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Berman's inequality under random scaling

Enkelejd Hashorva and Zhichao Weng*

Berman's inequality is the key for establishing asymptotic properties of maxima of Gaussian random sequences and supremum of Gaussian random fields. This contribution shows that, asymptotically an extended version of Berman's inequality can be established for randomly scaled Gaussian random vectors. Two applications presented in this paper demonstrate the use of Berman's inequality under random scaling.

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1. INTRODUCTION

In the analysis of extreme values of Gaussian processes and Gaussian random fields, Berman's inequality plays a crucial role. Essentially, for given two Gaussian distribution functions in $\mathbb{R}^d$ it bounds their difference by comparing the covariances. The key result that motivated this comparison method is Plackett's partial differential equation given in [27]. As explained in [20], it was then developed by Slepian [28], Berman [1, 2], Cramér [4], Piterbarg [25, 26] and then by Li and Shao [22]. Specifically, the developed results are summarised by Berman's inequality which we formulate below in the most general form derived in [22]. Let therefore $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be two Gaussian random vectors with $N(0,1)$ components and covariance matrices $\Lambda_1 = (\lambda_{ij}^{(1)})$ and $\Lambda_2 = (\lambda_{ij}^{(2)})$, respectively. For arbitrary constants $u_i, i \le n$, [22] obtained

$$\begin{aligned} &\mathbb{P}\left(X_i \le u_i, 1 \le i \le n\right) - \mathbb{P}\left(Y_i \le u_i, 1 \le i \le n\right) \\ \le\ & \frac{1}{2\pi} \sum_{1 \le i < j \le n} A_{ij} \exp\left(-\frac{u_i^2 + u_j^2}{2(1+\rho_{ij})}\right), \end{aligned}$$

where

$$\begin{aligned} &\rho_{ij} := \max(|\lambda_{ij}^{(1)}|, |\lambda_{ij}^{(2)}|), \\ &A_{ij} = \left|\arcsin(\lambda_{ij}^{(1)}) - \arcsin(\lambda_{ij}^{(2)})\right|. \end{aligned} \tag{1}$$

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*Corresponding author.

Clearly, for arbitrary constants $v_i, u_i, i \le n$, set $w := \min_{1 \le i \le n} \min(| u_i |, | v_i |)$

$$\begin{aligned} &\mathbb{P}\left(-v_i < X_i \le u_i, 1 \le i \le n\right) - \mathbb{P}\left(-v_i < Y_i \le u_i, 1 \le i \le n\right) \\ &\le \frac{2}{\pi} \sum_{1 \le i < j \le n} A_{ij} \exp\left(-\frac{w^2}{1+\rho_{ij}}\right), \end{aligned} \tag{2}$$

for a detailed proof see [21]. Berman's inequality can be applied also to non-Gaussian random vectors. For instance, consider two random vectors $\widetilde{\mathbf{X}} = (S_1X_1, \ldots, S_nX_n)$ and $\widetilde{\mathbf{Y}} = (S_1Y_1, \ldots, S_nY_n)$ with S, $S_i, i \le n$ some positive independent random variables with common distribution function G being further independent from $\mathbf{X}$ and $\mathbf{Y}$. In the special case G is the uniform distribution on $(0,1)$, the upper bound in (2) implies

$$\begin{aligned} & \Delta_{\mathbf{S}}(\mathbf{u}, \mathbf{v}) \\ := & \ \mathbb{P}\left(-v_i < S_iX_i \le u_i, 1 \le i \le n\right) \\ & - \mathbb{P}\left(-v_i < S_iY_i \le u_i, 1 \le i \le n\right) \\ \le & \ \frac{2}{\pi} \sum_{1 \le i < j \le n} \\ & A_{ij} \int_0^1 \int_0^1 \exp\left(-\frac{(w/s_i)^2 + (w/s_j)^2}{2(1+\rho_{ij})}\right) ds_i ds_j \\ \le & \ \frac{2}{\pi} \sum_{1 \le i < j \le n} A_{ij} \exp\left(-\frac{w^2}{1+\rho_{ij}}\right). \end{aligned} \tag{3}$$

Another tractable case is when $G(x) = 1 - \exp(-x), x > 0$ is the exponential distribution. Indeed, by (2) for all $0 < a, b < 1$ we have

$$\begin{aligned} & \Delta_{\mathbf{S}}(\mathbf{u}, \mathbf{v}) \\ \le & \ \frac{2}{\pi} \sum_{1 \le i < j \le n} A_{ij} \int_0^\infty \int_0^\infty \\ & \exp\left(-\frac{(w/s_i)^2 + (w/s_j)^2}{2(1+\rho_{ij})} - s_i - s_j\right) ds_i ds_j \\ = & \ \frac{2}{\pi} \sum_{1 \le i < j \le n} A_{ij} \int_0^\infty \int_0^\infty \\ & \exp\left(-\frac{(w/s_i)^2 + (w/s_j)^2}{2(1+\rho_{ij})} - as_i - bs_j\right) \\ & \times \exp\left(-(1-a)s_i - (1-b)s_j\right) ds_i ds_j \end{aligned} \tag{4}$$

$$\leq \frac{2}{\pi(1-a)(1-b)} \sum_{1\leq i<j\leq n} A_{ij} \exp\left(-\frac{3}{2}(a^{\frac{2}{3}}+b^{\frac{2}{3}})(1+\rho_{ij})^{-\frac{1}{3}} w^{\frac{2}{3}}\right).$$

Clearly, if we do not know the distribution function of S it is not possible to obtain an explicit upper bound for $\Delta_{\mathbf{S}}(\mathbf{u},\mathbf{v})$. Since for the analysis of extremes of Gaussian random sequences or processes Berman's inequality is applied for large values of the u_i's and v_i's (see e.g., [26]), in this paper we are concerned with the derivation of Berman's inequality for some general scaling random variable S and all u_i's and v_i's sufficiently large. We shall consider two particular cases for the random vector $\mathbf{S}=(S_1,\ldots,S_n)$, namely it has independent components, and it is comonotonic with $\mathbf{S}=(S,\ldots,S)=: S\mathbf{1}$. From the proofs it can be seen that the joint dependence of (S_i,S_j) for any pair (i,j) is crucial; our results can be in fact extended for certain tractable dependence models. We shall focus on simplicity only with these two cases.

Since random scaling is a natural phenomena related to the time-value of money in finance, measurement errors in experimental data, or physical constraints, the extension of Berman's inequality for inflated/deflated Gaussian random vectors is of certain interest also for statistical applications.

Of course, Berman's inequality alone is not enough for extending [17] to randomly scaled Gaussian triangular arrays; some additional results (see [15, 16]) which show that for some tractable tail assumptions on S the scaled random vector $\widetilde{\mathbf{X}}$ behaves similarly to $\mathbf{X}$ are also important. Specifically, we shall deal with two large classes of random scaling: a) S is a bounded random variable with a tractable tail behaviour at the right endpoint of its distribution function, including in particular the case that its survival function is regularly varying, and b) S is a Weibull-type random variable.

In view of our findings, several known results for Gaussian random sequences and processes can be extended to the scaled Gaussian framework; we shall demonstrate this with two representative applications.

Organisation of the rest of the paper: Section 2 presents Berman's inequality for scaled Gaussian random vectors. In Section 3 we display two applications, while the proofs are relegated to Section 4.

2. MAIN RESULTS

We consider first the case that S is non-negative with distribution function G which has right endpoint equal to 1. Intuitively, large values of S do not influence significantly large values of the product say SX if X is a Gaussian random variable being independent of S. It turns out that the following asymptotic upper bound

$$\mathbb{P}(S>1-1/u)\leq c_A u^{-\tau} \tag{5}$$

valid for all u large and some $c_A>0, \tau\geq 0$ is sufficient for the derivation of a useful upper bound for $\Delta_{\mathbf{S}}(\mathbf{u},\mathbf{v})$ defined above.

A canonical example of such S is the beta random variable, which is a special case of a power-tail random variable S, namely

$$\mathbb{P}(S>1-1/u)=(1+o(1))cu^{-\tau}, \quad u\to\infty \tag{6}$$

holds for some $c>0, \tau\geq 0$. Hereafter we set $w=\min_{1\leq i\leq n}\min(|\ u_i\ |, |\ v_i\ |)$ and write $\Delta_{S\mathbf{1}}(\mathbf{u},\mathbf{v})$ instead of $\Delta_{\mathbf{S}}(\mathbf{u},\mathbf{v})$ if $\mathbf{S}=(S,\ldots,S)$. Further write $\Delta_{\mathbf{S}}(u\mathbf{1})$ and $\Delta_{S\mathbf{1}}(u\mathbf{1})$ instead of $\Delta_{\mathbf{S}}(\mathbf{u},\mathbf{v})$ if all components of $\mathbf{v}$ equal $-\infty$, $\mathbf{u}=(u,\ldots,u)=: u\mathbf{1}$ and the covariance matrix Λ_2 of $\mathbf{Y}$ is identity matrix.

Theorem 2.1. *Let $\mathbf{X},\widetilde{\mathbf{X}},\mathbf{Y},\widetilde{\mathbf{Y}},S,S_i, i\leq n$ be as above. If (5) holds, then for all $u_i,v_i, 1\leq i\leq n$ large and $\epsilon>0$ we have*

$$\Delta_{\mathbf{S}}(\mathbf{u},\mathbf{v}) \leq (\mathbb{K}_A+\epsilon)w^{-4\tau}\sum_{1\leq i<j\leq n} A_{ij}(1+\rho_{ij})^{2\tau}\exp\left(-\frac{w^2}{1+\rho_{ij}}\right) \tag{7}$$

and

$$\Delta_{S\mathbf{1}}(\mathbf{u},\mathbf{v}) \leq (\mathbb{K}_A^*+\epsilon)w^{-2\tau}\sum_{1\leq i<j\leq n} A_{ij}(1+\rho_{ij})^{\tau}\exp\left(-\frac{w^2}{1+\rho_{ij}}\right), \tag{8}$$

where $\mathbb{K}_A=\frac{2}{\pi}c_A^2(\Gamma(\tau+1))^2$ and $\mathbb{K}_A^*=\frac{2^{1-\tau}}{\pi}c_A\Gamma(\tau+1)$.

Corollary 2.1. *Under the conditions of Theorem 2.1, for all u large and some positive constants $\mathcal{Q}$ we have*

$$\Delta_{\mathbf{S}}(u\mathbf{1}) \leq \mathcal{Q}u^{-4\tau}\sum_{1\leq i<j\leq n}|\lambda_{ij}^{(1)}|\exp\left(-\frac{u^2}{1+|\lambda_{ij}^{(1)}|}\right) \tag{9}$$

and

$$\Delta_{S\mathbf{1}}(u\mathbf{1}) \leq \mathcal{Q}u^{-2\tau}\sum_{1\leq i<j\leq n}|\lambda_{ij}^{(1)}|\exp\left(-\frac{u^2}{1+|\lambda_{ij}^{(1)}|}\right). \tag{10}$$

We shall investigate below the more difficult case that the scaling random variable S has distribution function with an infinite right endpoint. Motivated by the example of the exponential distribution in the Introduction, we shall assume that S has tail behaviour similar to a Weibull distribution. Specifically, for given constants $\alpha\in\mathbb{R}, c_B, L, p\in(0,\infty)$ suppose that

$$\mathbb{P}(S>u)=(1+o(1))c_B u^{\alpha}\exp(-Lu^p),\ u\to\infty \tag{11}$$

is valid. The class of distribution functions satisfying (11) is quite large. More importantly, under (11) SX has also a Weibull tail behaviour if X is a $N(0,1)$ random variable independent of S, see e.g., [16]. We state next our second result for Weibull-type random scaling.

Theorem 2.2. *Let $\mathbf{X}, \widetilde{\mathbf{X}}, \mathbf{Y}, \widetilde{\mathbf{Y}}, S, S_i, i \le n$ be as above. If (11) holds, then for all $u_i, v_i, 1 \le i \le n$ large and $\epsilon > 0$ we have*

$$
\begin{aligned}
(12)\qquad & \Delta_{\mathbf{S}}(\mathbf{u},\mathbf{v}) \\
& \le (\mathbb{K}_B+\epsilon) w^{\frac{4\alpha+2p}{2+p}} \sum_{1\le i<j\le n} A_{ij}(1+\rho_{ij})^{\frac{-2\alpha-p}{p+2}} \\
& \times \exp\left(-2(1+\rho_{ij})^{-\frac{p}{2+p}} T w^{\frac{2p}{2+p}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
(13)\qquad & \Delta_{S\mathbf{1}}(\mathbf{u},\mathbf{v}) \\
& \le (\mathbb{K}_B^*+\epsilon) w^{\frac{2\alpha+p}{2+p}} \sum_{1\le i<j\le n} A_{ij}(1+\rho_{ij})^{\frac{-2\alpha-p}{2(p+2)}} \\
& \times \exp\left(-(2(1+\rho_{ij})^{-1})^{\frac{p}{2+p}} T w^{\frac{2p}{2+p}}\right),
\end{aligned}
$$

where $T = L^{\frac{2}{p+2}} p^{-\frac{p}{p+2}} + (Lp)^{\frac{2}{p+2}} 2^{-1}$, $\mathbb{K}_B = 4c_B^2 \times (Lp)^{\frac{2(1-\alpha)}{p+2}}(p+2)^{-1}$ and $\mathbb{K}_B^* = 2^{\frac{3+2p+\alpha}{2+p}} \pi^{-\frac{1}{2}} c_B (Lp)^{\frac{1-\alpha}{p+2}}(p+2)^{-\frac{1}{2}}$.

Corollary 2.2. *Under the conditions of Theorem 2.2, for all u large and some positive constants $\mathcal{Q}$ we have*

$$
\begin{aligned}
(14)\qquad & \Delta_{\mathbf{S}}(u\mathbf{1}) \\
& \le \mathcal{Q} u^{\frac{4\alpha+2p}{2+p}} \sum_{1\le i<j\le n} |\lambda_{ij}^{(1)}| \\
& \times \exp\left(-2(1+|\lambda_{ij}^{(1)}|)^{-\frac{p}{2+p}} T u^{\frac{2p}{2+p}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
(15)\qquad & \Delta_{S\mathbf{1}}(u\mathbf{1}) \\
& \le \mathcal{Q} u^{\frac{2\alpha+p}{2+p}} \sum_{1\le i<j\le n} |\lambda_{ij}^{(1)}| \\
& \times \exp\left(-(2(1+|\lambda_{ij}^{(1)}|)^{-1})^{\frac{p}{2+p}} T u^{\frac{2p}{2+p}}\right).
\end{aligned}
$$

Remark 2.1. *a) Clearly, when S is uniformly distributed on $(0,1)$ then condition (5) holds with $c_A = \tau = 1$. For this case we have two results, the one derived in the Introduction and that given by (7). We see that the bound obtained by (7) with $c_A = \tau = 1$ is better due to the term $w^{-4\tau}$.*

b) Also for the case S is a unit exponential random variables we have two bounds, one which holds for all values of $u_i, v_i, i \le n$ and the asymptotic one given in Theorem 2.2. The bound implied by (12) with $c_B = 1, \alpha = 0, p = 1, L = 1$ is asymptotically better than that implied by (4).

3. APPLICATIONS

An important contribution in extreme value theory concerned with maxima of triangular arrays of Gaussian random variables is [17]. Motivated by the findings of Hüsler and Reiss in 1989 (see [18]) the aforementioned contribution considered a triangular array of $N(0,1)$ random variables $\{X_{n,i}, i, n \ge 1\}$ such that for each n, $\{X_{n,i}, i \ge 1\}$ is a stationary Gaussian random sequence. Assume that $\varrho_{n,j} = \mathbb{E}(X_{n,i}X_{n,i+j})$ satisfies for any $j \ge 1$

$$
(16)\quad \lim_{n\to\infty} (1-\varrho_{n,j}) \ln n = \delta_j \in (0,\infty), \quad \delta_0 := 0
$$

and for each n, $\varrho_{n,j}$ decays fast enough as j increases. Under some additional conditions (see Theorem 3.1 below) the deep contribution [17] shows that for the maxima $M_n = \max_{1\le i\le n} X_{n,i}$

$$
(17)\quad \lim_{n\to\infty} \mathbb{P}(M_n \le a_n x + b_n) = \exp(-\vartheta \exp(-x)), \ x \in \mathbb{R},
$$

where

$$
(18)\qquad
\begin{aligned}
a_n &= (2\ln n)^{-\frac{1}{2}}, \\
b_n &= (2\ln n)^{\frac{1}{2}} - \frac{1}{2}(2\ln n)^{-\frac{1}{2}}(\ln\ln n + \ln 4\pi)
\end{aligned}
$$

and

$$
\vartheta = \mathbb{P}\left(E/2 + \sqrt{\delta_{k-1}} W_k \le \delta_{k-1}, \quad \text{for all} \quad k \ge 2\right),
$$

with E a unit exponential random variable independent of W_k and $\{W_k, k \ge 2\}$ being jointly Gaussian with zero means and covariances

$$
\mathbb{E}(W_i W_j) = \frac{\delta_{i-1} + \delta_{j-1} - \delta_{|i-j|}}{2\sqrt{\delta_{i-1}\delta_{j-1}}}.
$$

The proof of (17) strongly relies on Berman's inequality. Hence, our first application extends the result of [17] to triangular arrays of randomly scaled Gaussian random variables. In the following we investigate the effect of a comonotonic random scaling considering a bounded S and thus $\mathbf{S} = S\mathbf{1}$.

Theorem 3.1. *Let $\{X_{n,i}, i, n \ge 1\}$ be a triangular array of standard Gaussian random variables defined as above satisfying (16), being further independent of the iid non-negative random variables $\{S_n, n \ge 1\}$ where S_1 satisfies (6). If there exist positive integers r_n, l_n such that*

$$
(19)\qquad \lim_{n\to\infty} \frac{l_n}{r_n} = 0, \quad \lim_{n\to\infty} \frac{r_n}{n} = 0,
$$

$$
(20)\quad \lim_{n\to\infty} \frac{n^2}{r_n} c_n^{-\tau} \sum_{j=l_n}^{n} \frac{|\varrho_{n,j}|(1+|\varrho_{n,j}|)^{\tau}}{\sqrt{1-\varrho_{n,j}^2}} \exp\left(-\frac{c_n}{1+|\varrho_{n,j}|}\right) = 0,
$$

with $c_n := 2\ln n - (2\tau+1)\ln\ln n$ *and further*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\sum_{j=m}^{r_n} n^{-\frac{1-\varrho_{n,j}}{1+\varrho_{n,j}}}(\ln n)^{\frac{\tau(1-\varrho_{n,j})-\varrho_{n,j}}{1+\varrho_{n,j}}}\frac{1}{\sqrt{1-\varrho_{n,j}^2}} = 0, \tag{21}$$

then for the maxima $M_n = \max_{1\le i\le n} S_n X_{n,i}$ *the result in (17) holds with* ϑ *defined as above and*

$$\begin{aligned} a_n &= (2\ln n)^{-1/2}, \\ b_n &= (2\ln n)^{1/2} + (2\ln n)^{-1/2} \\ &\quad\times\left(\ln(c(2\pi)^{-1/2}\Gamma(1+\tau)) - \frac{2\tau+1}{2}(\ln\ln n + \ln 2)\right). \end{aligned} \tag{22}$$

Remark 3.1. *Using similar arguments as in the proof of Theorem 3.1, the findings of the recent contribution [6] can also be extended by considering a randomly scaled Gaussian field on a lattice.*

In our second application we consider scaled Gaussian random vectors where the scaling vector $\mathbf{S}$ has independent components. Let $\{\mathbf{X}_{n,k} = (X_{n,k}^{(1)}, X_{n,k}^{(2)}), 1\le k\le n, n\ge 1\}$ be a triangular array of bivariate centered stationary Gaussian random vectors with unit-variance and correlation given by

$$\begin{aligned} corr\left(X_{n,k}^{(1)}, X_{n,k}^{(2)}\right) &= \lambda_0(n), \\ corr\left(X_{n,k}^{(i)}, X_{n,l}^{(j)}\right) &= \lambda_{ij}(|k-l|, n), \end{aligned}$$

where $1\le k\ne l\le n$ and $i,j\in\{1,2\}$. Further, let $\{\hat{\mathbf{X}}_{n,k} = (\hat X_{n,k}^{(1)}, \hat X_{n,k}^{(2)}), 1\le k\le n, n\ge 1\}$ denote the associated iid triangular array of $\{\mathbf{X}_{n,k}\}$, i.e., $corr(\hat X_{n,k}^{(1)}, \hat X_{n,k}^{(2)}) = \lambda_0(n)$ and $corr(\hat X_{n,k}^{(i)}, \hat X_{n,l}^{(j)}) = 0$, for $1\le k\ne l\le n$ and $i,j\in\{1,2\}$. Let $\{S_{n,k}, 1\le k\le n, n\ge 1\}$ be iid random variables being independent of $\{\mathbf{X}_{n,k}, 1\le k\le n, n\ge 1\}$ and $\{\hat{\mathbf{X}}_{n,k}, 1\le k\le n, n\ge 1\}$, respectively. Assume that the correlation $\lambda_0(n)$ satisfies

$$\lim_{n\to\infty}\frac{b_n}{a_n}(1-\lambda_0(n)) = 2\lambda^2 \quad \text{with} \quad \lambda\in[0,\infty], \tag{23}$$

where

$$a_n = \frac{1}{1-F(b_n)}\int_{b_n}^{\infty}(1-F(x))dx, \quad b_n = F^{-1}\left(1-\frac{1}{n}\right),$$

with F^{-1} the inverse of the df F of $S_{1,1}\hat X_{1,1}^{(1)}$. It is well-known (see e.g., [10]) that

$$\lim_{n\to\infty}\sup_{x,y\in\mathbb{R}}\left|\mathbb{P}\left(\max_{1\le k\le n} S_{n,k}\hat X_{n,k}^{(1)} \le u_n(x), \max_{1\le k\le n} S_{n,k}\hat X_{n,k}^{(2)} \le u_n(y)\right) - H_\lambda(x,y)\right| = 0,$$

where $u_n(z) = a_n z + b_n, z\in\mathbb{R}$ and the Hüsler-Reiss distribution function H_λ is given by

$$\begin{aligned} &H_\lambda(x,y) \\ &= \exp\left(-e^{-x}\Phi\left(\lambda+\frac{y-x}{2\lambda}\right) - e^{-y}\Phi\left(\lambda+\frac{x-y}{2\lambda}\right)\right), \end{aligned} \tag{24}$$

with Φ the distribution function of an $N(0,1)$ random variable.

In the following theorem, we are interested in the case where only a fraction of random vectors is observed. Assume therefore that $\varepsilon_{n,k}$ is an indicator random variable of the event that the random vector $\mathbf{X}_{n,k}$ is observed. Then $\Xi_n = \sum_{k=1}^n \varepsilon_{n,k}$ is the number of observed random vectors from the set $\{\mathbf{X}_{n,1},\ldots,\mathbf{X}_{n,n}\}$.

We shall impose the following condition:

Condition E. The indicator random variables $\varepsilon_{n,k}$ are independent of $\mathbf{X}_{n,k}$ and $S_{n,k}$. Further, the convergence in probability

$$\frac{\Xi_n}{n}\xrightarrow{P}\eta, \quad n\to\infty$$

holds with η some random variable taking values in $(0,1]$ almost surely.

For notational simplicity we set

$$\mathbf{M}_n(\varepsilon_n) := \begin{cases}\max\{S_{n,k}\mathbf{X}_{n,k}, 1\le k\le n, \varepsilon_{n,k}=1\}, & \textit{if } \sum_{k=1}^n\varepsilon_{n,k}\ge 1, \\ \inf\{\mathbf{x}|\mathbb{P}(S_{n,k}\mathbf{X}_{n,k}\le\mathbf{x})>\mathbf{0}\}, & \textit{otherwise},\end{cases}$$

$$\mathbf{m}_n(\varepsilon_n) := \begin{cases}\min\{S_{n,k}\mathbf{X}_{n,k}, 1\le k\le n, \varepsilon_{n,k}=1\}, & \textit{if } \sum_{k=1}^n\varepsilon_{n,k}\ge 1, \\ \inf\{\mathbf{x}|\mathbb{P}(S_{n,k}\mathbf{X}_{n,k}\le\mathbf{x})>\mathbf{0}\}, & \textit{otherwise}\end{cases}$$

and $\mathbf{M}_n = \max\{S_{n,k}\mathbf{X}_{n,k}, 1\le k\le n\}$, $\mathbf{m}_n = \min\{S_{n,k}\mathbf{X}_{n,k}, 1\le k\le n\}$.

For $S_{n,k}=1, 1\le k\le n$ almost surely, according to [12], under Condition $\mathbf{E}$ we have

$$\lim_{n\to\infty}\sup_{\substack{x_1,y_1\in\mathbb{R}\\ x_1\le y_1}}\left|\mathbb{P}\left(M_n^{(1)}(\varepsilon_n)\le u_n(x_1), M_n^{(1)}\le u_n(y_1)\right) - \mathbb{E}\left(\Lambda^\eta(x_1)\Lambda^{1-\eta}(y_1)\right)\right| = 0,$$

where $u_n(x) = a_n x + b_n$ with a_n and b_n defined in (18) and $\Lambda(x) = \exp(-e^{-x})$, $x\in\mathbb{R}$, provided that $\lim_{n\to\infty}\max_{l_n<k<n}\lambda_{11}(k,n)\ln n = 0$ with $l_n = [n^{\hat\beta}]$, $0<\hat\beta<(1-\hat\sigma)/(1+\hat\sigma)$ and $\hat\sigma = \max_{1\le k<n,n\ge 2}|\lambda_{11}(k,n)|$. Below we obtain a more general result for our 2-dimensional setup considering Weibull-type random scaling.

Theorem 3.2. *Let* $\{(X_{n,k}^{(1)}, X_{n,k}^{(2)}), 1\le k\le n, n\ge 1\}$ *be a bivariate triangular array of standard Gaussian random vectors defined as above. Let* $\{S_{n,k}, 1\le k\le n, n\ge 1\}$ *be iid* *random variables being independent of* $\{(X_{n,k}^{(1)}, X_{n,k}^{(2)}), 1 \le k \le n, n \ge 1\}$. *Suppose that the correlation* $\lambda_0(n)$ *satisfy* (23) *with* $\lambda \in (0,\infty)$ *and condition* $\mathbf{E}$ *holds. Let* β *be a constant satisfying* $0 < \beta < 2(1+\sigma)^{-\frac{p}{2+p}} - 1$ *with* $\sigma = \max_{\substack{1\le k<n, n\ge 2\\ i,j\in\{1,2\}}} |\lambda_{ij}(k,n)| < 1$, *and write* $\iota_n = [n^\beta]$. *If* (11) *holds and the covariance function satisfies*

$$\lim_{n\to\infty} \max_{\substack{\iota_n\le k<n\\ i,j\in\{1,2\}}} |\lambda_{ij}(k,n)| \ln n = 0,$$

then we have

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{\substack{x_i,y_i\in\mathbb{R}, i\le 4\\ x_1\le x_3, x_2\le x_4, y_1\le y_3, y_2\le y_4}} \\
&\Big|\mathbb{P}\Big(-u_n(y_1) < m_n^{(1)}(\varepsilon_n) \le M_n^{(1)}(\varepsilon_n) \le u_n(x_1),\\
&\quad -u_n(y_2) < m_n^{(2)}(\varepsilon_n) \le M_n^{(2)}(\varepsilon_n) \le u_n(x_2),\\
&\quad -u_n(y_3) < m_n^{(1)} \le M_n^{(1)} \le u_n(x_3),\\
&\quad -u_n(y_4) < m_n^{(2)} \le M_n^{(2)} \le u_n(x_4)\Big)\\
&\quad -\mathbb{E}\left(H_\lambda^\eta(x_1,x_2)H_\lambda^\eta(y_1,y_2)H_\lambda^{1-\eta}(x_3,x_4)H_\lambda^{1-\eta}(y_3,y_4)\right)\Big|\\
&= 0,
\end{aligned}$$

where H_λ *is defined in* (24) *and norming constants* a_n *and* b_n *satisfy*

$$\begin{aligned}
a_n &= \frac{2+p}{2p}T^{-\frac{2+p}{2p}}(\ln n)^{\frac{2-p}{2p}},\\
b_n &= \left(\frac{\ln n}{T}\right)^{\frac{2+p}{2p}} + \frac{2+p}{2p}T^{-\frac{2+p}{2p}}(\ln n)^{\frac{2-p}{2p}}\\
&\quad \times\left(\frac{\alpha}{p}\ln\ln n - \frac{\alpha}{p}\ln T + \ln \varpi_B\right)
\end{aligned} \tag{25}$$

with $T = 2^{-1}Q^2 + LQ^{-p}$, $\varpi_B = c_B(2+p)^{-\frac{1}{2}}Q^{-\alpha}$ *and* $Q = (pL)^{1/(2+p)}$.

4. PROOFS

Proof of Theorem 2.1 By the independence of $\mathbf{S}$ and $(\mathbf{X},\mathbf{Y})$ and the generalised Berman's inequality (see Theorem 2.1 in [22] and Lemma 11.1.2 in [21]), if (5) holds, then

$$\begin{aligned}
&\Delta_{\mathbf{S}}(\mathbf{u},\mathbf{v})\\
=\;& \mathbb{P}(-v_i < S_iX_i \le u_i, 1\le i\le n)\\
&-\mathbb{P}(-v_i < S_iY_i \le u_i, 1\le i\le n)\\
=\;& \int_{[0,1]^n}\left(\mathbb{P}\left(-\frac{v_i}{s_i} < X_i \le \frac{u_i}{s_i}, 1\le i\le n\right)\right.\\
&\left.-\mathbb{P}\left(-\frac{v_i}{s_i} < Y_i \le \frac{u_i}{s_i}, 1\le i\le n\right)\right) dG(s_1)\cdots dG(s_n)\\
\le\;& \frac{2}{\pi}\int_{[0,1]^n}\sum_{1\le i<j\le n} A_{ij}\\
&\exp\left(-\frac{(w/s_i)^2+(w/s_j)^2}{2(1+\rho_{ij})}\right) dG(s_1)\cdots dG(s_n)\\
=\;& \frac{2}{\pi}\sum_{1\le i<j\le n} A_{ij}\\
&\int_0^1\int_0^1 \exp\left(-\frac{(w/s)^2+(w/t)^2}{2(1+\rho_{ij})}\right) dG(s)dG(t),
\end{aligned}$$

where ρ_{ij} and A_{ij} are defined in (1) and $w = \min_{1\le i\le n}\min(|\ u_i\ |, |\ v_i\ |)$. Note that for $1\le i,j\le n$, $\varepsilon>0$

$$\begin{aligned}
&\int_0^1 \exp\left(-\frac{1}{2(1+\rho_{ij})}\left(\frac{w}{s}\right)^2\right) dG(s)\\
\sim&\int_{\frac{1}{\varepsilon+1}}^1 \exp\left(-\frac{1}{2(1+\rho_{ij})}\left(\frac{w}{s}\right)^2\right) dG(s)\\
=&\int_w^{w(1+\varepsilon)} \mathbb{P}\left(S > \frac{w}{s}\right) d\left(1-\exp\left(-\frac{1}{2(1+\rho_{ij})}s^2\right)\right)\\
=&\int_0^{\frac{\varepsilon}{1+\rho_{ij}}w^2} \mathbb{P}\left(S > \frac{w}{w+(1+\rho_{ij})w^{-1}t}\right)\left(1+\frac{1+\rho_{ij}}{w^2}t\right)\\
&\exp\left(-\frac{1}{2(1+\rho_{ij})}(w^2+2(1+\rho_{ij})t+(1+\rho_{ij})^2w^{-2}t^2)\right) dt\\
\sim&\int_0^{\frac{\varepsilon}{1+\rho_{ij}}w^2} \mathbb{P}\left(S > 1-\frac{1+\rho_{ij}}{w^2}t\right)\exp\left(-t-\frac{w^2}{2(1+\rho_{ij})}\right) dt\\
\le& c_A(1+\rho_{ij})^\tau w^{-2\tau}\exp\left(-\frac{w^2}{2(1+\rho_{ij})}\right)\\
&\times\int_0^{\frac{\varepsilon}{1+\rho_{ij}}w^2} t^\tau \exp(-t)\, dt\\
\sim& c_A\Gamma(\tau+1)(1+\rho_{ij})^\tau w^{-2\tau}\exp\left(-\frac{w^2}{2(1+\rho_{ij})}\right),
\end{aligned}$$

as $w\to\infty$. Consequently, for any $\epsilon>0$ and all large $u_i, v_i, i\le n$

$$\begin{aligned}
\Delta_{\mathbf{S}}(\mathbf{u},\mathbf{v}) \;\le\;& \frac{2}{\pi}(\Gamma(\tau+1))^2(c_A^2+\epsilon)w^{-4\tau}\\
&\sum_{1\le i<j\le n} A_{ij}(1+\rho_{ij})^{2\tau}\exp\left(-\frac{w^2}{1+\rho_{ij}}\right).
\end{aligned}$$

With similar arguments as above we have

$$\begin{aligned}
&\Delta_{S\mathbf{1}}(\mathbf{u},\mathbf{v})\\
=\;& \int_0^1\left(\mathbb{P}\left(-\frac{v_i}{s} < X_i \le \frac{u_i}{s}, 1\le i\le n\right)\right.\\
&\left.-\mathbb{P}\left(-\frac{v_i}{s} < Y_i \le \frac{u_i}{s}, 1\le i\le n\right)\right) dG(s)\\
\le\;& \frac{2}{\pi}\sum_{1\le i<j\le n} A_{ij}\int_0^1 \exp\left(-\frac{(w/s)^2}{1+\rho_{ij}}\right) dG(s)\\
\le\;& \frac{2^{1-\tau}}{\pi}\Gamma(\tau+1)(c_A+\epsilon)w^{-2\tau}
\end{aligned}$$

$$\sum_{1\le i<j\le n} A_{ij}(1+\rho_{ij})^{\tau}\exp\left(-\frac{w^2}{1+\rho_{ij}}\right),$$

hence the claim follows. $\square$

Proof of Theorem 2.2 According to the independence of the scaling factors with the Gaussian random variables and the generalised Berman's inequality (see Theorem 2.1 in [22] and Lemma 11.1.2 in [21]) again if (11) holds, then we have

$$\begin{aligned}
&\Delta_{\mathbf{S}}(\mathbf{u},\mathbf{v})\\
=\ & \int_{[0,\infty]^n}\left(\mathbb{P}\left(-\frac{v_i}{s_i}<X_i\le \frac{u_i}{s_i}, 1\le i\le n\right)\right.\\
&\left.-\ \mathbb{P}\left(-\frac{v_i}{s_i}<Y_i\le \frac{u_i}{s_i}, 1\le i\le n\right)\right) dG(s_1)\cdots dG(s_n)\\
\le\ & \frac{2}{\pi}\int_{[0,\infty]^n}\sum_{1\le i<j\le n} A_{ij}\exp\left(-\frac{(w/s_i)^2+(w/s_j)^2}{2(1+\rho_{ij})}\right)\\
&\qquad dG(s_1)\cdots dG(s_n)\\
=\ & \frac{2}{\pi}\sum_{1\le i<j\le n} A_{ij}\\
&\int_0^\infty\int_0^\infty \exp\left(-\frac{(w/s)^2+(w/t)^2}{2(1+\rho_{ij})}\right) dG(s)dG(t),
\end{aligned}$$

where ρ_{ij} and A_{ij} are defined in (1). Note that for $1\le i,j\le n$ and some positive constants c_1,c_2, using similar arguments as in the proof of Theorem 2.1 in [16], we have

$$\begin{aligned}
&\int_0^\infty \exp\left(-\frac{1}{2(1+\rho_{ij})}\left(\frac{w}{s}\right)^2\right) dG(s)\\
&\sim \int_{c_1 w^{\frac{2}{p+2}}}^{c_2 w^{\frac{2}{p+2}}} \exp\left(-\frac{1}{2(1+\rho_{ij})}\left(\frac{w}{s}\right)^2\right) dG(s)\\
&\sim c_B Lp\int_{c_1 w^{\frac{2}{p+2}}}^{c_2 w^{\frac{2}{p+2}}} s^{\alpha+p-1}\\
&\times \exp\left(-Ls^p-\frac{1}{2(1+\rho_{ij})}\left(\frac{w}{s}\right)^2\right) ds\\
&= c_B Lp\left(\frac{w^2}{Lp(1+\rho_{ij})}\right)^{\frac{\alpha+p}{p+2}}\int_{c_1(Lp(1+\rho_{ij}))^{\frac{1}{p+2}}}^{c_2(Lp(1+\rho_{ij}))^{\frac{1}{p+2}}} t^{\alpha+p-1}\\
&\times \exp\left(-Lp\left(\frac{w^2}{Lp(1+\rho_{ij})}\right)^{\frac{p}{p+2}}\left(p^{-1}t^p+2^{-1}t^{-2}\right)\right) dt\\
&\sim \sqrt{\frac{2\pi}{p+2}}c_B(Lp)^{\frac{1-\alpha}{p+2}}(1+\rho_{ij})^{\frac{-2\alpha-p}{2(p+2)}}w^{\frac{2\alpha+p}{p+2}}\\
&\times \exp\left(-(1+\rho_{ij})^{-\frac{p}{p+2}}\left(L^{\frac{2}{p+2}}p^{-\frac{p}{p+2}}+(Lp)^{\frac{2}{p+2}}2^{-1}\right)w^{\frac{2p}{p+2}}\right),
\end{aligned}$$

as $w\to\infty$. Hence for $\epsilon>0$ we have

$$\Delta_{\mathbf{S}}(\mathbf{u},\mathbf{v})\le \frac{4(c_B^2+\epsilon)(Lp)^{\frac{2(1-\alpha)}{p+2}}}{p+2}w^{\frac{4\alpha+2p}{2+p}}$$

$$\sum_{1\le i<j\le n} A_{ij}(1+\rho_{ij})^{\frac{-2\alpha-p}{p+2}}\exp\left(-2(1+\rho_{ij})^{-\frac{p}{2+p}}Tw^{\frac{2p}{2+p}}\right),$$

where $T=L^{\frac{2}{p+2}}p^{-\frac{p}{p+2}}+(Lp)^{\frac{2}{p+2}}2^{-1}$. Proceeding as above

$$\begin{aligned}
&\Delta_{S\mathbf{1}}(\mathbf{u},\mathbf{v})\\
=\ & \int_0^\infty\left(\mathbb{P}\left(-\frac{v_i}{s}<X_i\le\frac{u_i}{s}, 1\le i\le n\right)\right.\\
&\left.-\ \mathbb{P}\left(-\frac{v_i}{s}<Y_i\le\frac{u_i}{s}, 1\le i\le n\right)\right) dG(s)\\
\le\ & \frac{2}{\pi}\sum_{1\le i<j\le n} A_{ij}\int_0^\infty \exp\left(-\frac{(w/s)^2}{1+\rho_{ij}}\right) dG(s)\\
\le\ & 2^{\frac{3+2p+\alpha}{2+p}}\pi^{-\frac12}(c_B+\epsilon)(Lp)^{\frac{1-\alpha}{p+2}}(p+2)^{-\frac12}w^{\frac{2\alpha+p}{2+p}}\\
&\times\sum_{1\le i<j\le n} A_{ij}(1+\rho_{ij})^{\frac{-2\alpha-p}{2(p+2)}}\\
&\times\exp\left(-(2(1+\rho_{ij})^{-1})^{\frac{p}{2+p}}Tw^{\frac{2p}{2+p}}\right),
\end{aligned}$$

hence the proof is complete. $\square$

Lemma 4.1. *Under the conditions of Theorem 3.1, for any bounded set $K\subset\{2,3\ldots\}$ we have*

$$\begin{aligned}
&\lim_{n\to\infty}\mathbb{P}\left(S_nX_{n,k}\le u_n, k\in K|S_nX_{n,1}>u_n\right)\\
=\ & \mathbb{P}\left(E/2+\sqrt{\delta_{k-1}}W_k\le\delta_{k-1}, k\in K\right),
\end{aligned}$$

where E is a standard exponential random variable independent of $\{W_k, k\in K\}$ and the W_k have a jointly Gaussian distribution with mean zero and

$$\mathbb{E}(W_iW_j)=\frac{\delta_{i-1}+\delta_{j-1}-\delta_{|i-j|}}{2\sqrt{\delta_{i-1}\delta_{j-1}}},\quad i,j\in K.$$

Proof of Lemma 4.1 A centered Gaussian random vector $\mathbf{X}_n=(X_{n,k},k\in K\cup\{1\})^\top$ with covariance matrix $B_n^\top B_n=(\varrho_{n,|i-j|})_{i,j\in K\cup\{1\}}$ has stochastic representation

$$(X_{n,k},k\in K\cup\{1\})^\top\stackrel{d}{=}RB_n^\top\mathbf{U}_{m+1},$$

where m is the cardinality of set K, R is a positive random variable such that R^2 is chi-squared distributed with $m+1$ degrees of freedom and independent of $\mathbf{U}_{m+1}$ which is a random vector uniformly distributed on the unit sphere of $\mathbb{R}^{m+1}$. Since S_n is independent of $X_{n,k}$ using Corollary 5 in [3] we have (set $t_n(y):=u_n+y/u_n$)

$$(S_nX_{n,k},k\in K|S_nX_{n,1}=t_n(y))^\top\stackrel{d}{=}R_{m,y}\hat{B}_n^\top\mathbf{U}_m+t_n(y)\Sigma_{12},$$

where $\Sigma_{12}=(\varrho_{n,k-1},k\in K)^\top$, $\hat{B}_n^\top\hat{B}_n=(\varrho_{n,|i-j|}-\varrho_{n,i-1}\varrho_{n,j-1})_{i,j\in K}$ and $R_{m,y}$ is a positive random variable independent of $\mathbf{U}_m$ with distribution function $F_{m,y}$ defined by

$$F_{m,y}(x)=\frac{\int_{t_n(y)}^{((t_n(y))^2+x^2)^{1/2}}(s^2-(t_n(y))^2)^{\frac{m}{2}-1}s^{1-m}dF_1(s)}{\int_{t_n(y)}^{\infty}(s^2-(t_n(y))^2)^{\frac{m}{2}-1}s^{1-m}dF_1(s)},$$

$x > 0$, with F_1 the distribution function of $S_n R$. According to Theorem 3.1 in [11] F_1 in the Gumbel max-domain of attraction and

$$\lim_{n\to\infty} \frac{\mathbb{P}\left(S_n X_{n,1} > t_n(y)\right)}{\mathbb{P}\left(S_n X_{n,1} > u_n\right)} = e^{-y}, \qquad \forall y \in \mathbb{R}. \tag{26}$$

Hence, by Theorem 3.1 in [8]

$$\begin{aligned} p_{n,y} :=& \mathbb{P}\left(S_n X_{n,k} \le u_n, k \in K | S_n X_{n,1} = t_n(y)\right) \\ =& \mathbb{P}\left(\frac{u_n(1-\varrho_{n,k-1}^2)^{1/2}}{2} Z_{n,k} + \frac{\varrho_{n,k-1}}{2} y \right.\\ & \left. \le \frac{u_n^2(1-\varrho_{n,k-1})}{2}, k \in K\right) \\ \to & \mathbb{P}\left(\sqrt{\delta_{k-1}} W_k + \frac{y}{2} \le \delta_{k-1}, k \in K\right), \quad n \to \infty \end{aligned} \tag{27}$$

uniformly on compact sets of y, where

$$(Z_{n,k}, k \in K)^\top \stackrel{d}{=} R_{m,y} \tilde{B}_n^\top \mathbf{U}_m,$$

with

$$\tilde{B}_n^\top \tilde{B}_n = \left(\frac{\varrho_{n,|i-j|} - \varrho_{n,i-1}\varrho_{n,j-1}}{\sqrt{(1-\varrho_{n,i-1}^2)(1-\varrho_{n,j-1}^2)}} \right)_{i,j \in K}$$

and $\{W_k, k \in K\}$ being jointly Gaussian with zero means and covariances

$$\mathbb{E}\left(W_i W_j\right) = \frac{\delta_{i-1} + \delta_{j-1} - \delta_{|i-j|}}{2\sqrt{\delta_{i-1}\delta_{j-1}}}, \quad i, j \in K.$$

Since further

$$\begin{aligned} & \mathbb{P}\left(S_n X_{n,k} \le u_n, k \in K | S_n X_{n,1} > u_n\right) \\ = & \int_0^\infty p_{n,y} d \frac{\mathbb{P}\left(S_n X_{n,1} \le t_n(y)\right)}{\mathbb{P}\left(S_n X_{n,1} > u_n\right)} \end{aligned}$$

the proof is established by applying Lemma 4.4 in [8] (recall (26) and (27)). ☐

Proof of Theorem 3.1 According to (8), if $1 \le k_1 < \cdots < k_s \le n$ and $k = \min_{1 \le i < s}(k_{i+1} - k_i)$ then the joint distribution function $F_{k_1,\ldots,k_s}$ of $S_n X_{n,k_1}, \ldots, S_n X_{n,k_s}$ satisfies

$$\begin{aligned} & \left| F_{k_1,\ldots,k_s}(u_n) - \prod_{i=1}^s \mathbb{P}\left(S_n X_{n,k_i} \le u_n\right) \right| \\ \le & \mathcal{Q} u_n^{-2\tau} n \sum_{i=k}^n \frac{|\varrho_{n,i}|(1+|\varrho_{n,i}|)^\tau}{\sqrt{1-\varrho_{n,i}^2}} \exp\left(-\frac{u_n^2}{1+|\varrho_{n,i}|}\right). \end{aligned}$$

Suppose now that $1 \le i_1 < \cdots < i_p < j_1 < \cdots < j_{p'} \le n$ and $j_1 - i_p \ge l_n$. Identifying $\{k_1, \ldots, k_s\}$ in turn with $\{i_1, \ldots, i_p, j_1, \ldots, j_{p'}\}$, $\{i_1, \ldots, i_p\}$ and $\{j_1, \ldots, j_{p'}\}$, we thus have

$$\begin{aligned} & |F_{i_1,\ldots,i_p,j_1,\ldots,j_{p'}}(u_n) - F_{i_1,\ldots,i_p}(u_n) F_{j_1,\ldots,j_{p'}}(u_n)| \\ & \le 3\mathcal{Q} u_n^{-2\tau} n \sum_{i=l_n}^n \frac{|\varrho_{n,i}|(1+|\varrho_{n,i}|)^\tau}{\sqrt{1-\varrho_{n,i}^2}} \exp\left(-\frac{u_n^2}{1+|\varrho_{n,i}|}\right). \end{aligned}$$

By Example 1 in [9] and Table 3.4.4 in [5] we have

$$\lim_{n\to\infty} n\mathbb{P}\left(S_n X_{n,1} \ge u_n(x)\right) = e^{-x}, \quad \forall \ x \in \mathbb{R},$$

where $u_n(x) = a_n x + b_n$ with a_n and b_n defined in (22). Consequently, as $n \to \infty$

$$u_n^2(x) = 2\ln n - (2\tau + 1)\ln\ln n + O(1). \tag{28}$$

Hence, in view of (19) and (20), Theorem 2.1 in [23] implies

$$\begin{aligned} \lim_{n\to\infty} \Bigg[& \mathbb{P}\left(\max_{1 \le i \le n} S_n X_{n,i} \le u_n(x)\right) \\ & - \exp\Big(- n\mathbb{P}\left(S_n X_{n,1} > u_n(x)\right) \\ & \mathbb{P}\Big(\bigcap_{i=2}^{r_n} \{S_n X_{n,i} \le u_n(x)\} | S_n X_{n,1} > u_n(x)\Big)\Big)\Bigg] = 0. \end{aligned}$$

Note that for $m \le j \le r_n$ we have

$$\begin{aligned} & \mathbb{P}\left(W > u_n \sqrt{\frac{1-\varrho_{n,j}}{1+\varrho_{n,j}}} - \frac{y}{u_n} \frac{\varrho_{n,j}}{\sqrt{1-\varrho_{n,j}^2}}\right) \\ \le \quad & \mathcal{Q} n^{-\frac{1-\varrho_{n,j}}{1+\varrho_{n,j}}} \frac{(\ln n)^{\frac{\tau(1-\varrho_{n,j})-\varrho_{n,j}}{1+\varrho_{n,j}}}}{\sqrt{1-\varrho_{n,j}^2}}, \end{aligned}$$

where W is a $N(0,1)$ random variable. The claim can then be established by using similar arguments as in the proof of Theorem 2.1 in [17] making further use of (21) and Lemma 4.1. ☐

Next, for some index sets $I_n \subset N$ we define

$$\begin{aligned} & \widehat{\mathbf{M}}(I_n, \varepsilon_n) := \\ & \begin{cases} \max\{S_{n,k}\hat{\mathbf{X}}_{n,k}, k \in I_n, \varepsilon_{n,k} = 1\}, & \textit{if } \sum_{k \in I_n} \varepsilon_{n,k} \ge 1; \\ \inf\left\{\mathbf{x} | \mathbb{P}\left(S_{n,k}\hat{\mathbf{X}}_{n,k} \le \mathbf{x}\right) > \mathbf{0}\right\}, & \textit{otherwise}, \end{cases} \end{aligned}$$

$$\begin{aligned} & \widehat{\mathbf{m}}(I_n, \varepsilon_n) := \\ & \begin{cases} \min\{S_{n,k}\hat{\mathbf{X}}_{n,k}, k \in I_n, \varepsilon_{n,k} = 1\}, & \textit{if } \sum_{k \in I_n} \varepsilon_{n,k} \ge 1; \\ \inf\left\{\mathbf{x} | \mathbb{P}\left(S_{n,k}\hat{\mathbf{X}}_{n,k} \le \mathbf{x}\right) > \mathbf{0}\right\}, & \textit{otherwise}. \end{cases} \end{aligned}$$

For simplicity, we write $\widehat{\mathbf{M}}_n(\varepsilon_n) = \widehat{\mathbf{M}}(\{1, 2, \ldots, n\}, \varepsilon_n)$, $\widehat{\mathbf{M}}(I_n) = \max\{S_{n,k}\hat{\mathbf{X}}_{n,k}, k \in I_n\}$, $\widehat{\mathbf{M}}_n = \max\{S_{n,k}\hat{\mathbf{X}}_{n,k}, 1 \le k \le n\}$. Similarly we also define $\widehat{\mathbf{m}}_n(\varepsilon_n), \widehat{\mathbf{m}}(I_n), \widehat{\mathbf{m}}_n$.

Lemma 4.2. *Let $\{(\hat{X}^{(1)}_{n,i}, \hat{X}^{(2)}_{n,i}), 1 \le i \le n, n \ge 1\}$ be a triangular array of centered stationary Gaussian random vectors defined as above with the correlation $\lambda_0(n)$ satisfying (23) with $\lambda \in (0,\infty)$. Further let $\{S_{n,k}, 1 \le k \le n, n \ge 1\}$ be iid random variables being independent of $\{(\hat{X}^{(1)}_{n,i}, \hat{X}^{(2)}_{n,i}), 1 \le i \le n, n \ge 1\}$ and satisfying (11). Then we have*

$$\begin{aligned}&\lim_{n\to\infty} \mathbb{P}\Big(-u_n(y_1) < \widehat{m}^{(1)}_n \le \widehat{M}^{(1)}_n \le u_n(x_1),\\ &\qquad -u_n(y_2) < \widehat{m}^{(2)}_n \le \widehat{M}^{(2)}_n \le u_n(x_2)\Big)\\ =\ & H_\lambda(x_1,x_2)H_\lambda(y_1,y_2).\end{aligned}$$

Proof of Lemma 4.2 Our proof is similar to that of Theorem 2.1 in [14]. For any integer n we may write

$$\begin{aligned}& n\left(1 - P(n,x_1,x_2,y_1,y_2)\right)\\ =\ & nP_1(n,x_1,x_2) + nP_2(n,y_1,y_2)\\ & - nP_3(n,x_1,y_2) - nP_4(n,y_1,x_2),\end{aligned}$$

where

$$\begin{aligned}& P(n,x_1,x_2,y_1,y_2)\\ :=\ & \mathbb{P}\Big(-u_n(y_1) < S_{n,1}\hat{X}^{(1)}_{n,1} \le u_n(x_1),\\ &\qquad -u_n(y_2) < S_{n,1}\hat{X}^{(2)}_{n,1} \le u_n(x_2)\Big),\\ & P_1(n,x_1,x_2)\\ :=\ & \mathbb{P}\left(S_{n,1}\hat{X}^{(1)}_{n,1} > u_n(x_1)\right) + \mathbb{P}\left(S_{n,1}\hat{X}^{(2)}_{n,1} > u_n(x_2)\right)\\ & - \mathbb{P}\left(S_{n,1}\hat{X}^{(1)}_{n,1} > u_n(x_1), S_{n,1}\hat{X}^{(2)}_{n,1} > u_n(x_2)\right),\\ & P_2(n,y_1,y_2)\\ :=\ & \mathbb{P}\left(S_{n,1}\hat{X}^{(1)}_{n,1} \le -u_n(y_1)\right) + \mathbb{P}\left(S_{n,1}\hat{X}^{(2)}_{n,1} \le -u_n(y_2)\right)\\ & - \mathbb{P}\left(S_{n,1}\hat{X}^{(1)}_{n,1} \le -u_n(y_1), S_{n,1}\hat{X}^{(2)}_{n,1} \le -u_n(y_2)\right),\\ & P_3(n,x_1,y_2)\\ :=\ & \mathbb{P}\left(S_{n,1}\hat{X}^{(1)}_{n,1} > u_n(x_1), S_{n,1}\hat{X}^{(2)}_{n,1} \le -u_n(y_2)\right),\\ & P_4(n,y_1,x_2)\\ :=\ & \mathbb{P}\left(S_{n,1}\hat{X}^{(1)}_{n,1} \le -u_n(y_1), S_{n,1}\hat{X}^{(2)}_{n,1} > u_n(x_2)\right).\end{aligned}$$

The random vector $(\hat{X}^{(1)}_{n,1}, \hat{X}^{(2)}_{n,1})$ has the following stochastic representation

$$(\hat{X}^{(1)}_{n,1}, \hat{X}^{(2)}_{n,1}) \stackrel{d}{=} (R\cos\theta, R\cos(\theta-\psi_n)),$$

where R is a positive random variable being independent of the random variable θ which is uniformly distributed in $(-\pi,\pi)$ and $\psi_n = \arccos(\lambda_0(n))$. If $S_{n,1}$ satisfy (11) and is independent of $(\hat{X}^{(1)}_{n,1}, \hat{X}^{(2)}_{n,1})$, using Laplace approximation (see e.g.,[16]) we have that the distribution function of $S_{n,1}R$ is in the max-domain of attraction of the Gumbel distribution. Hence, according to Remark 2.2 in [13] we have

$$\lim_{n\to\infty} n\mathbb{P}\left(S_{n,1}\hat{X}^{(1)}_{n,1} > u_n(x)\right) = e^{-x}, \quad x \in \mathbb{R}, \tag{29}$$

where $u_n(x) = a_n x + b_n$ with a_n and b_n defined in (25). Moreover, by Theorem 2.1 in [7]

$$\begin{aligned}& \lim_{n\to\infty} nP_1(n,x_1,x_2)\\ =\ & \Phi\left(\lambda + \frac{x_1-x_2}{2\lambda}\right)e^{-x_2} + \Phi\left(\lambda + \frac{x_2-x_1}{2\lambda}\right)e^{-x_1}\\ =:\ & D(x_1,x_2)\end{aligned}$$

and since $(-S_{n,1}\hat{X}^{(1)}_{n,1}, -S_{n,1}\hat{X}^{(2)}_{n,1}) \stackrel{d}{=} (S_{n,1}\hat{X}^{(1)}_{n,1}, S_{n,1}\hat{X}^{(2)}_{n,1})$

$$\lim_{n\to\infty} nP_2(n,y_1,y_2) = D(y_1,y_2).$$

Since $\lim_{n\to\infty}\lambda_0(n) = 1$, $\lim_{n\to\infty}\psi_n = 0$ implying

$$\begin{aligned}& \lim_{n\to\infty} nP_3(n,x_1,y_2)\\ =\ & \lim_{n\to\infty} n\mathbb{P}\left(S_{n,1}R\cos(\theta) > u_n(x_1),\right.\\ & \left. S_{n,1}R\cos(\theta-\psi_n) \le -u_n(y_1)\right)\\ =\ & \lim_{n\to\infty} n\mathbb{P}\left(S_{n,1}R\cos(\theta) > u_n(x_1), \cos(\theta) > 0,\right.\\ & \left. S_{n,1}R\cos(\theta-\psi_n) \le -u_n(y_1), \cos(\theta-\psi_n) < 0\right)\\ =\ & \lim_{n\to\infty} n\mathbb{P}\Big(S_{n,1}R\cos(\theta) > u_n(x_1),\\ & S_{n,1}R\cos(\theta-\psi_n) \le -u_n(y_1),\\ & \max\left(-\frac{\pi}{2}, -\pi+\psi_n\right) < \theta < -\frac{\pi}{2} + \psi_n\Big)\\ =\ & 0.\end{aligned}$$

Similarly, we have $\lim_{n\to\infty} nP_4(n,y_1,x_2) = 0$. Hence for all $x_1,x_2,y_1,y_2 \in \mathbb{R}$

$$\begin{aligned}& \lim_{n\to\infty} \mathbb{P}\Big(-u_n(y_1) < \widehat{m}^{(1)}_n \le \widehat{M}^{(1)}_n \le u_n(x_1),\\ & -u_n(y_2) < \widehat{m}^{(2)}_n \le \widehat{M}^{(2)}_n \le u_n(x_2)\Big)\\ =\ & \lim_{n\to\infty} \left(P(n,x_1,x_2,y_1,y_2)\right)^n\\ =\ & \lim_{n\to\infty} \left(1 - (1 - P(n,x_1,x_2,y_1,y_2))\right)^n\\ =\ & \lim_{n\to\infty} \left(1 - \frac{D(x_1,x_2)+D(y_1,y_2)}{n} + o\left(\frac{1}{n}\right)\right)^n\\ =\ & \exp\left(-D(x_1,x_2) - D(y_1,y_2)\right)\\ =\ & H_\lambda(x_1,x_2)H_\lambda(y_1,y_2),\end{aligned}$$

hence the proof is complete. □

Lemma 4.3. *Under the conditions of Lemma 4.2, if the indicator random variables $\varepsilon_n = \{\varepsilon_{n,i}, 1 \le i \le n\}$ are independent of both $\{(\hat{X}^{(1)}_{n,i}, \hat{X}^{(2)}_{n,i}), 1 \le i \le n\}$ and $\{S_{n,i}, 1 \le i \le n\}$ and satisfying condition* **E**, *then*

$$\lim_{n\to\infty} \sup_{\substack{x_i, y_i \in \mathbb{R}, i=\{1,2,3,4\}\\ x_1 \le x_3, x_2 \le x_4, y_1 \le y_3, y_2 \le y_4}}$$

$$\Big|\mathbb{P}\Big(-u_n(y_1)<\widehat{m}_n^{(1)}(\varepsilon_n)\le\widehat{M}_n^{(1)}(\varepsilon_n)\le u_n(x_1),$$
$$-u_n(y_2)<\widehat{m}_n^{(2)}(\varepsilon_n)\le\widehat{M}_n^{(2)}(\varepsilon_n)\le u_n(x_2),$$
$$-u_n(y_3)<\widehat{m}_n^{(1)}\le\widehat{M}_n^{(1)}\le u_n(x_3),$$
$$-u_n(y_4)<\widehat{m}_n^{(2)}\le\widehat{M}_n^{(2)}\le u_n(x_4)\Big)$$
$$-\mathbb{E}\left(H_\lambda^\eta(x_1,x_2)H_\lambda^\eta(y_1,y_2)H_\lambda^{1-\eta}(x_3,x_4)H_\lambda^{1-\eta}(y_3,y_4)\right)\Big|$$
$$=0.$$

Proof of Lemma 4.3 Using similar arguments as for the derivation of [19], let $K_s=\{j:(s-1)\nu+1\le j\le s\nu\}, 1\le s\le l$, $\nu=[\frac{n}{l}]$, $\mathbf{x}=(x_1,x_2,x_3,x_4)$, $\mathbf{y}=(y_1,y_2,y_3,y_4)$ and $\beta_n=\{\beta_{n,k},1\le k\le n\}$ be a nonrandom triangular array consisting of 0's and 1's. For some random variable η such that $0\le\eta\le1$ a.s., write

$$B_{\mu,l}=\left\{\omega:\eta(\omega)\in\left\{\begin{array}{ll}[0,\frac{1}{2^l}], & \mu=0,\\ (\frac{\mu}{2^l},\frac{\mu+1}{2^l}], & 0<\mu\le 2^l-1\end{array}\right\}\right\},$$

$$B_{\mu,l,\beta_n}=\{\omega:\varepsilon_{n,k}(\omega)=\beta_{n,k},1\le k\le n\}\cap B_{\mu,l}.$$

Set

$$\begin{aligned}&P(K_s,\beta_n,\mathbf{x},\mathbf{y})\\ =\ &\mathbb{P}\Big(-u_n(y_1)<\widehat{m}^{(1)}(K_s,\beta_n)\le\widehat{M}^{(1)}(K_s,\beta_n)\le u_n(x_1),\\ &-u_n(y_2)<\widehat{m}^{(2)}(K_s,\beta_n)\le\widehat{M}^{(2)}(K_s,\beta_n)\le u_n(x_2),\\ &-u_n(y_3)<\widehat{m}^{(1)}(K_s)\le\widehat{M}^{(1)}(K_s)\le u_n(x_3),\\ &-u_n(y_4)<\widehat{m}^{(2)}(K_s)\le\widehat{M}^{(2)}(K_s)\le u_n(x_4)\Big)\end{aligned}$$

and

$$\begin{aligned}&P(n,\beta_n,\mathbf{x},\mathbf{y})\\ =\ &\mathbb{P}\Big(-u_n(y_1)<\widehat{m}_n^{(1)}(\beta_n)\le\widehat{M}_n^{(1)}(\beta_n)\le u_n(x_1),\\ &-u_n(y_2)<\widehat{m}_n^{(2)}(\beta_n)\le\widehat{M}_n^{(2)}(\beta_n)\le u_n(x_2),\\ &-u_n(y_3)<\widehat{m}_n^{(1)}\le\widehat{M}_n^{(1)}\le u_n(x_3),\\ &-u_n(y_4)<\widehat{m}_n^{(2)}\le\widehat{M}_n^{(2)}\le u_n(x_4)\Big).\end{aligned}$$

Using similar arguments as in the proof of Lemma 3.3 in [24] for n large we can choose a positive integer $\tilde{\nu}_n$ such that $l<\tilde{\nu}_n<\nu$ and $\tilde{\nu}_n=o(n)$, by (29) we have

$$\begin{aligned}(30)\quad&\left|P(n,\beta_n,\mathbf{x},\mathbf{y})-\prod_{s=1}^{l}P(K_s,\beta_n,\mathbf{x},\mathbf{y})\right|\\ &\le(4l+2)\tilde{\nu}_n\\ &\left(\mathbb{P}\left(S_{n,1}\hat{X}_{n,1}^{(1)}\le-u_n(y_1)\right)+\mathbb{P}\left(S_{n,1}\hat{X}_{n,1}^{(1)}>u_n(x_1)\right)\right.\\ &\left.+\mathbb{P}\left(S_{n,1}\hat{X}_{n,1}^{(2)}\le-u_n(y_2)\right)+\mathbb{P}\left(S_{n,1}\hat{X}_{n,1}^{(2)}>u_n(x_2)\right)\right)\\ &\to0,\quad n\to\infty.\end{aligned}$$

Note that

$$\begin{aligned}&1-\frac{\nu\mu}{2^l}\Sigma_1-\nu\left(1-\frac{\mu}{2^l}\right)\Sigma_2\\ &+\left(\frac{\sum_{j\in K_s}\beta_{nj}}{\nu}-\frac{\mu}{2^l}\right)\nu(\Sigma_2-\Sigma_1)\\ \le\ &P(K_s,\beta_n,\mathbf{x},\mathbf{y})\\ \le\ &1-\frac{\nu\mu}{2^l}\Sigma_1-\nu\left(1-\frac{\mu}{2^l}\right)\Sigma_2\\ &+\left(\frac{\sum_{j\in K_s}\beta_{nj}}{\nu}-\frac{\mu}{2^l}\right)\nu(\Sigma_2-\Sigma_1)+\nu\Sigma_3,\end{aligned}$$

where

$$\begin{aligned}\Sigma_1&=P_1(n,x_1,x_2)+P_2(n,y_1,y_2)\\ &\quad-P_3(n,x_1,y_2)-P_4(n,y_1,x_2),\\ \Sigma_2&=P_1(n,x_3,x_4)+P_2(n,y_3,y_4)\\ &\quad-P_3(n,x_3,y_4)-P_4(n,y_3,x_4)\end{aligned}$$

with $P_i(n,z_1,z_2)$'s defined in the proof of Lemma 4.2 and

$$\begin{aligned}\Sigma_3\ =\ &\sum_{i,j=\{1,2\}}\sum_{t=2}^{\nu}\Big(\mathbb{P}\Big(S_{n,1}\hat{X}_{n,(s-1)\nu+1}^{(i)}>u_n(x_i),\\ &S_{n,1}\hat{X}_{n,(s-1)\nu+t}^{(j)}>u_n(x_j)\Big)\\ &+\mathbb{P}\Big(S_{n,1}\hat{X}_{n,(s-1)\nu+1}^{(i)}>u_n(x_i),\\ &S_{n,1}\hat{X}_{n,(s-1)\nu+t}^{(j)}\le-u_n(y_j)\Big)\\ &+\mathbb{P}\Big(S_{n,1}\hat{X}_{n,(s-1)\nu+1}^{(i)}\le-u_n(y_i),\\ &S_{n,1}\hat{X}_{n,(s-1)\nu+t}^{(j)}>u_n(x_j)\Big)\\ &+\mathbb{P}\Big(S_{n,1}\hat{X}_{n,(s-1)\nu+1}^{(i)}\le-u_n(y_i),\\ &S_{n,1}\hat{X}_{n,(s-1)\nu+t}^{(j)}\le-u_n(y_j)\Big)\Big).\end{aligned}$$

Since $0\le1-\frac{\nu\mu}{2^l}\Sigma_1-\nu(1-\frac{\mu}{2^l})\Sigma_2\le1$ applying Lemma 3 in [19] we obtain

$$\begin{aligned}(31)\quad&\sum_{\mu=0}^{2^l-1}\sum_{\beta_n\in\{0,1\}^n}\mathbb{E}\left(\left|\prod_{s=1}^{l}P(K_s,\beta_n,\mathbf{x},\mathbf{y})\right.\right.\\ &\left.\left.-\prod_{s=1}^{l}\left[1-\frac{\frac{\mu}{2^l}n\Sigma_1-\left(1-\frac{\mu}{2^l}\right)n\Sigma_2}{l}\right]\right|\mathbb{I}\left(B_{\mu,l,\beta_n}\right)\right)\\ \le\ &\sum_{\mu=0}^{2^l-1}\sum_{\beta_n\in\{0,1\}^n}\mathbb{E}\left(\sum_{s=1}^{l}\left|P(K_s,\beta_n,\mathbf{x},\mathbf{y})\right.\right.\\ &\left.\left.-\left[1-\frac{\frac{\mu}{2^l}n\Sigma_1-\left(1-\frac{\mu}{2^l}\right)n\Sigma_2}{l}\right]\right|\mathbb{I}\left(B_{\mu,l,\beta_n}\right)\right)\\ \le\ &\sum_{\mu=0}^{2^l-1}\sum_{s=1}^{l}\mathbb{E}\left(\frac{\left|\frac{\sum_{j\in K_s}\varepsilon_{n,j}}{\nu}-\frac{\mu}{2^l}\right|}{l}\mathbb{I}\left(B_{\mu,l}\right)\right)n(\Sigma_1-\Sigma_2)\end{aligned}$$

$$
\begin{aligned}
&+n\Sigma_3 \\
\le\ & \sum_{s=1}^{l}\left[2(2s-1)\left(d\left(\frac{\Xi_{\nu s}}{\nu s},\eta\right)+d\left(\frac{\Xi_{\nu(s-1)}}{\nu(s-1)},\eta\right)\right)\right. \\
&\left.+\frac{1}{2^l}\right]\frac{n(\Sigma_1-\Sigma_2)}{l}+n\Sigma_3,
\end{aligned}
$$

where $d(X,Y)$ stands for Ky Fan metric, i.e., $d(X,Y)=\inf\{\varepsilon,\mathbb{P}(|X-Y|>\varepsilon)<\varepsilon\}$. Furthermore,

$$
\begin{aligned}
(32)\quad & \sum_{\mu=0}^{2^l-1}\sum_{\beta_n\in\{0,1\}^n}\mathbb{E}\left(\left|\prod_{s=1}^{l}\left[1-\frac{\frac{\mu}{2^l}n\Sigma_1-\left(1-\frac{\mu}{2^l}\right)n\Sigma_2}{l}\right]\right.\right. \\
&\left.\left.-\prod_{s=1}^{l}\left[1-\frac{\eta n\Sigma_1-(1-\eta)\,n\Sigma_2}{l}\right]\right|\mathbb{I}(B_{\mu,l,\beta_n})\right) \\
&\le\sum_{\mu=0}^{2^l-1}\sum_{s=1}^{l}\mathbb{E}\left(\left|\frac{\mu}{2^l}-\eta\right|\mathbb{I}(B_{\mu,l})\right)\frac{n}{l}(\Sigma_1+\Sigma_2) \\
&\le\frac{n(\Sigma_1+\Sigma_2)}{2^l}.
\end{aligned}
$$

By the fact that $\lim_{\nu\to\infty}d(\frac{\Xi_{\nu s}}{\nu s},\eta)=0$ and utilising (29)–(32), by passing to limit for $n\to\infty$ and then letting $\nu\to\infty$ we obtain

$$
\begin{aligned}
&\left|P(n,\varepsilon_n,\mathbf{x},\mathbf{y})-\mathbb{E}\left(1-\frac{\eta(D(x_1,x_2)+D(y_1,y_2))}{l}\right.\right. \\
&\left.\left.-\frac{(1-\eta)(D(x_3,x_4)+D(y_3,y_4))}{l}\right)^l\right| \\
\le\ & \frac{D(x_1,x_2)+D(y_1,y_2)}{2^{l-1}} \\
&+\frac{1}{l}(e^{-x_1}+e^{-y_1}+e^{-x_2}+e^{-y_2})^2.
\end{aligned}
$$

Next, letting $l\to\infty$ implies

$$
\begin{aligned}
&\lim_{n\to\infty}\sup_{\substack{x_i,y_i\in\mathbb{R},i=\{1,2,3,4\}\\ x_1\le x_3,x_2\le x_4,y_1\le y_3,y_2\le y_4}}\Big|P(n,\varepsilon_n,\mathbf{x},\mathbf{y}) \\
&-\mathbb{E}\left(H_\lambda^\eta(x_1,x_2)H_\lambda^\eta(y_1,y_2)H_\lambda^{1-\eta}(x_3,x_4)H_\lambda^{1-\eta}(y_3,y_4)\right)\Big| \\
&=0,
\end{aligned}
$$

hence the claim follows. $\square$

Proof of Theorem 3.2 If (11) holds, by (12) for some positive constant $\mathcal{Q}$ we have

$$
\begin{aligned}
&\Big|\mathbb{P}\Big(-u_n(y_1)<m_n^{(1)}(\varepsilon_n)\le M_n^{(1)}(\varepsilon_n)\le u_n(x_1), \\
&-u_n(y_2)<m_n^{(2)}(\varepsilon_n)\le M_n^{(2)}(\varepsilon_n)\le u_n(x_2), \\
&-u_n(y_3)<m_n^{(1)}\le M_n^{(1)}\le u_n(x_3), \\
&-u_n(y_4)<m_n^{(2)}\le M_n^{(2)}\le u_n(x_4)\Big) \\
&-\mathbb{P}\Big(-u_n(y_1)<\widehat{m}_n^{(1)}(\varepsilon_n)\le\widehat{M}_n^{(1)}(\varepsilon_n)\le u_n(x_1), \\
&-u_n(y_2)<\widehat{m}_n^{(2)}(\varepsilon_n)\le\widehat{M}_n^{(2)}(\varepsilon_n)\le u_n(x_2), \\
&-u_n(y_3)<\widehat{m}_n^{(1)}\le\widehat{M}_n^{(1)}\le u_n(x_3), \\
&-u_n(y_4)<\widehat{m}_n^{(2)}\le\widehat{M}_n^{(2)}\le u_n(x_4)\Big)\Big| \\
\le\ & \mathcal{Q}nw^{\frac{4\alpha+2p}{2+p}}\sum_{i,j=1,2}\sum_{k=1}^{n}|\lambda_{ij}(k,n)| \\
&\exp\left(-2(1+|\lambda_{ij}(k,n)|)^{-\frac{p}{2+p}}Tw^{\frac{2p}{2+p}}\right),
\end{aligned}
$$

where $w=\min(|u_n(x_i)|,|u_n(y_i)|,1\le i\le 4)$. In view of Lemma 3.3 in [13], the sum of the right side of the inequality tends to 0. Thus the claim follows by Lemma 4.3. $\square$

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Enkelejd Hashorva
University of Lausanne
Faculty of Business and Economics (HEC Lausanne)
Lausanne 1015
Switzerland
E-mail address: Enkelejd.Hashorva@unil.ch

Zhichao Weng
University of Lausanne
Faculty of Business and Economics (HEC Lausanne)
Lausanne 1015
Switzerland
E-mail address: zweng@unil.ch