpha\in\mathbb{R} \tag{1.5}$$

with $\lambda_n=(\frac{1}{2}b_n^2(1-\rho_n))^{1/2}$ and $\lambda\in(0,\infty)$. So far, there are no results in the literature concerning the uniform convergence rate of the distribution of normalized $\mathbf{M}_n$ to its ultimate extreme value distribution. The main goal of this paper is to derive such a result, filling the gap in the current literature. Our proofs show that, for the Hüsler-Reiss Gaussian triangular array, establishing the uniform convergence rate is more technical and complicated than the higher-order expansions of distribution of normalized $\mathbf{M}_n$.

The rest of this paper is organized as follows. In Section 2, we provide the main results, and all proofs are given in Section 3. Auxiliary lemmas and their proofs are deferred to Appendix A.

2. MAIN RESULTS

In this section, we provide the main results which show that the uniform convergence rate of $F^n(b_n+x/b_n, b_n+y/b_n)$ to its ultimate Hüsler-Reiss max-stable distribution is of order $O(1/\log n)$. For notational simplicity, let

$$\Delta(F^n,H_\lambda;x,y)=F^n(b_n+x/b_n,b_n+y/b_n)-H_\lambda(x,y).$$

For the case of $\lambda\in(0,\infty)$, the following theorem establishes the uniform convergence rate under the second-order Hüsler-Reiss condition (1.5).

Theorem 1. *For the triangular array of bivariate Gaussian random vectors with each vector following distribution F, assume that the second order Hüsler-Reiss condition (1.5) holds with $\lambda_n=(\frac{1}{2}b_n^2(1-\rho_n))^{1/2}$ and $\lambda\in(0,\infty)$. Then there exist absolute constants $0<D_1<D_2$ such that*

$$\frac{D_1}{\log n}<\sup_{(x,y)\in\mathbb{R}^2}\Big|\Delta(F^n,H_\lambda;x,y)\Big|<\frac{D_2}{\log n} \tag{2.1}$$

for $n\ge 2$.

Remark 1. *(i). Condition (1.5) is equivalent to*

$$\lim_{n\to\infty}(\log n)(\lambda_n-\lambda)=\alpha/2$$

since $b_n^2\sim 2\log n$ as $n\to\infty$ due to

$$b_n=(2\log n)^{1/2}-\frac{\log\log n+\log 4\pi}{2(2\log n)^{1/2}}+o\Big(\frac{1}{(\log n)^{1/2}}\Big)$$

by (1.3), see, e.g., [18, 22].

(ii). Let $\delta_n=(\lambda_n-\lambda)^{-1}$. If (1.5) does not converge but $|\delta_n|$ and b_n^2 are the same order, then (2.1) also holds. Proofs are similar, and details are omitted here.

(iii). If $\lim_{n\to\infty} b_n^2/|\delta_n|=\infty$, with arguments similar to that of Theorem 1, we can show that

$$\frac{D_3}{|\delta_n|}<\sup_{(x,y)\in\mathbb{R}^2}\Big|\Delta(F^n,H_\lambda;x,y)\Big|<\frac{D_4}{|\delta_n|} \tag{2.2}$$

for $n\ge 2$, where $0<D_3<D_4$ are absolute constants.

(iv). Conversely, for the bivariate Gaussian triangular arrays with correlations $\{\rho_n\}$ satisfying (1.2), we have the following assertions: (a). If (2.1) holds, then every subsequence of b_n^2/δ_n, denoted by $b_{n'}^2/\delta_{n'}$, satisfies (1.5), or $b_{n'}^2$ and $|\delta_{n'}|$ are the same order; (b). If (2.2) holds, then every subsequence of b_n^2/δ_n satisfies $\lim_{n\to\infty} b_{n'}^2/|\delta_{n'}|=\infty$, or $b_{n'}^2$ and $|\delta_{n'}|$ are the same order.

Remark 2. *(i). For the case of $\lambda\in(0,\infty)$, if (1.5) does not converge, and δ_n and b_n^2 are not the same order, there may be no convergence rates for the extremes. An example is: suppose that the bivariate Gaussian triangular arrays have correlations $\{\rho_n\}$ satisfying (1.2). Furthermore, assume that $\lim_{n\to\infty} b_{2n}^2/\delta_{2n}=0$ and $\lim_{n\to\infty} b_{2n+1}^2/\delta_{2n+1}=\infty$. Hence by Theorem 1 and Remark 1 (iii), we have*

$$\frac{D_1}{\log 2n}<\sup_{(x,y)\in\mathbb{R}^2}\Big|\Delta(F^{2n},H_\lambda;x,y)\Big|<\frac{D_2}{\log 2n}$$

and

$$\frac{D_3}{\delta_{2n+1}}<\sup_{(x,y)\in\mathbb{R}^2}\Big|\Delta(F^{2n+1},H_\lambda;x,y)\Big|<\frac{D_4}{\delta_{2n+1}}$$

for $n\ge 1$.

(ii). The situation that $\lim_{n\to\infty} b_n^2/|\delta_n|=\infty$ is the one that we are not so interested in since (2.2) shows that the convergence rate $1/|\delta_n|$ is related to correlation ρ_n and parameter λ.

Theorem 1 and the following remark show that the rate of convergence with norming constant b_n given by (1.3) is optimal comparing with that with norming constant β_n denoted by (2.3) even though $b_n-\beta_n=o(1/(\log n)^{1/2})$ by Remark 1.

Remark 3. *(i). Assume that the triangular array of bivariate Gaussian random vectors satisfies the second-order Hüsler-Reiss condition (1.5) with $\alpha\in\mathbb{R}$. If the norming constant b_n is replaced by β_n given by*

$$\beta_n=2(\log n)^{1/2}-\frac{\log\log n+\log 4\pi}{2(2\log n)^{1/2}}, \tag{2.3}$$

we can prove that

$$\begin{aligned}(2.4)\quad &\widetilde{\Delta}(F^n, H_\lambda; x, y)\\ &= F^n(\beta_n + x/\beta_n, \beta_n + y/\beta_n) - H_\lambda(x, y)\\ &\sim \left[\Phi\left(\lambda + \frac{x-y}{2\lambda}\right)e^{-y} + \Phi\left(\lambda + \frac{y-x}{2\lambda}\right)e^{-x}\right]\\ &\quad \times \frac{(\log\log n)^2}{16\log n} H_\lambda(x, y)\end{aligned}$$

as $n \to \infty$ for all $x, y \in \mathbb{R}$, from which shows that the convergence rate is no better than $(\log\log n)^2/(16\log n)$.

(ii). Under the second-order Hüsler-Reiss condition (1.5) with $\alpha = \pm\infty$, we have

$$\begin{aligned}(2.5)\quad &\widetilde{\Delta}(F^n, H_\lambda; x, y)\\ &= \left[\frac{(\log\log n)^2}{16\log n}\left(\Phi\left(\lambda + \frac{x-y}{2\lambda}\right)e^{-y}\right.\right.\\ &\quad \left. + \Phi\left(\lambda + \frac{y-x}{2\lambda}\right)e^{-x}\right)(1 + o(1))\\ &\quad \left. - (\lambda_n - \lambda)2e^{-x}\varphi\left(\lambda + \frac{x-y}{2\lambda}\right)(1 + o(1))\right] H_\lambda(x, y)\end{aligned}$$

as $n \to \infty$ for all $x, y \in \mathbb{R}$, where the norming constant β_n is given by (2.3). By (2.5), we can see that the convergence rate is no better than $\max\{(\log\log n)^2/(16\log n), |\lambda_n - \lambda|\}$.

For the two extreme cases $\lambda = 0$ and $\lambda = \infty$, we need to deal with them separately. For the case of $\lambda = \infty$, the results are stated as follows.

Theorem 2. *Let norming constant b_n be given by (1.3). For $\rho_n \in [-1, 1)$,*

(i). assertion (2.1) holds if $\rho_n \in [-1, 0]$.
(ii). if $\rho_n \in (0, 1)$, assume that (1.2) holds with $\lambda = \infty$ and $(\log b_n)/((1 - \rho_n)b_n^2) \to 0$ as $n \to \infty$, then (2.1) also holds.

For the case of $\lambda = 0$, we have the following results.

Theorem 3. *Let norming constant b_n be given by (1.3). For $\rho_n \in (0, 1]$,*

(i). assertion (2.1) holds if $\rho_n \equiv 1$ for all large n.
(ii). if $\rho_n \in (0, 1)$, assume that $b_n^{10}(1 - \rho_n) \to c \in [0, \infty)$ as $n \to \infty$, then (2.1) also holds.

Remark 4. *For the case of $\rho_n \in (0, 1)$, the proofs of Theorem 2 and Theorem 3 depend heavily on Berman's inequality. In order to derive the upper bound based on Berman's inequality, some sufficient conditions are needed. The condition in Theorem 2 requires that $(1 - \rho_n)b_n^2$ converges to infinity faster than $\log b_n$; The condition imposed on Theorem 3(ii) implies that (1.2) holds with $\lambda = 0$.*

3. PROOFS

The aim of this section is to prove our main results. In the sequel, we rewrite $H_\lambda(x, y)$ as

$$(3.1)\quad H_\lambda(x, y) = \exp\left(-e^{-x} - \int_y^\infty \Phi\left(\lambda + \frac{x-z}{2\lambda}\right)e^{-z}dz\right).$$

For notational simplicity, throughout this paper let

$$u_n(z) = b_n + z/b_n, \quad z \in \mathbb{R},$$
$$A_n = b_n^2\left(\lambda_n\left(1 - \frac{\lambda_n^2}{b_n^2}\right)^{-\frac{1}{2}} - \lambda\right),$$
$$B_n = \frac{1}{2}b_n^2\left(\frac{1}{\lambda_n}\left(1 - \frac{\lambda_n^2}{b_n^2}\right)^{-\frac{1}{2}} - \frac{1}{\lambda}\right)$$

and

$$C_n = \lambda_n\left(1 - \frac{\lambda_n^2}{b_n^2}\right)^{-\frac{1}{2}},$$

where the norming constant b_n is given by (1.3) and $\lambda_n = (b_n^2(1-\rho_n)/2)^{1/2}$. If the second-order Hüsler-Reiss condition (1.5) holds with $\lambda \in (0, \infty)$, it is easy to check that

$$(3.2)\quad A_n \to \frac{1}{2}\lambda^3 + \alpha, \quad B_n \to -\frac{1}{2}\alpha\lambda^{-2} + \frac{1}{4}\lambda, \quad C_n \to \lambda$$

as $n \to \infty$.

Proof of Theorem 1. By Lemma 1 and Lemma 3 in Appendix A and $b_n^2 \sim 2\log n$ as $n \to \infty$, for fixed $x, y \in \mathbb{R}$ we have

$$\begin{aligned}\Delta(F^n, H_\lambda; x, y) \sim &\left(e^{-x}\left(1 + x + \frac{1}{2}x^2\right)\right.\\ &\left. + \frac{1}{2}\kappa_3(x, y) + \kappa_1(x, y)\right)\\ &\times (2\log n)^{-1}H_\lambda(x, y)\end{aligned}$$

as $n \to \infty$, where $\kappa_1(x, y)$ and $\kappa_3(x, y)$ respectively are given by (A.4) and (A.13) in Appendix A. Hence there exists an absolute constant $D_1 > 0$ such that

$$\sup_{(x,y)\in\mathbb{R}^2} |\Delta(F^n, H_\lambda; x, y)| \geq \frac{D_1}{\log n}$$

for $n \geq 2$. Thus we need to show further that

$$(3.3)\quad \sup_{(x,y)\in\mathbb{R}^2} |\Delta(F^n, H_\lambda; x, y)| \leq \frac{D_2}{\log n}$$

for $n \geq 2$, where D_2 is an absolute constant. By Lemma 4 in Appendix A, it suffices to prove the following inequalities:

$$(3.4)\quad \sup_{(x,y)\in[-c_n,d_n]\times[-c_n,d_n]} |\Delta(F^n, H_\lambda; x, y)| \leq \mathbb{D}_2 b_n^{-2},$$

$$(3.5)\qquad \sup_{(x,y)\in[-c_n,d_n]\times[d_n,\infty)} |\Delta(F^n, H_\lambda; x,y)| \le \mathbb{D}_3 b_n^{-2},$$

$$(3.6)\qquad \sup_{(x,y)\in[d_n,\infty)\times[d_n,\infty)} |\Delta(F^n, H_\lambda; x,y)| \le \mathbb{D}_4 b_n^{-2}$$

for $n \ge n_0$ since both

$$(3.7)\qquad \sup_{(x,y)\in\mathbb{R}\times(-\infty,-c_n]} |\Delta(F^n, H_\lambda; x,y)| \le \mathbb{D}_1 b_n^{-2}$$

and

$$(3.8)\qquad \sup_{(x,y)\in[d_n,\infty)\times[-c_n,d_n]} |\Delta(F^n, H_\lambda; x,y)| \le \mathbb{D}_3 b_n^{-2}$$

also hold by the arguments similar to those used in (A.20) and (3.5), where $\mathbb{D}_i > 0$, $2 \le i \le 4$, are absolute constants, and c_n and d_n are given by Lemma 2 in Appendix A, i.e.,

$$c_n = \log\log b_n > 0, \quad d_n = -\log\log\frac{b_n^2}{b_n^2-1} > 0$$

for $n \ge n_0$. Note that $x \ge -c_n$ implies

$$u_n(x) \ge b_n - \frac{c_n}{b_n} = b_n\left(1 - \frac{\log\log b_n^2}{b_n^2}\right) > 0, \quad n \ge n_0.$$

So, the desired upper bound (3.3) can be obtained by (3.4)–(3.8) and (A.20).

For the rest of the proof, let $\mathbb{C}_i$, $7 \le i \le 13$, stand for absolute positive constants.

For $(x,y) \in [-c_n,\infty)\times[-c_n,\infty)$, let $\psi_n(x,y) = 1 - F(u_n(x), u_n(y))$, then

$$(3.9)\qquad n\log F(u_n(x),u_n(y)) = -n\psi_n(x,y) - R_n(x,y),$$

where

$$0 < R_n(x,y) < \frac{n\psi_n^2(x,y)}{2(1-\psi_n(x,y))}$$

due to

$$(3.10)\qquad -z - \frac{z^2}{2(1-z)} < \log(1-z) < -z, \quad 0<z<1.$$

By (1.3), (A.17) and (A.19),

$$\begin{aligned}
&\sup_{(x,y)\in[-c_n,\infty)\times[-c_n,\infty)} \psi_n(x,y)\\
&\le 1 - F(u_n(-c_n), u_n(-c_n))\\
&= n^{-1}\int_{-c_n}^{\infty} \Phi\left(\frac{u_n(-c_n) - \rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right) e^{-z}\exp\left(-\frac{z^2}{2b_n^2}\right)dz\\
&\quad + 1 - \Phi(u_n(-c_n))\\
&\le \frac{1}{\sqrt{2\pi}}\left(b_n - \frac{c_n}{b_n}\right)^{-1}\exp\left(-\frac{(b_n - \frac{c_n}{b_n})^2}{2}\right) + n^{-1}\int_{-c_n}^{\infty} e^{-z}dz\\
&\le n^{-1}(\log b_n^2)\left(1 + \left(1 - b_n^{-2}\log\log b_n^2\right)^{-1}\right)\\
&< \mathbb{C}_7 < 1
\end{aligned}$$

for $n \ge n_0$, which implies

$$\begin{aligned}
0 &< \sup_{(x,y)\in[-c_n,\infty)\times[-c_n,\infty)} R_n(x,y)\\
&< \sup_{(x,y)\in[-c_n,\infty)\times[-c_n,\infty)} \frac{n\psi_n^2(x,y)}{2(1-\psi_n(x,y))}\\
&< b_n^{-2}
\end{aligned}$$

for $n \ge n_0$. Hence by $e^x \ge 1+x, x\in\mathbb{R}$ we have

$$(3.11)\qquad 1 - \exp\Big(-R_n(x,y)\Big) \le R_n(x,y) < b_n^{-2}$$

for $n \ge n_0$. Hence by (3.11) we have

$$\begin{aligned}
(3.12)\quad |\Delta(F^n, H_\lambda; x,y)| &< H_\lambda(x,y)N_n(x,y)\,|Q_n(x,y)-1|\\
&\quad + H_\lambda(x,y)\,|N_n(x,y)-1|\\
&< H_\lambda(x,y)\,|Q_n(x,y)-1| + b_n^{-2}
\end{aligned}$$

for $n > n_0$, where

$$\begin{aligned}
Q_n(x,y) = \exp\Big(&-n\psi_n(x,y) + \Phi\left(\lambda + \frac{x-y}{2\lambda}\right)e^{-y}\\
&+ \Phi\left(\lambda + \frac{y-x}{2\lambda}\right)e^{-x}\Big)
\end{aligned}$$

and

$$N_n(x,y) = \exp(-R_n(x,y)).$$

Note that

$$(3.13)\quad -n\Big(1-\Phi(u_n(x))\Big) + e^{-x} = (1+b_n^{-2}x)^{-1}e^{-x}Z_n(x)$$

with

$$Z_n(x) = -\exp\left(-\frac{x^2}{2b_n^2}\right)\Big(1-\theta_n b_n^{-2}(1+b_n^{-2}x)^{-2}\Big) + 1 + b_n^{-2}x,$$

where $0 < \theta_n < 1$, cf., [7]. By arguments similar to those used in [7], we have

$$b_n^{-2}x < Z_n(x) < b_n^{-2}\Big(2^{-1}x^2 + (1+b_n^{-2}x)^{-2} + x\Big)$$

by $1 - z < e^{-z} < 1$ for $z > 0$, which implies

$$|Z_n(x)| < b_n^{-2}\Big(2^{-1}x^2 + (1+b_n^{-2}x)^{-2} + |x|\Big).$$

Combining with (3.13), we have

$$(3.14)\quad \left|-n\Big(1-\Phi(u_n(x))\Big) + e^{-x}\right| < \mathbb{C}_8 b_n^{-2} e^{-x}\left(\frac{x^2}{2} + |x| + \mathbb{C}_9\right)$$

for $n \ge n_0$, if $x \ge -c_n$.

Similarly, for $x \ge -c_n$ we have

$$(3.15)\quad \left|-\int_y^{\infty}\Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right)e^{-z}\exp\left(-\frac{z^2}{2b_n^2}\right)dz\right.$$

$$+\int_y^\infty \Phi\left(\lambda+\frac{x-z}{2\lambda}\right)e^{-z}dz\Bigg|$$

$$<\int_y^\infty\left|\Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right)-\Phi\left(\lambda+\frac{x-z}{2\lambda}\right)\right|e^{-z}dz$$

$$+b_n^{-2}e^{-y}\left(\frac{y^2}{2}+y+1\right)$$

by using $|e^{-x}-1|<x$ for $x>0$.

First, we prove (3.4). Combining (A.9), (3.14) and (3.15), we have

(3.16)

$$\left|-n\psi_n(x,y)+\Phi\left(\lambda+\frac{x-y}{2\lambda}\right)e^{-y}+\Phi\left(\lambda+\frac{y-x}{2\lambda}\right)e^{-x}\right|$$

$$\leq\left|-n\left(1-\Phi(u_n(x))\right)+e^{-x}\right|+\left|\int_y^\infty\Phi\left(\lambda+\frac{x-z}{2\lambda}\right)e^{-z}dz\right.$$

$$\left.-\int_y^\infty\Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right)e^{-z}\exp\left(-\frac{z^2}{2b_n^2}\right)dz\right|$$

$$\leq\mathbb{C}_8 b_n^{-2}e^{-x}\left(\frac{x^2}{2}+|x|+\mathbb{C}_9\right)+b_n^{-2}e^{-y}\left(\frac{y^2}{2}+y+1\right)$$

$$+b_n^{-2}\Big(e^{-y}(\mathbb{C}_1|y|+\mathbb{C}_2)+\mathbb{C}_3e^{-x}|x|+\mathbb{C}_4\Big)$$

$$\leq b_n^{-2}\left[\mathbb{C}_8e^{-x}\left(\frac{x^2}{2}+\left(1+\frac{\mathbb{C}_3}{\mathbb{C}_8}\right)|x|+\mathbb{C}_9\right)\right.$$

$$\left.+e^{-y}\left(\frac{y^2}{2}+(\mathbb{C}_1+1)|y|+\mathbb{C}_2+1\right)+\mathbb{C}_4\right]$$

for $n\geq n_0$. Note that

$$(3.17)\qquad e^{-x}x^2\leq 4,\quad e^{-x}x\leq 1,\quad \text{for } x>0$$

and by using (3.16) we have

$$\left|-n\psi_n(x,y)+\Phi\left(\lambda+\frac{x-y}{2\lambda}\right)e^{-y}+\Phi\left(\lambda+\frac{y-x}{2\lambda}\right)e^{-x}\right|$$

$$\leq\mathbb{C}_{10}b_n^{-2}\leq 1$$

for $n\geq n_0$ and any $(x,y)\in[0,d_n]\times[0,d_n]$. Hence for $(x,y)\in[0,d_n]\times[0,d_n]$,

$$(3.18)\qquad H_\lambda(x,y)\left|Q_n(x,y)-1\right|$$

$$\leq H_\lambda(x,y)\Bigg|-n\psi_n(x,y)+\Phi\left(\lambda+\frac{x-y}{2\lambda}\right)e^{-y}$$

$$+\Phi\left(\lambda+\frac{y-x}{2\lambda}\right)e^{-x}\Bigg|\Bigg(1+\exp\Bigg|-n\psi_n(x,y)$$

$$+\Phi\left(\lambda+\frac{x-y}{2\lambda}\right)e^{-y}+\Phi\left(\lambda+\frac{y-x}{2\lambda}\right)e^{-x}\Bigg|\Bigg)$$

$$\leq\mathbb{C}_{10}(e+1)b_n^{-2}$$

since $|e^x-1|\leq|x|(e^{|x|}+1), x\in\mathbb{R}$.

For the case of $(x,y)\in[-c_n,0]\times[-c_n,0]$, (3.16) implies

$$\left|-n\psi_n(x,y)+\Phi\left(\lambda+\frac{x-y}{2\lambda}\right)e^{-y}+\Phi\left(\lambda+\frac{y-x}{2\lambda}\right)e^{-x}\right|\leq 1$$

for $n\geq n_0$. For $(x,y)\in[-c_n,0]\times[-c_n,0]$,

$$(3.19)\quad H_\lambda(x,y)\left|Q_n(x,y)-1\right|$$

$$\leq(e+1)b_n^{-2}\left[\mathbb{C}_8e^{-1-\frac{x^2}{2}}\left(\frac{x^2}{2}+\left(1+\frac{\mathbb{C}_3}{\mathbb{C}_8}\right)|x|+\mathbb{C}_9\right)\right.$$

$$\left.+e^{-1-\frac{y^2}{2}}\left(\frac{y^2}{2}+(\mathbb{C}_1+1)y+\mathbb{C}_2+1\right)+\mathbb{C}_4\right]$$

$$\leq\mathbb{C}_{10}(e+1)b_n^{-2}$$

for $n\geq n_0$ by noting that $e^{-t}>1-t+\frac{t^2}{2}$ for $t<0$, and $\frac{t^2}{2}\exp(-\frac{t^2}{2})\leq 1$ and $t\exp(-\frac{t^2}{2})\leq 1$ for $t>0$.

By arguments similar to those used in (3.18) and (3.19), for $n\geq n_0$ we have

$$(3.20)\quad H_\lambda(x,y)\left|Q_n(x,y)-1\right|\leq\mathbb{C}_{10}(e+1)b_n^{-2}$$

if $(x,y)\in[0,d_n]\times[-c_n,0]$ or $(x,y)\in[-c_n,0]\times[0,d_n]$.

Combining (3.18)–(3.20) and (3.12), we get

$$\sup_{(x,y)\in[-c_n,d_n]\times[-c_n,d_n]}\left|F^n(u_n(x),u_n(y))-H_\lambda(x,y)\right|\leq\mathbb{D}_2b_n^{-2},$$

which completes the proof of (3.4).

Second, we consider the case in which $(x,y)\in[-c_n,d_n]\times[d_n,\infty)$. By (A.10), (3.14) and (3.15), for all $y\in[d_n,\infty)$ we have

$$\left|-n\psi_n(x,y)+\Phi\left(\lambda+\frac{x-y}{2\lambda}\right)e^{-y}+\Phi\left(\lambda+\frac{y-x}{2\lambda}\right)e^{-x}\right|$$

$$\leq b_n^{-2}\left(\mathbb{C}_8e^{-x}\left(\frac{x^2}{2}+|x|+\mathbb{C}_9\right)+(\mathbb{C}_5+4)\right)$$

for $n\geq n_0$. Thus by the arguments similar to those used in (3.18) and (3.19), we have

$$(3.21)\qquad H_\lambda(x,y)\left|Q_n(x,y)-1\right|\leq\mathbb{C}_{11}b_n^{-2}$$

for $n\geq n_0$ if $(x,y)\in[0,d_n]\times[d_n,\infty)$, and

$$(3.22)\qquad H_\lambda(x,y)\left|Q_n(x,y)-1\right|\leq\mathbb{C}_{12}b_n^{-2}$$

for $n\geq n_0$ if $(x,y)\in[-c_n,0]\times[d_n,\infty)$. Combining (3.12), (3.21) and (3.22), we can get (3.5).

Finally, we prove (3.6). Note that $d_n=-\log\log\frac{b_n^2}{b_n^2-1}$, we have

$$(3.23)\qquad \sup_{(x,y)\in[d_n,\infty)\times[d_n,\infty)}\Big(1-H_\lambda(x,y)\Big)$$

$$\leq 1-H_\lambda(d_n,d_n)$$

$$\leq b_n^{-2}+1-\exp\left(-\int_{d_n}^\infty e^{-z}dz\right)=2b_n^{-2}.$$

By using (3.11) and $e^z > 1+z, z\in\mathbb{R}$, we have

$$\text{(3.24)}\qquad \begin{aligned}&\sup_{(x,y)\in[d_n,\infty)\times[d_n,\infty)}\Big(1-F^n(u_n(x),u_n(y))\Big)\\ &\le n\psi_n(d_n,d_n)+R_n(d_n,d_n)\\ &\le (1+\mathbb{C}_{13})b_n^{-2},\end{aligned}$$

where the last inequality is due to

$$\begin{aligned}&n\psi_n(d_n,d_n)\\ &=\int_{d_n}^{\infty}\Phi\left(\frac{u_n(d_n)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right)e^{-z}\exp\left(-\frac{z^2}{2b_n^2}\right)dz\\ &\quad+n\Big(1-\Phi(u_n(d_n))\Big)\\ &<n\left(b_n+\frac{d_n}{b_n}\right)^{-1}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(b_n+\frac{d_n}{b_n})^2}{2}\right)+\int_{d_n}^{\infty}e^{-z}dz\\ &=\left(1+\frac{d_n}{b_n^2}\right)^{-1}e^{-d_n}\exp\left(-\frac{d_n^2}{2b_n^2}\right)+e^{-d_n}\\ &<\mathbb{C}_{13}b_n^{-2}\end{aligned}$$

for $n\ge n_0$ by using (A.17), (A.19) and

$$\begin{aligned}b_n^2e^{-d_n}&=b_n^2\left(-\log(1-b_n^{-2})\right)\\ &<b_n^2\left(b_n^{-2}+\frac{b_n^{-4}}{2(1-b_n^{-2})}\right)\\ &=1+\frac{1}{2(b_n^2-1)}\end{aligned}$$

as $-z-\frac{z^2}{2(1-z)}<\log(1-z), 0<z<1$.

Combining (3.23) with (3.24), we have

$$\sup_{(x,y)\in[d_n,\infty)\times[d_n,\infty)}\Big|\Delta(F^n,H_\lambda;x,y)\Big|\le \mathbb{D}_4 b_n^{-2}$$

for $n\ge n_0$, which completes the proof of (3.6). The proof of Theorem 1 is complete. □

Proof of Theorem 2. (i). For the case of $\rho_n\in[-1,0]$, we first consider two special cases, $\rho_n\equiv -1$ and $\rho_n\equiv 0$ respectively, then extend the result to the general case $\rho_n\in[-1,0]$ by using Slepian's Lemma.

Note that, for $\rho_n\equiv-1$ and $\rho_n\equiv 0$, for the upper bound of (2.1), by Lemma 4 in Appendix A we only need to check that

$$\text{(3.25)}\qquad \sup_{(x,y)\in[-c_n,\infty)\times[-c_n,\infty)}\Big|\Delta(F^n,H_\infty;x,y)\Big|<\mathbb{D}_5 b_n^{-2}$$

for large n, where $c_n=\log\log b_n^2$, and $\mathbb{D}_5$ is an absolute positive constant.

Let (ξ,η) be a bivariate Gaussian random vector with correlation $\rho_n\equiv 0$. By (A.17) and (A.19), for $(x,y)\in[-c_n,\infty)\times[-c_n,\infty)$ we have

$$\text{(3.26)}\qquad \begin{aligned}&n\,\mathbb{P}(\xi>u_n(x),\eta>u_n(y))\\ &<\frac{1}{\sqrt{2\pi}b_n\exp(\frac{b_n^2}{2})}\left(1+\frac{x}{b_n^2}\right)^{-1}\left(1+\frac{y}{b_n^2}\right)^{-1}e^{-x-y}\\ &<b_n^{-2}\frac{b_n(\log b_n^2)^2(1-\frac{\log\log b_n^2}{b_n^2})^{-2}}{\sqrt{2\pi}\exp(\frac{b_n^2}{2})}\\ &<b_n^{-2}\end{aligned}$$

for large n.

By using (3.26) and (3.14) and arguments similar to those used in [7], for $(x,y)\in[-c_n,\infty)\times[-c_n,\infty)$ we have

$$\text{(3.27)}\qquad \begin{aligned}&\left|-n(1-F(u_n(x),u_n(y)))+e^{-x}+e^{-y}\right|\\ &\le\left|-n(1-\Phi(u_n(x)))+e^{-x}\right|+\left|-n(1-\Phi(u_n(y)))+e^{-y}\right|\\ &\quad+n\,\mathbb{P}(\xi>u_n(x),\eta>u_n(y))\\ &\le b_n^{-2}\left(1+\mathbb{C}_8e^{-x}\left(x^2/2+|x|+\mathbb{C}_9\right)\right.\\ &\quad\left.+\mathbb{C}_8e^{-y}\left(y^2/2+|y|+\mathbb{C}_9\right)\right)\end{aligned}$$

for large n. Note that, by (3.27) and (3.17), for large n we have

$$\begin{aligned}&\sup_{(x,y)\in[-c_n,0]\times[0,\infty)}\left|-n(1-F(u_n(x),u_n(y)))+e^{-x}+e^{-y}\right|\\ &\le b_n^{-2}\Big(\mathbb{C}_8(\log b_n^2)\left((\log\log b_n^2)^2/2+\log\log b_n^2+\mathbb{C}_9\right)\\ &\quad+\mathbb{C}_8\left(3+\mathbb{C}_9\right)+1\Big)\le 1.\end{aligned}$$

Obviously, for $\rho_n\equiv 0$,

$$\begin{aligned}Q_n(x,y)=\exp\Big(&-n(1-\Phi(u_n(x)))+e^{-x}-n(1-\Phi(u_n(y)))\\ &+e^{-y}+n\,\mathbb{P}(\xi>u_n(x),\eta>u_n(y))\Big).\end{aligned}$$

So, for $(x,y)\in[-c_n,0)\times[0,\infty)$,

$$\begin{aligned}&H_\infty(x,y)\left|Q_n(x,y)-1\right|\\ &\le H_\infty(x,y)\left|-n(1-F(u_n(x),u_n(y)))+e^{-x}+e^{-y}\right|\\ &\quad\times\left(\exp\left|-n(1-F(u_n(x),u_n(y)))+e^{-x}+e^{-y}\right|+1\right)\\ &\le b_n^{-2}(e+1)\Big(1+\mathbb{C}_8\left(3+\mathbb{C}_9\right)\\ &\quad+\mathbb{C}_8\exp\left(-e^{-x}-x\right)\left(x^2/2+|x|+\mathbb{C}_9\right)\Big)\\ &\le b_n^{-2}(e+1)\Big(1+\mathbb{C}_8\left(3+\mathbb{C}_9\right)\\ &\quad+\mathbb{C}_8\exp\left(-1-x^2/2\right)\left(x^2/2+|x|+\mathbb{C}_9\right)\Big)\\ &\le(5\mathbb{C}_8+2\mathbb{C}_8\mathbb{C}_9+1)\,(e+1)b_n^{-2}\end{aligned}$$

for large n. Similarly, for large n we have

$$\begin{aligned}&\sup_{(x,y)\in[0,\infty)\times[-c_n,0]}H_\infty(x,y)\left|Q_n(x,y)-1\right|\\ &\le(5\mathbb{C}_8+2\mathbb{C}_8\mathbb{C}_9+1)\,(e+1)b_n^{-2},\end{aligned}$$

$$\sup_{(x,y)\in[0,\infty)\times[0,\infty)} H_\infty(x,y)\,|Q_n(x,y)-1| \le (6\mathbb{C}_8+2\mathbb{C}_8\mathbb{C}_9+1)\,(e+1)b_n^{-2}$$

and

$$\sup_{(x,y)\in[-c_n,0]\times[-c_n,0]} H_\infty(x,y)\,|Q_n(x,y)-1| \le (4\mathbb{C}_8+2\mathbb{C}_8\mathbb{C}_9+1)\,(e+1)b_n^{-2}.$$

Combining above with (3.12), we can get (3.25), hence the upper bound in (2.1) is derived. For the lower bound of (2.1) as $\rho_n \equiv 0$, by (A.16) and arguments similar to those used in [7], we have

$$\begin{aligned}
&1-F(u_n(x),u_n(y))\\
&=n^{-1}\left[e^{-x}+e^{-y}-b_n^{-2}\left(e^{-x}\left(x^2/2+x+1\right)\right.\right.\\
&\quad\left.\left.+e^{-y}\left(y^2/2+y+1\right)+O(b_n^{-2})\right)\right]
\end{aligned}$$

for large n, where x, y are fixed real constants. Hence,

$$\begin{aligned}
&F^n(u_n(x),u_n(y))-H_\infty(x,y)\\
&=b_n^{-2}H_\infty(x,y)\left(e^{-x}\left(x^2/2+x+1\right)\right.\\
&\quad\left.+e^{-y}\left(y^2/2+y+1\right)+O(b_n^{-2})\right)
\end{aligned}\tag{3.28}$$

for large n, which implies the left hand side inequality in (2.1). From whence (2.1) is derived for $\rho_n \equiv 0$.

Next we consider the case of $\rho_n \equiv -1$. For $(x,y) \in [-c_n,\infty)\times[-c_n,\infty)$, noting that for large n we have

$$\begin{aligned}
Q_n(x,y)=\exp\big(&-n(1-\Phi(u_n(x)))+e^{-x}\\
&-n(1-\Phi(u_n(y)))+e^{-y}\big)
\end{aligned}$$

as $\mathbb{P}(\xi > u_n(x), \eta > u_n(y)) = 0$ for large n. By arguments similar to that of the case of $\rho_n \equiv 0$, we can derive (3.25). Finally the lower bound of (2.1) can be derived by noting that (3.28) also holds if $\rho_n \equiv -1$.

Let us now turn to the general case of $\rho_n \in [-1, 0]$. We just proved that (2.1) holds for $\rho_n \equiv -1$ and $\rho_n \equiv 0$, respectively. Hence by Slepian's Lemma, one can check that (2.1) also holds if $\rho_n \in [-1, 0]$.

(ii). For the case of $\rho_n \in (0, 1)$. If $(x,y) \in [-c_n,\infty)\times[-c_n,\infty)$, by Berman's inequality in [18], we have

$$\begin{aligned}
&|F^n(u_n(x),u_n(y))-(\Phi(u_n(x))\Phi(u_n(y)))^n|\\
&\le \mathbb{C}_{14}b_n^{-2}\exp\left(-\frac{(1-\rho_n)b_n^2}{4}+3\log b_n+2\log\log b_n^2\right)\\
&\le b_n^{-2}
\end{aligned}\tag{3.29}$$

for large n due to $\lim_{n\to\infty}(1-\rho_n)^{-1}b_n^{-2}\log b_n = 0$, where $\mathbb{C}_{14}$ is an absolute positive constant.

From the proof of (A.21), it shows that

$$\sup_{(x,y)\in(-\infty,-c_n]\times\mathbb{R}} F^n(u_n(x),u_n(y)) < \mathbb{C}_6 b_n^{-2}\tag{3.30}$$

for large n if $\rho_n \in [-1, 1]$, hence

$$\sup_{(x,y)\in(-\infty,-c_n]\times\mathbb{R}} \left|F^n(u_n(x),u_n(y))-(\Phi(u_n(x))\Phi(u_n(y)))^n\right| < 2\mathbb{C}_6 b_n^{-2}\tag{3.31}$$

for large n. Similarly,

$$\sup_{(x,y)\in\mathbb{R}\times(-\infty,-c_n]} \left|F^n(u_n(x),u_n(y))-(\Phi(u_n(x))\Phi(u_n(y)))^n\right| < 2\mathbb{C}_6 b_n^{-2}\tag{3.32}$$

for large n. Combining (3.29), (3.31), (3.32), and (2.1) for the case of $\rho_n \equiv 0$, the upper bound of (2.1) is derived if $\rho_n \in (0, 1)$.

Next we derive the lower bound in (2.1). For fixed $x, y \in \mathbb{R}$, by Mills' ratio we have

$$1-\Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right) < \frac{\exp\left(-\frac{(1-\rho_n)b_n^2}{4}-\frac{x}{1+\rho_n}+\frac{\rho_n z}{1+\rho_n}\right)}{\sqrt{2\pi}\left(\lambda_n+\frac{x-z}{2\lambda_n}+\frac{\lambda_n z}{b_n^2}\right)}\tag{3.33}$$

if $y < z < 4\log b_n$. Hence,

$$\int_y^{4\log b_n}\left(1-\Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right)\right)e^{-z}dz = O(b_n^{-4})\tag{3.34}$$

for large n. Similarly, for large n we have

$$\int_y^{4\log b_n}\left(1-\Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right)\right)e^{-z}z^2dz = O(b_n^{-4}).\tag{3.35}$$

Note that

$$\int_{4\log b_n}^{\infty}\Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right)e^{-z}\left(1-\frac{z^2}{2b_n^2}\right)dz = O(b_n^{-4})$$

for large n. Hence by (3.34) and (3.35), for large n we have

$$\begin{aligned}
&\int_y^{\infty}\Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right)e^{-z}\exp\left(-\frac{z^2}{2b_n^2}\right)dz\\
&=\int_y^{4\log b_n}\left(\Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right)-1\right)e^{-z}dz\\
&\quad+2^{-1}b_n^{-2}\int_y^{4\log b_n}\left(1-\Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right)\right)e^{-z}z^2dz\\
&\quad+\int_{4\log b_n}^{\infty}\Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right)e^{-z}\left(1-\frac{z^2}{2b_n^2}\right)dz\\
&\quad+\int_y^{4\log b_n}e^{-z}\left(1-\frac{z^2}{2b_n^2}\right)dz+O(b_n^{-4})
\end{aligned}$$

$$= e^{-y} - b_n^{-2} e^{-y} \left(\frac{y^2}{2} + y + 1 \right) + O(b_n^{-4}).$$

Combining with (A.16), we have

$$\begin{aligned} &1 - F(u_n(x), u_n(y)) \\ &= n^{-1} \int_y^\infty \Phi \left(\frac{u_n(x) - \rho_n u_n(z)}{\sqrt{1-\rho_n^2}} \right) e^{-z} \exp\left(-\frac{z^2}{2b_n^2} \right) dz \\ &\quad + 1 - \Phi(u_n(x)) \\ &= n^{-1} \left[e^{-x} + e^{-y} - b_n^{-2} \left(e^{-x} \left(\frac{x^2}{2} + x + 1 \right) \right.\right. \\ &\quad \left.\left. + e^{-y} \left(\frac{y^2}{2} + y + 1 \right) + O(b_n^{-2}) \right) \right] \end{aligned}$$

for large n, so that the lower bound in (2.1) can be derived if $\rho_n \in (0,1)$, which completes the proof. $\square$

Proof of Theorem 3. (i). For the case of $\rho_n \equiv 1$. Note that

$$F^n(u_n(x), u_n(y)) = \Phi^n(b_n + \min(x,y)/b_n)$$

and $H_0(x,y) = \Lambda(\min(x,y))$. Hence by the arguments provided by [7], we can derive (2.1) if $\rho_n \equiv 1$.

(ii). If $\rho_n \in (0,1)$ such that $b_n^{10}(1-\rho_n) \to c \in [0,\infty)$ as $n \to \infty$, this implies that (1.2) holds with $\lambda = 0$. First, note that, for $(x,y) \in [-c_n,\infty) \times [-c_n,\infty)$ and large n, by Berman's inequality in [20], we have

$$\begin{aligned} &|F^n(u_n(x), u_n(y)) - \Phi^n(b_n + \min(x,y)/b_n)| \\ &< \mathbb{C}_{15} n \left(\frac{\pi}{2} - \arcsin \rho_n \right) \exp\left(-\frac{u_n^2(x) + u_n^2(y)}{4} \right) \\ &< 2\sqrt{2}\mathbb{C}_{15} n (1-\rho_n)^{\frac{1}{2}} \exp\left(-\frac{b_n^2 + x + y}{2} \right) \\ &< 8\sqrt{\pi}\mathbb{C}_{15} b_n^{-2} \left(b_n^3 (\log b_n)(1-\rho_n)^{\frac{1}{2}} \right) \\ &< b_n^{-2} \end{aligned}$$

due to $\lim_{\rho_n \to 1}(1-\rho_n)^{-1/2}(\pi/2 - \arcsin(\rho_n)) = \sqrt{2}$ and $\lim_{n\to\infty} b_n^{10}(1-\rho_n) = c$, where $\mathbb{C}_{15}$ is an absolute positive constant. Combining (i) with (A.21), we can obtain the upper bound in (2.1).

Finally, we consider the lower bound in (2.1). For fixed $x, y \in \mathbb{R}$, if $\max(x,y) < z < 4\log b_n$ we have

$$\begin{aligned} &\Phi\left(\frac{u_n(\min(x,y)) - \rho_n u_n(z)}{\sqrt{1-\rho_n^2}} \right) \\ &< \frac{\exp(-\frac{b_n^2(1-\rho_n)}{4} - \frac{\min(x,y)}{1+\rho_n} + \frac{\rho_n z}{1+\rho_n})}{\sqrt{2\pi}(\frac{z-\min(x,y)}{2\lambda_n} - \lambda_n - \frac{\lambda_n z}{b_n^2})} \end{aligned}$$

for large n. So, for $\max(x,y) < z < 4\log b_n$,

(3.36)
$$\begin{aligned} &\int_{\max(x,y)}^{4\log b_n} \Phi\left(\frac{u_n(\min(x,y)) - \rho_n u_n(z)}{\sqrt{1-\rho_n^2}} \right) e^{-z} \exp\left(-\frac{z^2}{2b_n^2} \right) dz \\ &< \frac{b_n\sqrt{1-\rho_n}\exp(-\frac{b_n^2(1-\rho_n)}{4} - \frac{\min(x,y)}{1+\rho_n} - \frac{\max(x,y)}{1+\rho_n})}{\sqrt{\pi}(\max(x,y) - \min(x,y) - 2\lambda_n^2 - \frac{8\lambda_n^2 \log b_n}{b_n^2})} \\ &= O(b_n^{-4}) \end{aligned}$$

for large n due to $\lim_{n\to\infty} b_n^{10}(1-\rho_n) = c$.

Note that

(3.37)
$$\begin{aligned} &\int_{4\log b_n}^{\infty} \Phi\left(\frac{u_n(\min(x,y)) - \rho_n u_n(z)}{\sqrt{1-\rho_n^2}} \right) e^{-z} \exp\left(-\frac{z^2}{2b_n^2} \right) dz \\ &= O(b_n^{-4}) \end{aligned}$$

for large n. By (A.16), (3.36) and (3.37), we have

$$\begin{aligned} &1 - F\Big(u_n(x), u_n(y)\Big) \\ &= n^{-1} \int_{\max(x,y)}^{\infty} \Phi\left(\frac{u_n(\min(x,y)) - \rho_n u_n(z)}{\sqrt{1-\rho_n^2}} \right) e^{-z} \\ &\quad \times \exp\left(-\frac{z^2}{2b_n^2} \right) dz + 1 - \Phi\Big(u_n(\min(x,y))\Big) \\ &= n^{-1} \Big(- b_n^{-2} e^{-\min(x,y)} \left((\min(x,y))^2/2 + \min(x,y) + 1 \right) \\ &\quad + e^{-\min(x,y)} + O(b_n^{-4}) \Big), \end{aligned}$$

which implies

$$\begin{aligned} &F^n\Big(u_n(x), u_n(y)\Big) - H_0(x,y) \\ &= b_n^{-2} \left(\frac{(\min(x,y))^2}{2} + \min(x,y) + 1 + O(b_n^{-2}) \right) \\ &\quad \times H_0(x,y) e^{-\min(x,y)} \end{aligned}$$

for large n. Hence the lower bound in (2.1) is obtained if $\rho_n \in (0,1)$ such that $\lim_{n\to\infty} b_n^{10}(1-\rho_n) = c$.

The proof is complete. $\square$

APPENDIX A

Auxiliary lemmas used in the proofs of the main results are given in this appendix.

Lemma 1. *Let norming constant b_n be given by (1.3). Under the second-order Hüsler-Reiss condition (1.5), for fixed $x, y \in \mathbb{R}$ we have*

$$b_n^2 \int_y^\infty \left(\Phi\left(\frac{u_n(x) - \rho_n u_n(z)}{\sqrt{1-\rho_n^2}} \right) - \Phi\left(\lambda + \frac{x-z}{2\lambda} \right) \right) e^{-z} dz$$

(A.1)

$$= (A_n + B_n x) \int_y^\infty \varphi\left(\lambda + \frac{x-z}{2\lambda}\right) e^{-z} dz + (C_n - B_n) \int_y^\infty \varphi\left(\lambda + \frac{x-z}{2\lambda}\right) z e^{-z} dz + O(b_n^{-2})$$

(A.2)

$$\to -\kappa_1(x,y)$$

as $n \to \infty$, and

$$b_n^2 \int_y^\infty \left(\Phi\left(\frac{u_n(x) - \rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right) - \Phi\left(\lambda + \frac{x-z}{2\lambda}\right)\right) z^2 e^{-z} dz$$
$$= (A_n + B_n x) \int_y^\infty \varphi\left(\lambda + \frac{x-z}{2\lambda}\right) z^2 e^{-z} dz + (C_n - B_n) \int_y^\infty \varphi\left(\lambda + \frac{x-z}{2\lambda}\right) z^3 e^{-z} dz + O(b_n^{-2})$$

(A.3)

$$\to -\kappa_2(x,y)$$

as $n \to \infty$, where $\varphi(x)$ denotes the standard Gaussian density function, and $\kappa_1(x,y)$, $\kappa_2(x,y)$ are respectively given by

$$\kappa_1(x,y) = (2\lambda^4 - 2\lambda^2 x) e^{-x} \left(1 - \Phi\left(\lambda + \frac{y-x}{2\lambda}\right)\right) - (2\alpha + 3\lambda^3) e^{-x} \varphi\left(\lambda + \frac{y-x}{2\lambda}\right) \tag{A.4}$$

and

$$\kappa_2(x,y) = \big(8\lambda^8 + 32\lambda^6 - 16\lambda^6 x + 10\lambda^4 x^2 - 20\lambda^4 x + 16\alpha\lambda^3 - 2\lambda^2 x^3 - 8\alpha\lambda x\big) e^{-x} \left(1 - \Phi\left(\lambda + \frac{y-x}{2\lambda}\right)\right) - \big(8\lambda^7 + 24\lambda^5 - 4\lambda^5 y - 12\lambda^5 x + 4\lambda^3 x^2 + 4\lambda^3 xy + 3\lambda^3 y^2 + 16\alpha\lambda^2 + 2\alpha y^2\big) e^{-x} \varphi\left(\lambda + \frac{y-x}{2\lambda}\right).$$

Proof: By Taylor expansion with the Lagrange remainder term, we have

$$\Phi\left(\frac{u_n(x) - \rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right) = \Phi\left(\lambda + \frac{x-z}{2\lambda}\right) + \varphi\left(\lambda + \frac{x-z}{2\lambda}\right)\left(\frac{u_n(x) - \rho_n u_n(z)}{\sqrt{1-\rho_n^2}} - \lambda - \frac{x-z}{2\lambda}\right) + \frac{1}{2}\nu_n \varphi(\nu_n)\left(\frac{u_n(x) - \rho_n u_n(z)}{\sqrt{1-\rho_n^2}} - \lambda - \frac{x-z}{2\lambda}\right)^2, \tag{A.5}$$

where $\min\{\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}, \lambda + \frac{x-z}{2\lambda}\} < \nu_n < \max\{\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}, \lambda + \frac{x-z}{2\lambda}\}$. Note that

$$\frac{u_n(x) - \rho_n u_n(z)}{\sqrt{1-\rho_n^2}} = \lambda_n \left(1 - \frac{\lambda_n^2}{b_n^2}\right)^{-\frac{1}{2}} + \frac{x-z}{2\lambda_n}\left(1 - \frac{\lambda_n^2}{b_n^2}\right)^{-\frac{1}{2}} + \frac{\lambda_n z}{b_n^2}\left(1 - \frac{\lambda_n^2}{b_n^2}\right)^{-\frac{1}{2}}$$

and

$$\int_y^\infty \varphi(\lambda + \frac{x-z}{2\lambda}) e^{-z} dz = 2\lambda e^{-x}\left(1 - \Phi\left(\lambda + \frac{y-x}{2\lambda}\right)\right),$$
$$\int_y^\infty \varphi(\lambda + \frac{x-z}{2\lambda}) z e^{-z} dz = 4\lambda^2 e^{-x} \varphi\left(\lambda + \frac{y-x}{2\lambda}\right) + (2\lambda x - 4\lambda^3) e^{-x}\left(1 - \Phi\left(\lambda + \frac{y-x}{2\lambda}\right)\right).$$

Combining with (3.2), we have

$$b_n^2 \int_y^\infty \left(\frac{u_n(x) - \rho_n u_n(z)}{\sqrt{1-\rho_n^2}} - \lambda - \frac{x-z}{2\lambda}\right) \times \varphi\left(\lambda + \frac{x-z}{2\lambda}\right) e^{-z} dz$$

(A.6)

$$= (A_n + B_n x) \int_y^\infty \varphi(\lambda + \frac{x-z}{2\lambda}) e^{-z} dz + (C_n - B_n) \int_y^\infty \varphi(\lambda + \frac{x-z}{2\lambda}) z e^{-z} dz$$
$$\to \left(\alpha + \frac{1}{2}\lambda^3 - \frac{1}{2}\alpha\lambda^{-2} x + \frac{1}{4}\lambda x\right) \int_y^\infty \varphi(\lambda + \frac{x-z}{2\lambda}) e^{-z} dz + \left(\frac{3}{4}\lambda + \frac{1}{2}\alpha\lambda^{-2}\right) \int_y^\infty \varphi(\lambda + \frac{x-z}{2\lambda}) z e^{-z} dz$$

(A.7)

$$= (2\lambda^2 x - 2\lambda^4) e^{-x}\left(1 - \Phi\left(\lambda + \frac{y-x}{2\lambda}\right)\right) + (2\alpha + 3\lambda^3) e^{-x} \varphi\left(\lambda + \frac{y-x}{2\lambda}\right)$$

as $n \to \infty$. Similarly,

(A.8)

$$b_n^2 \int_y^\infty \left(\frac{u_n(x) - \rho_n u_n(z)}{\sqrt{1-\rho_n^2}} - \lambda - \frac{x-z}{2\lambda}\right)^2 \nu_n \varphi(\nu_n) e^{-z} dz = O(b_n^{-2}).$$

Combining with (A.6)–(A.8), we can derive (A.1) and (A.2).

By arguments similar to that of the first assertion, we can derive (A.3). The proof is complete. □

Lemma 2. *For large n, let*

$$c_n := \log\log b_n^2 > 0, \quad d_n := -\log\log \frac{b_n^2}{b_n^2 - 1} > 0$$

with b_n given by (1.3). Then under the second-order Hüsler-Reiss condition (1.5), with absolute positive constants $\mathbb{C}_i, 1 \le i \le 5$, we have

(1) for large n, the following inequality

(A.9)
$$\int_y^\infty \left|\Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right) - \Phi\left(\lambda + \frac{x-z}{2\lambda}\right)\right| e^{-z}dz \le b_n^{-2}\left(e^{-y}(\mathbb{C}_1|y| + \mathbb{C}_2) + \mathbb{C}_3 e^{-x}|x| + \mathbb{C}_4\right)$$

holds uniformly for all $(x,y) \in [-c_n, d_n] \times [-c_n, d_n]$.

(2) for large n, the following inequality

(A.10)
$$\int_y^\infty \left|\Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right) - \Phi\left(\lambda + \frac{x-z}{2\lambda}\right)\right| e^{-z}dz \le \mathbb{C}_5 b_n^{-2}$$

holds uniformly for all $(x,y) \in [-c_n, d_n] \times [d_n, \infty)$.

Proof: By the Taylor expansion with the Lagrange remainder, we have

(A.11)
$$\begin{aligned}
&\int_y^\infty \left|\Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right) - \Phi\left(\lambda + \frac{x-z}{2\lambda}\right)\right| e^{-z}dz \\
&= \int_y^\infty \left|\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}} - \lambda - \frac{x-z}{2\lambda}\right| \varphi(\nu_n)e^{-z}dz \\
&\le b_n^{-2}\left(|A_n|\int_y^\infty \varphi(\nu_n)e^{-z}dz + |B_n|\,|x|\int_y^\infty \varphi(\nu_n)e^{-z}dz \right.\\
&\qquad \left. + (|B_n| + |C_n|)\int_y^\infty \varphi(\nu_n)|z|e^{-z}dz\right) \\
&\le b_n^{-2}\left(|A_n|\int_y^\infty e^{-z}dz + |B_n|\,|x|\int_y^{d_n} \varphi(\nu_n)e^{-z}dz \right.\\
&\qquad \left. + |B_n|\,|x|\int_{d_n}^\infty e^{-z}dz + (|B_n| + |C_n|)\int_y^\infty |z|e^{-z}dz\right) \\
&\le 2b_n^{-2}\left(\left|\frac{1}{2}\alpha\lambda^{-2} - \frac{1}{4}\lambda\right||x|\int_y^{d_n}\varphi(\nu_n)e^{-z}dz \right.\\
&\quad + \left(\left|\frac{1}{2}\alpha\lambda^{-2} - \frac{1}{4}\lambda\right| + \lambda\right)\left(2-(y+1)e^{-y}\right) \\
&\quad \left. + \left|\frac{1}{2}\lambda^3 + \alpha\right|e^{-y} + \left|\frac{1}{2}\alpha\lambda^{-2} - \frac{1}{4}\lambda\right||x|e^{-d_n}\right)
\end{aligned}$$

for large n, where $\min(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}, \lambda + \frac{x-z}{2\lambda}) < \nu_n < \max(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}, \lambda + \frac{x-z}{2\lambda})$. Note that, with the second-order Hüsler-Reiss condition (1.5), we have

$$\begin{aligned}
&\left|\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}} - \lambda - \frac{x-z}{2\lambda}\right| \\
&\le (c_n+d_n)\left|\frac{1}{\lambda_n}\left(1+\frac{\lambda_n^2}{2b_n^2} + O(b_n^{-4})\right) - \frac{1}{\lambda}\right| \\
&\quad + \left|\lambda_n\left(1-\frac{\lambda_n^2}{b_n^2}\right)^{-\frac{1}{2}} - \lambda\right| + \frac{\lambda_n(c_n+d_n)}{b_n^2}\left(1-\frac{\lambda_n^2}{b_n^2}\right)^{-\frac{1}{2}} \\
&\le \left|\lambda_n\left(1-\frac{\lambda_n^2}{b_n^2}\right)^{-\frac{1}{2}} - \lambda\right| + \frac{\lambda_n(c_n+d_n)}{b_n^2}\left(1-\frac{\lambda_n^2}{b_n^2}\right)^{-\frac{1}{2}} \\
&\quad + (c_n+d_n)\frac{|\lambda-\lambda_n|}{\lambda\lambda_n} + \frac{(c_n+d_n)\lambda_n}{2b_n^2} + O\left(\frac{c_n+d_n}{b_n^4}\right) \\
&\to 0
\end{aligned}$$

uniformly for all $(x,z) \in [-c_n, d_n] \times [-c_n, d_n]$ as $n \to \infty$, which implies that $\varphi(\nu_n)$ converges to $\varphi\left(\lambda + \frac{x-z}{2\lambda}\right)$ uniformly for $(x,z) \in [-c_n, d_n] \times [-c_n, d_n]$ since $|\varphi(x) - \varphi(y)| < |x-y|$ for all $x, y \in \mathbb{R}$. Hence,

$$\begin{aligned}
\int_y^{d_n} \varphi(\nu_n)e^{-z}dz &< 2\int_y^{d_n} \varphi(\lambda + \frac{x-z}{2\lambda})e^{-z}dz \\
&= 2e^{-x}\int_y^{d_n}\varphi(\lambda + \frac{z-x}{2\lambda})dz \\
&< 4\lambda e^{-x}
\end{aligned}$$

for $(x,y) \in [-c_n, d_n] \times [-c_n, d_n]$. Combining with (A.11), we have

$$\begin{aligned}
&\int_y^\infty \left|\Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho^2}}\right) - \Phi\left(\lambda + \frac{x-z}{2\lambda}\right)\right| e^{-z}dz \\
&\le b_n^{-2}\left((\mathbb{C}_1 + \mathbb{C}_2|y|)e^{-y} + \mathbb{C}_3|x|e^{-x} + \mathbb{C}_4\right)
\end{aligned}$$

for large n, which completes the proof of (A.9).

For $(x,y) \in [-c_n, d_n] \times [d_n, \infty]$, we can derive the desired result immediately by (A.11) since

$$\int_y^\infty \left|\Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho^2}}\right) - \Phi\left(\lambda + \frac{x-z}{2\lambda}\right)\right| e^{-z}dz \le \mathbb{C}_5 b_n^{-2}$$

for large n. The proof is complete. □

Lemma 3. *Let norming constant b_n be defined by (1.3). Assume that the second-order Hüsler-Reiss condition (1.5) holds. Then for fixed $x, y \in \mathbb{R}$ and sufficiently large n, we have*

(A.12)
$$\begin{aligned}
&F^n(u_n(x), u_n(y)) - H_\lambda(x,y) \\
&= H_\lambda(x,y)b_n^{-2}\left(e^{-x}\left(1 + x + \frac{1}{2}x^2\right) + \frac{1}{2}\kappa_3(x,y) \right.\\
&\quad - (A_n + B_n x)\int_y^\infty \varphi\left(\lambda + \frac{x-z}{2\lambda}\right)e^{-z}dz \\
&\quad \left. + (B_n - C_n)\int_y^\infty \varphi\left(\lambda + \frac{x-z}{2\lambda}\right)ze^{-z}dz + O(b_n^{-2})\right),
\end{aligned}$$

where

(A.13)

$$\kappa_3(x,y) = \left(y^2+2y+2\right)e^{-y}\Phi\left(\lambda+\frac{x-y}{2\lambda}\right)$$
$$-\left(4\lambda^4-4\lambda^2x+x^2+2x+2\right)e^{-x}\left(1-\Phi\left(\lambda+\frac{y-x}{2\lambda}\right)\right)$$
$$+\left(4\lambda^3-2\lambda x-2\lambda y-4\lambda\right)e^{-x}\varphi\left(\lambda+\frac{y-x}{2\lambda}\right).$$

Proof: First note that

(A.14)

$$\int_y^\infty \Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right)e^{-z}\exp\left(-\frac{z^2}{2b_n^2}\right)dz$$
$$=\int_y^\infty \Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right)e^{-z}\left(1-\frac{z^2}{2b_n^2}\right)dz+O(b_n^{-4})$$

holds for large n due to $|e^{-x}-(1-x)|<x^2/2$ for $x>0$.

Noting that

$$\kappa_3(x,y)=\int_y^\infty \Phi\left(\lambda+\frac{x-z}{2\lambda}\right)z^2e^{-z}dz$$

and by (A.1) and (A.3), we have

(A.15)

$$\int_y^\infty \Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right)e^{-z}\left(1-\frac{z^2}{2b_n^2}\right)dz$$
$$=\int_y^\infty \Phi\left(\lambda+\frac{x-z}{2\lambda}\right)e^{-z}dz-2^{-1}b_n^{-2}\kappa_3(x,y)$$
$$+\int_y^\infty\left(\Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right)-\Phi\left(\lambda+\frac{x-z}{2\lambda}\right)\right)e^{-z}dz$$
$$-\frac{b_n^{-2}}{2}\int_y^\infty\left(\Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right)-\Phi\left(\lambda+\frac{x-z}{2\lambda}\right)\right)z^2e^{-z}dz$$
$$=\int_y^\infty \Phi\left(\lambda+\frac{x-z}{2\lambda}\right)e^{-z}dz-2^{-1}b_n^{-2}\kappa_3(x,y)$$
$$+b_n^{-2}(A_n+B_nx)\int_y^\infty\varphi\left(\lambda+\frac{x-z}{2\lambda}\right)e^{-z}dz$$
$$+b_n^{-2}(C_n-B_n)\int_y^\infty\varphi\left(\lambda+\frac{x-z}{2\lambda}\right)ze^{-z}dz+O(b_n^{-4})$$

for large n.

It follows from [7] that

(A.16)

$$1-\Phi(u_n(x))=n^{-1}e^{-x}\left[1-b_n^{-2}\left(1+x+\frac{1}{2}x^2\right)+O(b_n^{-4})\right]$$

for large n. Combining (1.3) and (A.14)–(A.16), we have

$$1-F(u_n(x),u_n(y))$$
$$=n^{-1}\int_y^\infty \Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right)e^{-z}\exp\left(-\frac{z^2}{2b_n^2}\right)dz+1-\Phi(u_n(x))$$
$$=n^{-1}\Big(\int_y^\infty \Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right)e^{-z}\left(1-\frac{z^2}{2b_n^2}\right)dz+O(b_n^{-4})\Big)+1-\Phi(u_n(x))$$
$$=n^{-1}\Big[e^{-x}-b_n^{-2}e^{-x}\left(1+x+\frac{1}{2}x^2\right)+\int_y^\infty\Phi\left(\lambda+\frac{x-z}{2\lambda}\right)e^{-z}dz-2^{-1}b_n^{-2}\kappa_3(x,y)$$
$$+b_n^{-2}(A_n+B_nx)\int_y^\infty\varphi\left(\lambda+\frac{x-z}{2\lambda}\right)e^{-z}dz$$
$$+b_n^{-2}(C_n-B_n)\int_y^\infty\varphi\left(\lambda+\frac{x-z}{2\lambda}\right)ze^{-z}dz+O(b_n^{-4})\Big]$$

for large n, which implies the desired result (A.12). The proof is complete. □

Before proving Lemma 4, we cite the following distributional tail decomposition of standard Gaussian distribution in [1], i.e., for $x>0$,

$$1-\Phi(x)=x^{-1}\varphi(x)-r(x) \tag{A.17}$$
$$=x^{-1}\left(1-x^{-2}\right)\varphi(x)+s(x), \tag{A.18}$$

where

$$0<r(x)<x^{-3}\varphi(x),\quad 0<s(x)<3x^{-5}\varphi(x). \tag{A.19}$$

Lemma 4. *Let norming constant b_n be defined by (1.3). For sufficiently large n, we have*

$$\sup_{(x,y)\in(-\infty,-c_n]\times\mathbb{R}}|\Delta(F^n,H_\lambda;x,y)|<\mathbb{D}_1b_n^{-2}, \tag{A.20}$$

where $c_n=\log\log b_n^2$ and $\mathbb{D}_1$ is an absolute positive constant.

Proof: By (1.3), (A.18), (A.19) and $e^z>1+z, z\in\mathbb{R}$, for all $y\in\mathbb{R}$ we have

$$1-F(u_n(-c_n),u_n(y))\geq 1-\Phi(u_n(-c_n))$$
$$>n^{-1}(\log b_n^2)\left(1-\frac{(\log\log b_n^2)^2}{2b_n^2}-b_n^{-2}\left(1-\frac{\log\log b_n^2}{b_n^2}\right)^{-2}\right)$$

for large n, which implies that

(A.21)

$$\sup_{(x,y)\in(-\infty,-c_n]\times\mathbb{R}}F^n(u_n(x),u_n(y))$$

$$
\begin{aligned}
&\le \sup_{y\in\mathbb{R}} F^n(u_n(-c_n),u_n(y)) \\
&\le \left[1-\frac{\log b_n^2}{n}\left(1-\frac{(\log\log b_n^2)^2}{2b_n^2}-\frac{1}{b_n^2(1-\frac{\log\log b_n^2}{b_n^2})^2}\right)\right]^n \\
&\le b_n^{-2}\exp\left(\frac{(\log b_n^2)(\log\log b_n^2)^2}{2b_n^2}+\frac{\log b_n^2}{b_n^2(1-\frac{\log\log b_n^2}{b_n^2})^2}\right) \\
&< \mathbb{C}_6 b_n^{-2}
\end{aligned}
$$

for large n, where $\mathbb{C}_6$ is an absolute positive constant. Hence,

$$
\left|F^n(u_n(x),u_n(y))-H_\lambda(x,y)\right| < (\mathbb{C}_6+1)b_n^{-2}=\mathbb{D}_1 b_n^{-2}
$$

uniformly for all $(x,y)\in(-\infty,-c_n]\times\mathbb{R}$ for large n, which is the desired result. □

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