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The tail behavior of randomly weighted sums of dependent random variables

Xuan Leng and Taizhong Hu*,†

Consider dependent random variables $X_1,\dots,X_d$ with a common distribution function F and denote by ω_F the right endpoint of the support of F. Let $\Theta_1,\dots,\Theta_d$ be non-negative random variables, independent of $\boldsymbol{X}=(X_1,\dots,X_d)$ and satisfying certain moment conditions if necessary. Under the assumption that $\boldsymbol{X}$ is in the maximum domain of attraction of a multivariate extreme value distribution, we establish the asymptotic behaviors of randomly weighted sums: there exist limiting constants $q^{\mathrm{F}}_{\boldsymbol{\theta}}$, $q^{\mathrm{W}}_{\boldsymbol{\theta}}$ and $q^{\mathrm{G}}_{\boldsymbol{\theta}}$ such that for large t, $\mathsf{P}(\sum_{i=1}^d \Theta_i X_i > t) \sim \mathsf{E}\, q^{\mathrm{F}}_{\boldsymbol{\Theta}} \cdot \mathsf{P}(X_1>t)$, $\mathsf{P}(\sum_{i=1}^d \Theta_i(\omega_F - X_i) < 1/t) \sim \mathsf{E}\, q^{\mathrm{W}}_{\boldsymbol{\Theta}} \cdot \mathsf{P}(X_1 > \omega_F - 1/t)$, and for $\sum_{i=1}^d \Theta_i = 1$ and t approaching to ω_F, $\mathsf{P}(\sum_{i=1}^d \Theta_i X_i > t) \sim \mathsf{E}\, q^{\mathrm{G}}_{\boldsymbol{\Theta}} \cdot \mathsf{P}(X_1>t)$ according to F belonging to the maximum domains of attraction of the Fréchet, Weibull and Gumbel distributions, respectively. Moreover, some basic properties of the proportionality factor $\mathsf{E}\, q^{\mathrm{F}}_{\boldsymbol{\Theta}}$ are presented.

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1. INTRODUCTION

In modern insurance mathematics and finance one of the main issues is to model and compute the aggregation effects of different risks. Consider dependent risks (random variables) $X_1,\dots,X_d$ with a common distribution function F, and let ω_F be the right endpoint of the support of F. A way to describe the dependence structure of risks is the copula approach. Under the assumption that $\boldsymbol{X}=(X_1,\dots,X_d)$ has an Archimedean survival copula with generator ψ which is regularly varying at zero with index $-\alpha<0$, Wüthrich (2003) and Alink et al. (2004) presented the tail behaviors of the following three types:

- If F belongs to the maximum domain of attraction (MDA) of the Fréchet distribution Φ_β for some $\beta>0$, then

$$\lim_{u\to\infty}\frac{1}{\overline{F}(u)}\mathsf{P}\left(\sum_{i=1}^d X_i > u\right) = q_d^{\mathrm{F}}(\alpha,\beta); \tag{1}$$

- If F belongs to the MDA of the Weibull distribution Ψ_β for some $\beta>0$, then

$$\lim_{u\to\infty}\frac{1}{\overline{F}(\omega_F-1/u)}\mathsf{P}\left(\sum_{i=1}^d X_i > d\omega_F-\frac{1}{u}\right) = q_d^{\mathrm{W}}(\alpha,\beta); \tag{2}$$

- If F belongs to the MDA of the Gumbel distribution Λ, then

$$\lim_{u\uparrow\omega_F}\frac{1}{\overline{F}(u)}\mathsf{P}\left(\sum_{i=1}^d X_i > du\right) = q_d^{\mathrm{G}}(\alpha). \tag{3}$$

Here, the positive constants $q_d^{\mathrm{F}}(\alpha,\beta)$, $q_d^{\mathrm{W}}(\alpha,\beta)$ and $q_d^{\mathrm{G}}(\alpha)$ quantify the diversification effect between those risks. From (1) to (3), we see that the probability of a large aggregate risk $\sum_{i=1}^d X_i$ scales like the probability of a large individual risk X_1, times a proportionality factor. Properties (monotonicity and boundary values) of these proportionality factors were investigated by Embrechts et al. (2009) and Chen et al. (2012). Alink et al. (2005) studied the asymptotic behavior of expected shortfall of $\sum_{i=1}^d X_i$ for the Fréchet and Gumbel cases. Barbe et al. (2006) extended (1) from the Archimedean dependence structure to the one with the property of multivariate regular variation. Therefore, the proportionality factor $q_d^{\mathrm{F}}(\alpha,\beta)$ arises naturally in multivariate extreme-value theory. As shown by these authors, the asymptotic Value-at-Risk can be obtained from such analysis of tail probabilities of the aggregate risks.

The form of the randomly weighted sum $\sum_{i=1}^d \Theta_i X_i$ is usually encountered in many cases. For example, consider a global macro-strategy investment portfolio consisting of d assets over one period and X_i is regarded as a potential loss of the asset i at the terminal time while the corresponding discount factor over the period is Θ_i. The randomness of the discount factors may result from the stochastic interest rates or random return on investment. Then $\sum_{i=1}^d \Theta_i X_i$ is the discounted losses from the portfolio.

In the present paper, we discuss the tail behaviors of randomly weighted sums where $\boldsymbol{X}$ is in the MDA of a multivariate extreme value distribution G. More specifically, let

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†Corresponding author.

$\boldsymbol{\Theta} = (\Theta_1, \dots, \Theta_d)$ be a vector of non-negative random variables, independent of $\boldsymbol{X}$ and satisfying certain moment conditions if necessary. Then there exist proportionality factors $q^{\mathrm{F}}_{\boldsymbol{\theta}}(\beta)$, $q^{\mathrm{W}}_{\boldsymbol{\theta}}(\beta)$ and $q^{\mathrm{G}}_{\boldsymbol{\theta}}$ such that

- If F belongs to the MDA of the Fréchet distribution Φ_β for some $\beta > 0$, then

$$\lim_{t\to\infty} \frac{1}{\overline{F}(t)} \mathsf{P}\left(\sum_{i=1}^d \Theta_i X_i > t\right) = \mathsf{E}\left[q^{\mathrm{F}}_{\boldsymbol{\Theta}}(\beta)\right];$$

- If F belongs to the MDA of the Weibull distribution Ψ_β for some $\beta > 0$, then

$$\lim_{t\to\infty} \frac{1}{\overline{F}(\omega_F - 1/t)} \mathsf{P}\left(\sum_{i=1}^d \Theta_i(\omega_F - X_i) < \frac{1}{t}\right) = \mathsf{E}\left[q^{\mathrm{W}}_{\boldsymbol{\Theta}}(\beta)\right];$$

- If F belongs to the MDA of the Gumbel distribution Λ, and if $\sum_{i=1}^d \Theta_i = 1$, then

$$\lim_{t\uparrow\omega_F} \frac{1}{\overline{F}(t)} \mathsf{P}\left(\sum_{i=1}^d \Theta_i X_i > t\right) = \mathsf{E}\left[q^{\mathrm{G}}_{\boldsymbol{\Theta}}\right].$$

Here, $q^{\mathrm{F}}_{\boldsymbol{\theta}}(\beta)$, $q^{\mathrm{W}}_{\boldsymbol{\theta}}(\beta)$ and $q^{\mathrm{G}}_{\boldsymbol{\theta}}$ are represented by the integral with respect to the spectral measure of G. Moreover, some properties of $\mathsf{E}\left[q^{\mathrm{F}}_{\boldsymbol{\Theta}}(\beta)\right]$ are given. The main results in this paper extend some known results in Alink et al. (2004) and Barbe et al. (2006).

This paper is organized as follows. Section 2 recalls the concept of multivariate regular variation and its basic properties. The main results on the asymptotic behavior for dependent random variables in terms of randomly weighted sums are given in Section 3. Section 4 is devoted to the study of basic properties of $\mathsf{E}\left[q^{\mathrm{F}}_{\boldsymbol{\Theta}}(\beta)\right]$. Section 5 is an appendix.

Throughout, for any increasing function h, define its generalized inverse $h^{\leftarrow}$ by $h^{\leftarrow}(u) = \inf\{x : h(x) \ge u\}$; and for any decreasing function h, define its generalized inverse $h^{\leftarrow}$ by $h^{\leftarrow}(u) = \sup\{x : h(x) \ge u\}$.

2. MULTIVARIATE REGULAR VARIATION

An $\mathbb{R}^d$-valued random vector $\boldsymbol{X}$ or its distribution is said to be of multivariate regularly variation (MRV) if there exists a nonzero Radon measure μ on the Borel σ-field $\mathscr{B}(\overline{\mathbb{R}}^d \backslash \{\mathbf{0}\})$ and a sequence of positive constants $\{a_n\}$, $a_n \to \infty$ as $n \to \infty$, such that

$$n\,\mathsf{P}\left(\frac{\boldsymbol{X}}{a_n} \in \cdot\right) \xrightarrow{\nu} \mu(\cdot), \quad n \to \infty. \tag{4}$$

Here $\xrightarrow{\nu}$ refers to vague convergence on $\mathscr{B}(\overline{\mathbb{R}}^d \backslash \{\mathbf{0}\})$. If $\boldsymbol{X}$ takes values in $[\mathbf{0}, \boldsymbol{\infty}]$, the MRV in this special case can be defined by restricting vague convergence in (4) on $\mathscr{B}([\mathbf{0}, \boldsymbol{\infty}] \backslash \{\mathbf{0}\})$. We write $\boldsymbol{X} \in \mathrm{MRV}_d(-\beta)$ since μ necessarily satisfies $\mu(tB) = t^{-\beta}\mu(B)$ for some $\beta > 0$ and all $t > 0$ and $B \in \mathscr{B}(\overline{\mathbb{R}}^d \backslash \{\mathbf{0}\})$. μ is called the limit Radon measure of $\boldsymbol{X}$. For more on MRV, we refer to Resnick (1987, 2007) and Lindskog (2004).

Lemma 2.1. (Basrak et al., 2002, Proposition A.1) *Assume that $\boldsymbol{X} \in \mathrm{MRV}_d(-\beta)$ in the sense of (4) with $\beta > 0$, and $\boldsymbol{A}$ is a random $q \times d$ matrix, independent of $\boldsymbol{X}$. If $0 < \mathsf{E}\,|\boldsymbol{A}|^\gamma < \infty$ for some $\gamma > \beta$ and an arbitrary matrix norm $|\cdot|$, then*

$$n\,\mathsf{P}(a_n^{-1}\boldsymbol{A}\boldsymbol{X} \in \cdot) \xrightarrow{\nu} \mathsf{E}\left[\mu \circ \boldsymbol{A}^{-1}(\cdot)\right]. \tag{5}$$

where, for any $B \in \mathscr{B}(\overline{\mathbb{R}}^q \backslash \{\mathbf{0}\})$,

$$\boldsymbol{A}^{-1}(B) = \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{A}\boldsymbol{x} \in B\}.$$

Let $\boldsymbol{X} \in \mathrm{MRV}_d(-\beta)$, $\beta > 0$, have a distribution function K with the limit Radon measure μ. Then K is in the MDA of G, where

$$G(\boldsymbol{x}) = \exp\{-\mu([-\boldsymbol{\infty}, \boldsymbol{x}]^c)\} \text{ for } \boldsymbol{x} \in \mathbb{R}^d.$$

Define

$$\boldsymbol{Z} = \left(\frac{1}{\overline{K}_1(X_1)}, \dots, \frac{1}{\overline{K}_d(X_d)}\right), \tag{6}$$

where $K_1, \dots, K_d$ are the univariate margins of K. Then $t\,\mathsf{P}(\boldsymbol{Z}/t \in \cdot) \xrightarrow{\nu} \mu_*(\cdot)$ on $\mathscr{B}([\mathbf{0}, \boldsymbol{\infty}] \backslash \{\mathbf{0}\})$ as $t \to \infty$, implying $\boldsymbol{Z} \in \mathrm{MRV}_d(-1)$. Let G_* be a multivariate extreme value (MEV) distribution function with canonical exponent measure μ_*, that is,

$$-\log G_*(\boldsymbol{x}) = \mu_*([\mathbf{0}, \boldsymbol{\infty}] \setminus [\mathbf{0}, \boldsymbol{x}])$$

for $\boldsymbol{x} \in [\mathbf{0}, \boldsymbol{\infty}]$. It is seen that the margins of G_* are the standard Fréchet distribution. Now define the spectral measure $S_{\|\cdot\|}$ on $\aleph_+ = \{\boldsymbol{x} \in \mathbb{R}^d_+ : \|\boldsymbol{x}\| = 1\}$ by

$$S_{\|\cdot\|}(B) = \mu_*\left(\left\{\boldsymbol{x} \in \mathbb{R}^d_+ : \|\boldsymbol{x}\| \ge 1, \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \in B\right\}\right)$$

for $B \in \mathscr{B}(\aleph_+)$ and $\|\cdot\|$ is an arbitrary norm. The most popular choice for the norm $\|\cdot\|$ is the sum-norm or ℓ_1-norm, $\|\boldsymbol{x}\|_{\mathbb{1}} = \sum_{i=1}^d |x_i|$ with $\mathbb{1} = (1, \cdots, 1)$. Throughout, the spectral measure $S_{\|\cdot\|_{\mathbb{1}}}$ is typically denoted by H on $\mathbb{S}_{d-1} = \{\boldsymbol{\omega} \in \mathbb{R}^d_+ : \|\boldsymbol{\omega}\|_{\mathbb{1}} = 1\}$.

Lemma 2.2. (Resnick, 1987, Proposition 5.15) *Let $\boldsymbol{X}$ be a d-dimensional random vector with distribution function K, and let $\boldsymbol{Z}$ be as defined by (6). If $K \in \mathrm{MDA}(G)$, then $\boldsymbol{Z} \in \mathrm{MRV}_d(-1)$.*

3. MAIN RESULTS

In this section, main results are given according to the cases that the marginal distributions of the underlying random vector belong to the MDA of the Fréchet, Weibull and Gumbel distributions, respectively.

3.1 The Fréchet case

Theorem 3.1. *Let $\boldsymbol{X} \in \mathrm{MRV}_d(-\beta)$ be a non-negative random vector with a common univariate marginal distribution F, where $\beta > 0$, and let $\boldsymbol{\Theta} = (\Theta_1, \cdots, \Theta_d)$ be another non-negative random vector, independent of $\boldsymbol{X}$. If $\mathsf{E}[\Theta_k^\gamma] < \infty$ for some $\gamma > \beta$ and each k, then*

$$\lim_{t\to\infty} \frac{1}{\overline{F}(t)} \mathsf{P}\left(\sum_{i=1}^d \Theta_i X_i > t\right) = \mathsf{E}\left[\int_{\mathbb{S}_{d-1}} \left(\sum_{i=1}^d \Theta_i \omega_i^{1/\beta}\right)^\beta H(d\boldsymbol{\omega})\right],$$

where H is the spectral measure corresponding to the distribution function of $\boldsymbol{X}$ with respect to the ℓ_1-norm. In particular, for any $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d) \in \mathbb{R}_+^d$,

$$\lim_{t\to\infty} \frac{1}{\overline{F}(t)} \mathsf{P}\left(\sum_{i=1}^d \theta_i X_i > t\right) = \int_{\mathbb{S}_{d-1}} \left(\sum_{i=1}^d \theta_i \omega_i^{1/\beta}\right)^\beta H(d\boldsymbol{\omega}). \tag{7}$$

Proof. First, $\boldsymbol{X} \in \mathrm{MRV}_d(-\beta)$ with a common univariate marginal distribution F implies that

$$\frac{1}{\overline{F}(t)} \mathsf{P}\left(\frac{\boldsymbol{X}}{t} \in \cdot\right) \xrightarrow{\nu} \mu(\cdot), \quad t \to \infty.$$

Then, applying Lemma 2.1 yields that

$$\lim_{t\to\infty} \frac{\mathsf{P}\left(\sum_{i=1}^d \Theta_i X_i > t\right)}{\overline{F}(t)} = \mathsf{E}\left[\mu(A_{\boldsymbol{\Theta}})\right], \tag{8}$$

where $A_{\boldsymbol{\theta}} = \{\boldsymbol{x} \in \mathbb{R}_+^d : \sum_{i=1}^d \theta_i x_i > 1\}$ for each $\boldsymbol{\theta}$. On the other hand, note that

$$t\, \mathsf{P}\left(\frac{\boldsymbol{Y}}{t} \in \cdot\right) \xrightarrow{\nu} \mu_*(\cdot), \quad t \to \infty,$$

where

$$\boldsymbol{Y} = \left(\frac{1}{\overline{F}(X_1)}, \ldots, \frac{1}{\overline{F}(X_d)}\right).$$

Then, based on a similar argument to that in Section 2 of Barbe et al. (2006), we have

$$\mu(A_{\boldsymbol{\theta}}) = \mu_*\left(A_{\boldsymbol{\theta}}^\beta\right), \tag{9}$$

where $A_{\boldsymbol{\theta}}^\beta = \{\boldsymbol{x} \in \mathbb{R}_+^d : \sum_{i=1}^d \theta_i x_i^{1/\beta} > 1\}$ for any $\boldsymbol{\theta} \in \mathbb{R}_+^d$. Therefore, by the spectral decomposition of μ_*, (8) and (9), we get that

$$\begin{aligned}\lim_{t\to\infty} \frac{\mathsf{P}\left(\sum_{i=1}^d \Theta_i X_i > t\right)}{\overline{F}(t)} &= \mathsf{E}\left[\mu_*\left(A_{\boldsymbol{\Theta}}^\beta\right)\right] \\ &= \mathsf{E}\left[\int_{\Omega_{\boldsymbol{\Theta}}^*} r^{-2} dr\, H(d\boldsymbol{\omega})\right] \\ &= \mathsf{E}\left[\int_{\mathbb{S}_{d-1}} \left(\sum_{i=1}^d \Theta_i \omega_i^{1/\beta}\right)^\beta H(d\boldsymbol{\omega})\right],\end{aligned}$$

where

$$\Omega_{\boldsymbol{\theta}}^* = \left\{(r, \boldsymbol{\omega}) \in \mathbb{R}_+ \times \mathbb{S}_{d-1} : r > \left(\sum_{i=1}^d \theta_i \omega_i^{1/\beta}\right)^{-\beta}\right\}.$$

This completes the proof. □

We call a non-negative multivariate regularly varying random vector asymptotically independent if the spectral measure $S_{\|\cdot\|}$ is concentrated on the points $\boldsymbol{e}_j/\|\boldsymbol{e}_j\|$, $j = 1, \cdots, d$, where $\boldsymbol{e}_j$ denotes the jth unit vector in $\mathbb{R}^d$, that is, the jth coordinate of $\boldsymbol{e}_j$ is one and all other coordinates are zero; it is called asymptotically fully dependent if the spectral measure $S_{\|\cdot\|}$ is concentrated on $\mathbb{1}/\|\mathbb{1}\|$; see Resnick (2004).

Corollary 3.2. *Under the conditions of Theorem 3.1, if $\boldsymbol{X}$ is asymptotically independent, then*

$$\lim_{t\to\infty} \frac{1}{\overline{F}(t)} \mathsf{P}\left(\sum_{i=1}^d \Theta_i X_i > t\right) = \sum_{j=1}^d \mathsf{E}\, \Theta_j^\beta;$$

and if $\boldsymbol{X}$ is asymptotically fully dependent, then

$$\lim_{t\to\infty} \frac{1}{\overline{F}(t)} \mathsf{P}\left(\sum_{i=1}^d \Theta_i X_i > t\right) = \mathsf{E}\left(\sum_{i=1}^d \Theta_i\right)^\beta.$$

Proof. For the asymptotic independence case, the spectral measure H consists of point masses of size 1 at the points $\boldsymbol{e}_j$'s. Then

$$\int_{\mathbb{S}_{d-1}} \left(\sum_{i=1}^d \Theta_i \omega_i^{1/\beta}\right)^\beta H(d\boldsymbol{\omega}) = \sum_{j=1}^d \Theta_j^\beta H(\boldsymbol{e}_j) = \sum_{j=1}^d \Theta_j^\beta.$$

For the asymptotic full dependence case, H collapses to a single point mass of size d at the point $\mathbb{1}/d$. Then

$$\begin{aligned}&\int_{\mathbb{S}_{d-1}} \left(\sum_{i=1}^d \Theta_i \omega_i^{1/\beta}\right)^\beta H(d\boldsymbol{\omega}) \\ &\quad = \frac{\left(\sum_{i=1}^d \Theta_i\right)^\beta}{d} \cdot H\left(\frac{1}{d}\mathbb{1}\right) = \left(\sum_{i=1}^d \Theta_i\right)^\beta.\end{aligned}$$

This completes the proof. □

Corollary 3.3. *Let $\boldsymbol{X}$ be a d-dimensional random vector with a common univariate marginal distribution F and a joint distribution function $K \in \mathrm{MDA}(G)$, where G is an*

MEV *distribution with univariate margins being the Weibull distribution* Ψ_β, $\beta > 0$. *Let* $\boldsymbol{\Theta} = (\Theta_1, \ldots, \Theta_d)$ *be a non-negative random vector, independent of* $\boldsymbol{X}$. *If* $\mathsf{E}[\Theta_k^\gamma] < \infty$ *for some* $\gamma > \beta$ *and each* k, *then*

$$\lim_{t\to\infty} \frac{1}{\overline{F}(\omega_F - 1/t)} \mathsf{P}\left(\sum_{i=1}^d \frac{\Theta_i}{\omega_F - X_i} > t\right) = \mathsf{E}\left[\int_{\mathbb{S}_{d-1}} \left(\sum_{i=1}^d \Theta_i \omega_i^{1/\beta}\right)^\beta H(d\boldsymbol{\omega})\right],$$

where H *is the spectral measure of the distribution of* $(1/\overline{F}(X_1), \ldots, 1/\overline{F}(X_d))$ *with respect to the* ℓ_1*-norm.*

Proof. Define

$$\boldsymbol{Z} = \left(\frac{1}{\omega_F - X_1}, \ldots, \frac{1}{\omega_F - X_d}\right).$$

Then $\boldsymbol{Z} \in \mathrm{MRV}_d(-\beta)$ since $\boldsymbol{X}$ and $\boldsymbol{Z}$ have the same copula and $\overline{F}(\omega_F - 1/\cdot) \in \mathrm{RV}_{-\beta}$ (see Embrechts et al., 1997, p. 136). Applying Theorem 3.1 to $\boldsymbol{Z}$ yields that

$$\lim_{t\to\infty} \frac{1}{\overline{F}(\omega_F - 1/t)} \mathsf{P}\left(\sum_{i=1}^d \frac{\Theta_i}{\omega_F - X_i} > t\right) = \mathsf{E}\left[\int_{\mathbb{S}_{d-1}} \left(\sum_{i=1}^d \Theta_i \omega_i^{1/\beta}\right)^\beta H(d\boldsymbol{\omega})\right].$$

This completes the proof. □

In Theorem 3.1, MRV of $\boldsymbol{X}$ implies that its components belong to the MDA of the Fréchet distribution. Motivated by this, we will consider the other two cases that $\boldsymbol{X}$ belongs to the MDA of an MEV distribution with the univariate margins being the Weibull and Gumbel distributions, respectively.

3.2 The Weibull case

To prove the main result, we need the following lemma.

Lemma 3.4. (Pratt, 1960) *Let* X_n *and* U_n *be two sequences of random variables. Assume* $|X_n| \le U_n$, *a.s.,* $U_n \xrightarrow{\mathsf{P}} U$, $\mathsf{E}\, U_n \to \mathsf{E}\, U$ *and* U *is integrable. If* $X_n \xrightarrow{\mathsf{P}} X$, *then*

$$\mathsf{E}\,|X_n - X| \to 0$$

and, hence, $\mathsf{E}\, X_n \to \mathsf{E}\, X$.

Theorem 3.5. *Let* $\boldsymbol{X}$ *be a* d*-dimensional random vector with a common univariate marginal distribution* F *and a joint distribution function* $K \in \mathrm{MDA}(G)$, *where* G *is an* MEV *distribution with univariate margins being the Weibull distribution* Ψ_β, $\beta > 0$. *Let* $\boldsymbol{\Theta} = (\Theta_1, \ldots, \Theta_d)$ *be a non-negative random vector, independent of* $\boldsymbol{X}$. *If* $\mathsf{E}[\Theta_k^{-\gamma}] < \infty$ *for some* $\gamma > \beta$ *and each* k, *then*

$$\lim_{t\to\infty} \frac{1}{\overline{F}(\omega_F - 1/t)} \mathsf{P}\left(\sum_{i=1}^d \Theta_i(\omega_F - X_i) < \frac{1}{t}\right) = \mathsf{E}\left[\int_{\mathbb{S}_{d-1}} \left(\sum_{i=1}^d \Theta_i \omega_i^{-1/\beta}\right)^{-\beta} H(d\boldsymbol{\omega})\right], \tag{10}$$

where H *is the spectral measure of of the distribution of* $(1/\overline{F}(X_1), \ldots, 1/\overline{F}(X_d))$ *with respect to the* ℓ_1*-norm. In particular, for any* $\boldsymbol{\theta} \in \mathbb{R}_+^d$,

$$\lim_{t\to\infty} \frac{1}{\overline{F}(\omega_F - 1/t)} \mathsf{P}\left(\sum_{i=1}^d \theta_i(\omega_F - X_i) < \frac{1}{t}\right) = \int_{\mathbb{S}_{d-1}} \left(\sum_{i=1}^d \theta_i \omega_i^{-1/\beta}\right)^{-\beta} H(d\boldsymbol{\omega}). \tag{11}$$

Proof. First, we prove (11). A similar argument to that in Section 2 of Barbe et al. (2006) is used here to prove the desired result. Denote by $g(x) = 1/\mathsf{P}(\omega_F - X_i < x) = 1/\overline{F}(\omega_F - x)$ and $b(x) = g^{\leftarrow}(x)$. Define

$$\begin{aligned} \boldsymbol{Y} &= (g(\omega_F - X_1), \ldots, g(\omega_F - X_d)) \\ &= \left(\frac{1}{\overline{F}(X_1)}, \ldots, \frac{1}{\overline{F}(X_d)}\right). \end{aligned}$$

Observe that $\boldsymbol{X}$ and $\boldsymbol{Y}$ have the same copula since the copula is invariant under increasing transformations of the margins. Also, $(\omega_F - X_1, \ldots, \omega_F - X_d)$ and $(b(Y_1), \ldots, b(Y_d))$ have the same distribution. Then

$$\sum_{i=1}^d \theta_i\big(\omega_F - X_i\big) < s \Longleftrightarrow \sum_{i=1}^d \theta_i b\big(Y_i\big) < s$$

or, equivalently,

$$\frac{\boldsymbol{Y}}{g(s)} \in \Omega(s) \equiv \left\{\boldsymbol{z} \in \mathbb{R}_+^d : \sum_{i=1}^d \theta_i b(g(s)z_i) < s\right\}. \tag{12}$$

By Lemma 2.3(ii) in Lv et al. (2012),

$$\begin{aligned} \overline{F}(\omega_F - 1/\cdot) \in \mathrm{RV}_{-\beta} &\implies g \in \mathrm{RV}_{-\beta}(0^+) \\ &\implies b \in \mathrm{RV}_{-1/\beta}, \end{aligned}$$

implying that

$$\lim_{s\to 0} \frac{b(g(s)z)}{s} = \lim_{s\to 0} \frac{b(g(s)z)}{b(g(s))} = \lim_{t\to\infty} \frac{b(tz)}{b(t)} = z^{-1/\beta},$$

and

$$\lim_{s\to 0} \Omega(s) = \Omega = \left\{\boldsymbol{z} \in \mathbb{R}_+^d : \sum_{i=1}^d \theta_i z_i^{-1/\beta} < 1\right\}.$$

By choosing the ℓ_1-norm and using a polar coordinate transformation T, $T(\boldsymbol{x}) = (\|\boldsymbol{x}\|, \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}) =: (r, \boldsymbol{\omega})$, we have

$$T(\Omega) = \Omega^* = \left\{ (r, \boldsymbol{\omega}) \in \mathbb{R}_+ \times \mathbb{S}_{d-1} : r > \left(\sum_{i=1}^d \theta_i \omega_i^{-1/\beta} \right)^\beta \right\}.$$

From Lemma 2.2, it follows that $\boldsymbol{Y} \in \mathrm{MRV}_d(-1)$ and, hence,

$$\begin{aligned} \lim_{t\to\infty} t \, \mathsf{P}\left(\frac{\boldsymbol{Y}}{t} \in \Omega \right) &= \mu_*(\Omega) = \int_{\Omega^*} r^{-2} dr \, H(d\boldsymbol{\omega}) \\ &= \int_{\mathbb{S}_{d-1}} \left(\sum_{i=1}^d \theta_i \omega_i^{-1/\beta} \right)^{-\beta} H(d\boldsymbol{\omega}). \end{aligned}$$

Therefore,

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{\overline{F}(\omega_F - 1/t)} \mathsf{P}\left(\sum_{i=1}^d \theta_i (\omega_F - X_i) < \frac{1}{t} \right) \\ &= \lim_{t\to\infty} g(1/t) \, \mathsf{P}\left(\frac{\boldsymbol{Y}}{g(1/t)} \in \Omega(1/t) \right) \\ &= \lim_{t\to\infty} t \, \mathsf{P}\left(\frac{\boldsymbol{Y}}{t} \in \Omega \right) \\ &= \int_{\mathbb{S}_{d-1}} \left(\sum_{i=1}^d \theta_i \omega_i^{-1/\beta} \right)^{-\beta} H(d\boldsymbol{\omega}). \end{aligned}$$

This proves (11).

Next, we turn to prove (10). Set $\Theta_{\min} = \wedge_{i=1}^d \Theta_i$, and define

$$\begin{aligned} U_t &= \frac{1}{\overline{F}(\omega_F - 1/t)} \mathsf{P}\left(\sum_{i=1}^d \Theta_i (\omega_F - X_i) < \frac{1}{t} \middle| \boldsymbol{\Theta} \right), \\ U &= \int_{\mathbb{S}_{d-1}} \left(\sum_{i=1}^d \Theta_i \omega_i^{-1/\beta} \right)^{-\beta} H(d\boldsymbol{\omega}), \\ Z_t &= \frac{1}{\overline{F}(\omega_F - 1/t)} \mathsf{P}\left(\sum_{i=1}^d (\omega_F - X_i) < \frac{1}{t\Theta_{\min}} \middle| \boldsymbol{\Theta} \right), \end{aligned}$$

and

$$c_\beta = \int_{\mathbb{S}_{d-1}} \left(\sum_{i=1}^d \omega_i^{-1/\beta} \right)^{-\beta} H(d\boldsymbol{\omega}).$$

From (11), it follows that $U_t \to U$ as $t \to \infty$ and that

$$\mathsf{P}\left(\sum_{i=1}^d (\omega_F - X_i) < \frac{1}{\cdot} \right) \in \mathrm{RV}_{-\beta}.$$

Hence,

$$\begin{aligned} Z_t &= \frac{\mathsf{P}\left(\frac{1}{\sum_{i=1}^d (\omega_F - X_i)} > t\Theta_{\min} \middle| \boldsymbol{\Theta} \right)}{\mathsf{P}\left(\frac{1}{\sum_{i=1}^d (\omega_F - X_i)} > t \right)} \\ &\quad \times \frac{\mathsf{P}\left(\frac{1}{\sum_{i=1}^d (\omega_F - X_i)} > t \right)}{\overline{F}(\omega_F - 1/t)} \\ &\to c_\beta (\Theta_{\min})^{-\beta}, \quad t \to \infty. \end{aligned}$$

Clearly, $U_t \le Z_t$ and $\mathsf{E}\left[(\Theta_{\min})^{-\gamma}\right] \le \sum_{i=1}^d \mathsf{E}\left[\Theta_i^{-\gamma}\right] < \infty$. Moreover, by Breiman's theorem (Breiman, 1965), we have

$$\begin{aligned} &\lim_{t\to\infty} \mathsf{E}\, Z_t \\ &= \lim_{t\to\infty} \frac{1}{\overline{F}(\omega_F - 1/t)} \mathsf{P}\left((\Theta_{\min})^{-1} \frac{1}{\sum_{i=1}^d (\omega_F - X_i)} > t \right) \\ &= \lim_{t\to\infty} \mathsf{E}\left[(\Theta_{\min})^{-\beta} \right] \frac{\mathsf{P}\left(\frac{1}{\sum_{i=1}^d (\omega_F - X_i)} > t \right)}{\overline{F}(\omega_F - 1/t)} \\ &= c_\beta \, \mathsf{E}\left[(\Theta_{\min})^{-\beta} \right]. \end{aligned}$$

Therefore, by Lemma 3.4, we conclude that $\mathsf{E}\, U_t \to \mathsf{E}\, U$ as $t \to \infty$, which implies (10). This completes the proof. □

3.3 The Gumbel case

Theorem 3.6. *Let $\boldsymbol{X}$ be a d-dimensional random vector with a common univariate marginal distribution F and a joint distribution function $K \in \mathrm{MDA}(G)$, where G is an* MEV *distribution with univariate margins being the Gumbel distribution Λ. Let $\boldsymbol{\Theta} = (\Theta_1, \ldots, \Theta_d)$ be a non-negative random vector, independent of $\boldsymbol{X}$, such that $\sum_{i=1}^d \Theta_i = 1$. Then*

(13)
$$\lim_{t\uparrow\omega_F} \frac{1}{\overline{F}(t)} \mathsf{P}\left(\sum_{i=1}^d \Theta_i X_i > t \right) = \mathsf{E}\left[\int_{\mathbb{S}_{d-1}} \prod_{i=1}^d {\omega_i}^{\Theta_i} H(d\boldsymbol{\omega}) \right].$$

where H is the spectral measure of of the distribution of $(1/\overline{F}(X_1), \ldots, 1/\overline{F}(X_d))$ with respect to the ℓ_1-norm. In particular, for any $\boldsymbol{\theta} \in \mathbb{R}_+^d$ such that $\sum_{i=1}^d \theta_i = 1$,

$$\lim_{t\uparrow\omega_F} \frac{1}{\overline{F}(t)} \mathsf{P}\left(\sum_{i=1}^d \theta_i X_i > t \right) = \int_{\mathbb{S}_{d-1}} \prod_{i=1}^d {\omega_i}^{\theta_i} H(d\boldsymbol{\omega}). \tag{14}$$

Proof. We first prove (14) by using a similar argument to that in the proof of Theorem 3.5. Denote by $g(t) = 1/\overline{F}(t)$ and $b(t) = g^{\leftarrow}(t)$. By Theorem 3.3.27 in Embrechts et al. (1997), there exists some positive function $a(t)$ such that

$$\lim_{t\uparrow\omega_F} \frac{\overline{F}(t)}{\overline{F}(t + a(t)u)} = e^u \quad u \in \mathbb{R}.$$

Thus, by Lemma 1.1.1 in de Haan and Ferreira (2006), we have

$$\lim_{t\uparrow\omega_F}\frac{b\big(g(t)y\big)-t}{a(t)}=\log y\quad y>0. \tag{15}$$

Define $\boldsymbol{Y}=(g(X_1),\ldots,g(X_d))$, and observe that $\boldsymbol{X}$ and $\boldsymbol{Y}$ have the same copula. Also, $\boldsymbol{X}$ and $(b(Y_1),\ldots,b(Y_d))$ have the same distribution. Then

$$\sum_{i=1}^d\theta_iX_i>t\Longleftrightarrow\sum_{i=1}^d\theta_i(b(Y_i)-t)>0$$

or, equivalently,

$$\frac{\boldsymbol{Y}}{g(t)}\in\Omega(t)\equiv\left\{\boldsymbol{z}\in\mathbb{R}_+^d:\sum_{i=1}^d\frac{\theta_i(b(g(t)z_i)-t)}{a(t)}>0\right\}.$$

From (15), it follows that

$$\lim_{t\uparrow\omega_F}\Omega(t)=\Omega=\left\{\boldsymbol{z}\in\mathbb{R}_+^d:\prod_{i=1}^d z_i^{\theta_i}>1\right\}.$$

Then

$$\begin{aligned}&\lim_{t\uparrow\omega_F}\frac{1}{\overline{F}(t)}\,\mathsf{P}\left(\sum_{i=1}^d\theta_iX_i>t\right)\\ &=\lim_{t\uparrow\omega_F}g(t)\,\mathsf{P}\left(\frac{\boldsymbol{Y}}{g(t)}\in\Omega(t)\right)=\lim_{t\to\infty}t\,\mathsf{P}\left(\frac{\boldsymbol{Y}}{t}\in\Omega\right).\end{aligned}\tag{16}$$

By choosing the ℓ_1-norm and using a polar coordinate transformation T, we have

$$T(\Omega)=\Omega^*=\left\{(r,\boldsymbol{\omega})\in\mathbb{R}_+\times\mathbb{S}_{d-1}:r>\prod_{i=1}^d\omega_i^{-\theta_i}\right\}.$$

From Lemma 2.2, it follows that $\boldsymbol{Y}\in\mathrm{MRV}_d(-1)$ and, hence,

$$\begin{aligned}&\lim_{t\to\infty}t\,\mathsf{P}\left(\frac{\boldsymbol{Y}}{t}\in\Omega\right)=\mu_*(\Omega)\\ &=\int_{\Omega^*}r^{-2}dr\,H(d\boldsymbol{\omega})=\int_{\mathbb{S}_{d-1}}\prod_{i=1}^d{\omega_i}^{\theta_i}H(d\boldsymbol{\omega}).\end{aligned}\tag{17}$$

Therefore, (14) follows from (16) and (17).

To prove (13), note that

$$\begin{aligned}\frac{1}{\overline{F}(t)}\mathsf{P}\left(\sum_{i=1}^d\Theta_iX_i>t\,\middle|\,\boldsymbol{\Theta}\right)&\le\frac{1}{\overline{F}(t)}\mathsf{P}\left(\bigcup_{i=1}^d\{X_i>t\}\,\middle|\,\boldsymbol{\Theta}\right)\\ &\le\frac{1}{\overline{F}(t)}\sum_{i=1}^d\mathsf{P}(X_i>t)=d.\end{aligned}$$

Then, applying the dominated convergence theorem and (14) yields that

$$\begin{aligned}&\lim_{t\uparrow\omega_F}\frac{1}{\overline{F}(t)}\mathsf{P}\left(\sum_{i=1}^d\Theta_iX_i>t\right)\\ &=\mathsf{E}\left[\lim_{t\uparrow\omega_F}\frac{1}{\overline{F}(t)}\mathsf{P}\left(\sum_{i=1}^d\Theta_iX_i>t\,\middle|\,\boldsymbol{\Theta}\right)\right]\\ &=\mathsf{E}\left[\int_{\mathbb{S}_{d-1}}\prod_{i=1}^d\omega_i^{\Theta_i}H(d\boldsymbol{\omega})\right].\end{aligned}$$

This completes the proof of the theorem. ☐

4. BASIC PROPERTIES OF A LIMITING CONSTANT

The main result in Section 3 states that the probability of a large loss of $\sum_{i=1}^d\Theta_iX_i$ scales like the probability of a large individual loss of X_1, times the proportionality factor. In terms of Value-at-Risk, the approximation of large quantiles of the distribution of $\sum_{i=1}^d\Theta_iX_i$ is allowed through those of the individual claim of X_i. So the study of these proportionality factors is of obvious interest. In this section, denote by $q_{\boldsymbol{\theta}}^{\mathrm{F}}(\beta)$ the right-hand side of (7), and we will give some basic properties of $\mathsf{E}\left[q_{\boldsymbol{\Theta}}^{\mathrm{F}}(\beta)\right]$.

Proposition 4.1. *Under the conditions of Theorem 3.1, we have*

$$\mathsf{E}\left[q_{\boldsymbol{\Theta}}^{\mathrm{F}}(\beta)\right]\le\mathsf{E}\left(\sum_{i=1}^d\Theta_i\right)^\beta,\quad\beta\ge1;$$

$$\mathsf{E}\left[q_{\boldsymbol{\Theta}}^{\mathrm{F}}(\beta)\right]\ge\mathsf{E}\left(\sum_{i=1}^d\Theta_i\right)^\beta,\quad0<\beta<1.$$

Proof. Since $H(\mathbb{S}_{d-1})=d$, H is a finite measure on $\mathscr{B}(\mathbb{S}_{d-1})$. Then, applying Minkowski's inequality for $\beta\ge1$, we have

$$\left[q_{\boldsymbol{\Theta}}^{\mathrm{F}}(\beta)\right]^{1/\beta}\le\sum_{i=1}^d\left(\int_{\mathbb{S}_{d-1}}\left(\Theta_i\omega_i^{1/\beta}\right)^\beta H(d\boldsymbol{\omega})\right)^{1/\beta}=\sum_{i=1}^d\Theta_i,$$

where the last equation follows from the fact that

$$\int_{\mathbb{S}_{d-1}}\omega_j\,H(d\boldsymbol{\omega})=1,\quad j=1,\cdots,d;$$

see, e.g., Beirlant et al. (2004, p. 260). For $0<\beta<1$, the above inequality is reversed. The desired results follow. ☐

Proposition 4.2. *Under the conditions of Theorem 3.1, we have*

$$\lim_{\beta\to\infty}\mathsf{E}\left[\frac{q_{\boldsymbol{\Theta}}^{\mathrm{F}}(\beta)}{\left(\sum_{i=1}^d\Theta_i\right)^\beta}\right]$$

$$= \mathsf{E}\left(\int_{\mathbb{S}_{d-1}}\left(\prod_{i=1}^{d}\omega_i^{\Theta_i}\right)^{1/\sum_{i=1}^{d}\Theta_i} H(d\boldsymbol{\omega})\right)$$

and

$$\lim_{\beta\to 0} q_{\boldsymbol{\theta}}^{\mathrm{F}}(\beta) = \int_{\mathbb{S}_{d-1}} \max(\omega_1,\ldots,\omega_d)\, H(d\boldsymbol{\omega}).$$

Proof. First, denote

$$\lambda_{\boldsymbol{\theta}}(\beta) = \left(\sum_{i=1}^{d}\theta_i\omega_i^{1/\beta}\right)^{\beta}.$$

Then $q_{\boldsymbol{\theta}}^{\mathrm{F}}(\beta) = \int_{\mathbb{S}_{d-1}}\lambda_{\boldsymbol{\theta}}(\beta)\, H(d\boldsymbol{\omega})$. It is shown in the appendix that

$$\lim_{\beta\to 0}\lambda_{\boldsymbol{\theta}}(\beta) = \max(\omega_1,\cdots,\omega_d), \tag{18}$$

and

$$\lim_{\beta\to\infty}\frac{\lambda_{\boldsymbol{\theta}}(\beta)}{\left(\sum_{i=1}^{d}\theta_i\right)^{\beta}} = \left(\prod_{j=1}^{d}\omega_j^{\theta_j}\right)^{1/\sum_{i=1}^{d}\theta_i}. \tag{19}$$

Since $0\le\omega_j\le 1$ for $j=1,\ldots,d$, we have

$$\frac{\lambda_{\boldsymbol{\theta}}(\beta)}{\left(\sum_{i=1}^{d}\theta_i\right)^{\beta}} \le 1. \tag{20}$$

Then, for $\beta\ge 1$, by (19) and (20), applying the dominated convergence theorem twice yields that

$$\begin{aligned}\lim_{\beta\to\infty}\frac{q_{\boldsymbol{\Theta}}^{\mathrm{F}}(\beta)}{\left(\sum_{i=1}^{d}\Theta_i\right)^{\beta}} &= \int_{\mathbb{S}_{d-1}}\lim_{\beta\to\infty}\frac{\lambda_{\boldsymbol{\Theta}}(\beta)}{\left(\sum_{i=1}^{d}\Theta_i\right)^{\beta}} H(d\boldsymbol{\omega})\\ &= \int_{\mathbb{S}_{d-1}}\left(\prod_{i=1}^{d}\omega_i^{\Theta_i}\right)^{1/\sum_{i=1}^{d}\Theta_i} H(d\boldsymbol{\omega}),\end{aligned}$$

and

$$\begin{aligned}\lim_{\beta\to\infty}\mathsf{E}\left(\frac{q_{\boldsymbol{\Theta}}^{\mathrm{F}}(\beta)}{\left(\sum_{i=1}^{d}\Theta_i\right)^{\beta}}\right) &= \mathsf{E}\left(\lim_{\beta\to\infty}\frac{q_{\boldsymbol{\Theta}}^{\mathrm{F}}(\beta)}{\left(\sum_{i=1}^{d}\Theta_i\right)^{\beta}}\right)\\ &= \mathsf{E}\left(\int_{\mathbb{S}_{d-1}}\left(\prod_{i=1}^{d}\omega_i^{\Theta_i}\right)^{1/\sum_{i=1}^{d}\Theta_i} H(d\boldsymbol{\omega})\right).\end{aligned}$$

Next, by (18) and (20), applying the dominated convergence theorem yields that

$$\begin{aligned}\lim_{\beta\to 0} q^{\mathrm{F}}(\beta,\boldsymbol{\Theta}) &= \lim_{\beta\to 0}\int_{\mathbb{S}_{d-1}}\frac{\lambda_{\boldsymbol{\Theta}}(\beta)}{\left(\sum_{i=1}^{d}\Theta_i\right)^{\beta}} H(d\boldsymbol{\omega})\\ &= \int_{\mathbb{S}_{d-1}}\lim_{\beta\to 0}\frac{\lambda_{\boldsymbol{\Theta}}(\beta)}{\left(\sum_{i=1}^{d}\Theta_i\right)^{\beta}} H(d\boldsymbol{\omega})\\ &= \int_{\mathbb{S}_{d-1}}\lim_{\beta\to 0}\lambda_{\boldsymbol{\Theta}}(\beta)\, H(d\boldsymbol{\omega})\\ &= \int_{\mathbb{S}_{d-1}}\max(\omega_1,\ldots,\omega_d)\, H(d\boldsymbol{\omega})\end{aligned}$$

This completes the proof. □

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APPENDIX A

Proof of Eq. (18): Denote $\omega_{i_0} = \max(\omega_1,\ldots,\omega_d)$. Then,

$$\begin{aligned}&\lim_{\beta\to 0}\log\lambda_{\boldsymbol{\theta}}(\beta)\\ &= \lim_{\beta\to 0}\beta\log\left(\omega_{i_0}^{1/\beta}\sum_{i=1}^{d}\theta_i\left(\frac{\omega_i}{\omega_{i_0}}\right)^{1/\beta}\right)\\ &= \log\omega_{i_0} + \lim_{\beta\to 0}\beta\log\left(\theta_{i_0}\left(1+\sum_{i\ne i_0}\frac{\theta_i}{\theta_{i_0}}\left(\frac{\omega_i}{\omega_{i_0}}\right)^{1/\beta}\right)\right)\\ &= \log\omega_{i_0},\end{aligned}$$

implying (18). □

Proof of Eq. (19): For $\boldsymbol{\theta} > \mathbf{0}$,

$$\begin{aligned}&\lim_{\beta\to\infty}\log\left(\frac{\lambda_{\boldsymbol{\theta}}(\beta)}{\left(\sum_{i=1}^{d}\theta_i\right)^{\beta}}\right)\\ &\quad= \lim_{\beta\to\infty}\beta\log\left(1+\frac{1}{\sum_{i=1}^{d}\theta_i}\sum_{j=1}^{d}\theta_j(\omega_j^{1/\beta}-1)\right).\end{aligned}$$

Note that

$$\lim_{\beta\to\infty}\beta\bigl(\omega_j^{1/\beta}-1\bigr) = \log\omega_j$$

and

$$\lim_{t\to\infty} t\ \log\bigl(1+c_t/t\bigr) = c \text{ whenever } c_t\to c.$$

Then

$$\lim_{\beta\to\infty}\log\left(\frac{\lambda_{\boldsymbol{\theta}}(\beta)}{\left(\sum_{i=1}^{d}\theta_i\right)^{\beta}}\right) = \frac{1}{\sum_{i=1}^{d}\theta_i}\sum_{j=1}^{d}\theta_j\log\omega_j,$$

implying (19). □

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Xuan Leng
Department of Statistics and Finance
School of Management
University of Science and Technology of China
Hefei, Anhui 230026
China
E-mail address: lxuan10@mail.ustc.edu.cn

Taizhong Hu
Department of Statistics and Finance
School of Management
University of Science and Technology of China
Hefei, Anhui 230026
China
E-mail address: thu@ustc.edu.cn