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Pricing synthetic CDO with MGB2 distribution

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In this paper we apply MGB2 distribution to price synthetic CDO. MGB2 distribution has flexible dependence structure and it is suitable to model extreme risk. The monotonicity of the spread of equity tranche with respect to some parameter is shown. We compare our model with the one-factor Gaussian, Clayton and double t models. Although our proposed MGB2 model is not flexible enough to produce the implied compound correlation smile, it is much more flexible to produce the patterns of base correlation curve than the others. Besides, concerning base correlation, MGB2 and double t model match the market data better than the Gaussian and Clayton copula models.

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1. INTRODUCTION

When we come to the pricing of the credit derivative on a portfolio of reference entities, the critical point is how to model the default correlation or default dependence of the different reference names. Among all kinds of the pricing methodologies, factor model is one of the most popular. The factor model approach for modeling correlated defaults assumes that conditional on the common factors, the defaults are independent,which helps simplify the calculation of the loss distribution. A modeler is then left to choose the number of the factors and specify the distributions of them so that the model can fit the market prices well. For more details about factor model for multi-name credit derivatives pricing, we refer the readers to [9, 10].

The most well-known factor model is one-factor Gaussian model, in which there is only one common factor and one idiosyncratic factor for each entity and all the factors independently follow Gaussian distributions. It is initially proposed by [11] for the pricing of multi-name credit derivations and since then it has been the industry standard and benchmark. However, as has been extensively indicated (see, for example, [3, 6, 14]), the one-factor Gaussian model fails to simultaneously fit the market prices of the different tranches of a synthetic CDO which leads to the implied correlation smile. In order to fit the market data better, many other factor models based on different distributions or copulas have recently been proposed. [1, 4] extended one-factor Gaussian model by replacing the constant correlation with stochastic correlation. [7] argued if the common factors have heavy tails, there will be a greater possibility of default clustering which may explain the phenomenon of correlation smile, so it extended one-factor Gaussian model simply by replacing the Gaussian distributions with t-distributions. It also suggested double t-distribution copula with 4 degrees of freedom to fit the market price. However, since t-distribution is not stable under convolution, the distributions of default risk factors do not have analytic or semi-analytic formulas, thus the computation will be time-consuming by Monte Carlo method, as suggested by [8] where the normal inverse Gaussian (NIG) distribution is introduced to price synthetic CDO. [5] compared some popular CDO pricing models and suggested the best criteria to assess a model's ability for pricing CDO is the difference in implied correlation. Nevertheless, the existence of implied correlation is not guaranteed, see [15]. To overcome it, [13] proposed a substitute of implied correlation, the base correlation. In general, an eligible factor model for the pricing of multi-name credit derivatives is on all accounts obliged to be flexible in modeling dependence structure, efficient in large-scale computation and capable of producing implied correlation smile or proper base correlation trend curve.

In this paper, we apply MGB2 distribution to price CDOs. MGB2 introduced recently by [16] is a scale mixture of generalized gamma distributions with inverse generalized gamma weights. It can produce heavy tails, skewness and flexible dependence structures. In particular, it is appropriate to model extreme events like default clustering and extreme risks in catastrophic insurance. As shown later, the proposed factor model based on MGB2 distribution possesses the properties an eligible multi-name credit derivations pricing model should have.

The remainder of this paper is organized as follows. Section 2 introduces one-factor model and MGB2 distribution, which is then applied to modeling the joint distribution of default times. Section 3 presents how to price the synthetic CDOs and shows that the spread of equity tranche with respect to some parameter is monotonic. Section 4 calibrates the proposed model with market data, and then evaluates the applicability and flexibility of our model. Section 5 concludes.

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2. ONE-FACTOR MODEL AND MGB2 DISTRIBUTION

2.1 One-factor model

Consider a portfolio consisting of n credit-risky assets. The default times of the obligors are denoted by $(\tau_1, \ldots, \tau_n)$ with marginal distribution functions F_i's. The one-factor model can be written as

$$X_i = \rho_i M + \sqrt{1-\rho_i^2} Z_i, \quad i = 1, \ldots, n, \tag{1}$$

where M and Z_i's have independent zero-mean unit-variance distributions. Conditional on M, the X_i's are independent and the conditional cumulative distribution function (c.d.f.) of X_i is

$$P(X_i \le x_i | M) = P\left(Z_i \le \frac{x_i - \rho_i M}{\sqrt{1-\rho_i^2}} | M \right).$$

In Gaussian model M and Z_i's follow independent Gaussian distributions, and in double t model they follow independent normalized t distributions. One-factor portfolio credit risk model means that the default times are given by

$$\tau_i = F_i^{-1}\big(F_{X_i}(X_i)\big), \quad i = 1, \ldots, n, \tag{2}$$

where F_{X_i} is the distribution function of X_i. Hence τ_i's and X_i's have the same copula structure. It follows from (1) and (2) that

$$P(\tau_i \le t_i | M) = P\left(Z_i \le \frac{F_{X_i}^{-1}(F_i(t_i)) - \rho_i M}{\sqrt{1-\rho_i^2}} | M \right),$$

which indicates the default times are correlated by the common factor M and they are independent conditioning on M. That is why factor model is also called conditional independence model. Note that in practice $F_i(t_i)$ is calibrated to CDS market quotes for single names, then they can give the corresponding x_i by $x_i = F_{X_i}^{-1}(F_i(t_i))$.

Another one-factor structure can be formulated as follows. Let V be a positive random variable and $\psi(s)$ the Laplace transform of V, i.e., $\psi(s) = E[e^{-sV}]$. Define the latent variables X_i's as

$$X_i = \psi(-\ln U_i / V), \quad i = 1, \ldots, n, \tag{3}$$

where $U_1, \ldots, U_n$ are independent uniform random variables and independent from V. Then the default times are given by

$$\tau_i = F_i^{-1}(X_i), \quad i = 1, \ldots, n. \tag{4}$$

From (3) and (4), we have

$$P(\tau_i \le t_i | V) = \exp\big(-V\psi^{-1}\big(F_i(t_i)\big)\big), \tag{5}$$

and default times are independent given V. The joint distribution function can be written as

$$F(t_1, \ldots, t_n) = \psi\left(\sum_{i=1}^n \psi^{-1}\big(F_i(t_i)\big) \right).$$

A typical example is the Clayton copula with $\psi(s) = (1+s)^{-1/\theta}, \theta > 0$.

As can be seen later, our proposed model takes an approach closer to the latter one, but cannot be generated by either of these one-factor structures.

2.2 MGB2 distribution

Let $\mathbf{X} = (X_1, \ldots, X_n)^T$ be a real n-dimensional random vector on $(0, \infty]^n$ such that given θ each X_i follows a generalized gamma distribution $GG(a_i, b_i\theta^{1/a_i}, p_i)$ with probability density function

$$f_{X_i|\theta}(x_i) = \frac{a_i}{\Gamma(p_i) x_i \theta^{p_i}} (x_i/b_i)^{a_i p_i} e^{-(x_i/b_i)^{a_i}/\theta}, \quad x_i > 0.$$

Moreover, if θ follows an inverse gamma distribution with shape parameter q and unit scale, i.e., $\theta \sim \text{InvGa}(q, 1)$,

$$f_\theta(\theta) = \frac{1}{\Gamma(q)} \theta^{-q-1} e^{-1/\theta},$$

then $(X_1, \ldots, X_n)$ is said to follow multivariate GB2 distribution (MGB2); see [16] for details. The marginal distribution is written as $\text{GB2}(a_i, b_i, p_i, q)$.

Notice that the generalized gamma distribution $GG(a_i, b_i\theta^{1/a_i}, p_i)$ with $p_i = 1$ transforms to Weibull distribution and Fréchet distribution with scale parameter $b_i\theta^{1/a_i}$ and shape parameter $|a_i|$ when $a_i > 0$ and $a_i < 0$ respectively. In particular, if in addition $a_i = 1$, the generalized gamma distribution $GG(a_i, b_i\theta^{1/a_i}, p_i)$ becomes the exponential distribution with mean $b_i\theta$. Here θ can be understood as the common component of the market, and p_i, a_i, b_i contain the idiosyncratic component of the X_i.

The conditional c.d.f. of X_i given θ is

$$\begin{aligned} P(X_i \le x_i | \theta) &= \int_0^{x_i} \frac{a_i}{\Gamma(p_i) t \theta^{p_i}} (t/b_i)^{a_i p_i} e^{-(t/b_i)^{a_i}/\theta} dt \\ &= G_p\big((x_i/b_i)^{a_i}\theta^{-1}\big), \end{aligned} \tag{6}$$

where $G_p(z) = \frac{1}{\Gamma(p)} \int_0^z t^{p-1} e^{-t} dt, z > 0$ and $\Gamma(p)$ is the gamma function. Recall the relation between a GB2 variable and a standard beta variable: if X is a $\text{GB2}(a, b, p, q)$, then the transformed variable $(X/b)^a / [1 + (X/b)^a]$ follows a beta distribution $B(p, q)$. Thus, the c.d.f. of X_i is given by

$$F_{X_i}(x_i) = B_{p_i,q}\big(x_i^{a_i} / \big(x_i^{a_i} + b_i^{a_i}\big)\big)$$

where $B_{p_i,q}$ is the c.d.f of a standard beta variable with parameters p_i and q, i.e. $B_{p_i,q}(z) = \frac{1}{B(p_i,q)} \int_0^z t^{p_i-1}(1-t)^{q-1} dt, 0 \le z \le 1$.

The joint conditional c.d.f. of $\mathbf{X}$ given θ is

$$F_{\mathbf{X}|\theta}(x_1,\ldots,x_n) = \prod_{i=1}^n G_{p_i}\big((x_i/b_i)^{a_i}\theta^{-1}\big).$$

Then the unconditional c.d.f. of $\mathbf{X}$ is

$$\begin{aligned} F_{\mathbf{X}}(x_1,\ldots,x_n) &= E_\theta\big[F_{\mathbf{X}|\theta}(x_1,\ldots,x_n)\big] \\ &= \int_0^\infty \prod_{i=1}^n G_{p_i}\big((x_i/b_i)^{a_i}\theta^{-1}\big) f_\theta(\theta)d\theta. \end{aligned}$$

Moreover, let $S_n = \sum_{i=1}^n p_i$, then the probability density function of $\mathbf{X}$ is

$$\begin{aligned} &f_{\mathbf{X}}(x_1,\ldots,x_n) = \int_0^\infty f_\theta(\theta)\prod_{i=1}^n f_{X_i|\theta}(x_i|\theta)d\theta \\ &= \left(\prod_{i=1}^n \frac{a_i}{\Gamma(p_i)x_i}\left(\frac{x_i}{b_i}\right)^{a_ip_i}\right)\frac{1}{\Gamma(q)}\int_0^\infty \theta^{-q-1-S_n} e^{\frac{\sum_{i=1}^n(\frac{x_i}{b_i})^{a_i}+1}{\theta}}d\theta \\ &= \left(\prod_{i=1}^n \frac{a_i}{\Gamma(p_i)x_i}\left(\frac{x_i}{b_i}\right)^{a_ip_i}\right)\frac{1}{\Gamma(q)}\frac{\Gamma(S_n+q)}{(\sum_{i=1}^n(x_i/b_i)^{a_i}+1)^{S_n+q}} \\ &= \frac{\Gamma(S_n+q)}{\Gamma(q)\prod_{i=1}^n\Gamma(p_i)x_i}\frac{\prod_{i=1}^n a_i(x_i/b_i)^{a_ip_i}}{[1+\sum_{i=1}^n(x_i/b_i)^{a_i}]^{S_n+q}}. \end{aligned}$$

The n-dimensional MGB2 copula is defined by

$$\begin{aligned} (7)\quad & C^{\mathrm{MGB2}}_{p_1,\ldots,p_n,q}(u_1,\ldots,u_n) = F_{\mathbf{X}}\big(F^{-1}_{X_1}(u_1),\ldots,F^{-1}_{X_n}(u_n)\big) \\ &= \int_0^\infty \prod_{i=1}^n G_{p_i}\left(\frac{B^{-1}_{p_i,q}(u_i)}{(1-B^{-1}_{p_i,q}(u_i))\theta}\right)\frac{\theta^{-(q+1)}e^{-1/\theta}}{\Gamma(q)}d\theta, \end{aligned}$$

where $(u_1,\ldots,u_n)\in[0,1]^n$. As shown in [16], MGB2 copula is very flexible in modeling tail dependence and it includes some other copulas as its limit cases. To name a few, independence copula, Fréchet-Hoeffding upper bound and Gaussian copula can be approached by MGB2 copula. What is more, bivariate MGB2 copula is lower-tail independent and upper-tail dependent.

2.3 Joint distribution of default times

For pricing multi-name credit derivatives, we need to model the joint distribution function of default times τ_i's. Under the framework of intensity model, it is often the case that the distribution function of default time τ_i is assumed to be $p_i(t) = P(\tau_i \le t) = 1-\exp(-\int_0^t \lambda_i(s)ds)$, where $\lambda_i(t)$ is deterministic non-negative process and called intensity process, then copulas are used to generate the joint distribution; see [11]. [12] evaluated two extreme value distributions Weibull and Fréchet distributions for the time to default and found the former fits the market data better. There they used one-factor Gaussian and Clayton copulas to link the marginal distributions for pricing basket default swaps. In this paper, we choose MGB2 distribution as the joint distribution of default times. Our model can be treated like this: each default time follows GB2 distribution, which is a mixture of generalized gamma distribution, i.e., $\tau_i \sim GB2(a_i,b_i,p_i,q)$, and their dependence structure is formed by MGB2 copula. Besides, because Weibull distribution is a special case of generalized gamma distribution, the proposed model is more general than that in [12], to a large extent.

3. PRICING OF SYNTHETIC CDO

3.1 Distributions of number of default and time of mth default

Throughout this paper, Q and $\tilde{E}$ represent the risk-neutral probability measure and the expectation under this measure, respectively. The calculation of premium payment leg involves the distribution of counting process $N(t)$ defined as $N(t) = \sum_{i=1}^n \mathbf{1}_{\tau_i\le t}$, namely, $N(t)$ is the number of defaults at time t. $N_i(t) = \mathbf{1}_{\tau_i\le t}$ is defined as the indicator of default of name i. We now compute the probability of k defaults at time t, i.e. $Q(N(t)=k)$ for $k=0,\ldots,n$, using the probability generating function.

The probability generating function of $N(t)$ is

$$(8)\qquad \psi_{N(t)}(u) = \tilde{E}\big[u^{N(t)}\big] = \sum_{k=0}^n Q\big(N(t)=k\big)u^k.$$

On the other hand, $N_i(t), i=1,\ldots,n$, are conditionally independent given θ. Write $p_{i|\theta}(t) = Q(\tau_i\le t|\theta)$. We have

$$\begin{aligned} (9)\qquad \tilde{E}\big[u^{N(t)}\big] &= \tilde{E}\big[\tilde{E}\big[u^{N(t)}|\theta\big]\big] = \tilde{E}\left[\prod_{i=1}^n \tilde{E}\big[u^{N_i(t)}|\theta\big]\right] \\ &= \int f_\theta(\theta)\prod_{i=1}^n\big[1-p_{i|\theta}(t)+p_{i|\theta}(t)u\big]d\theta. \end{aligned}$$

Note that the above procedure for MGB2 distribution is still applicable for any other one-factor model, in which θ is replaced by the common factor. The probability mass function of $N(t)$ is given by matching the coefficient of the polynomial terms of the same order in (8) and (9).

Similarly, the conditional distribution of $N(t)$ given θ can be obtained by matching the coefficients of the following two polynomials:

$$(10)\qquad \psi_{N(t)|\theta}(u) = \tilde{E}\big[u^{N(t)}|\theta\big] = \sum_{k=0}^n Q\big(N(t)=k|\theta\big)u^k,$$

(11)

$$\tilde{E}\big[u^{N(t)}|\theta\big] = \prod_{i=1}^n \tilde{E}\big[u^{N_i(t)}|\theta\big] = \prod_{i=1}^n\big[1-p_{i|\theta}(t)+p_{i|\theta}(t)u\big].$$

Therefore,

$$Q\big(N(t)=k|\theta\big) = \sum_{i_1<\cdots<i_k}\prod_{j=1}^k p_{i_j|\theta}(t)\prod_{j\notin\{i_1,\ldots,i_k\}}\big(1-p_{j|\theta}(t)\big),$$

which in case of identical marginal distribution, i.e. $p_{i|\theta}(t) = p_\theta(t)$, simplifies to

$$Q\big(N(t)=k|\theta\big) = \binom{n}{k} p_\theta^k(t)\big(1-p_\theta(t)\big)^{n-k}.$$

We consider the distribution of the mth order statistic $\tau^{(m)}$ of $\tau_i, i=1,\dots,n$. The conditional survival function of $\tau^{(m)}$ is

(12)
$$Q\big(\tau^{(m)}>t|\theta\big) = Q\big(N(t)<m|\theta\big) = \sum_{k=0}^{m-1} Q\big(N(t)=k|\theta\big).$$

To complete the calculation we introduce the default counting processes $N^{(-i)}(t)$ that count the number of default excluding name i, i.e., $N^{(-i)}(t) = \sum_{j\neq i} \mathbf{1}_{\tau_j\le t}$. To obtain the probability density function of $\tau^{(m)}$, take derivative of (10) and (11) with respect to t,

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{\partial Q(N(t)=k|\theta)}{\partial t} u^k \\
&= \sum_{i=1}^{n} (u-1)\frac{\partial p_{i|\theta}(t)}{\partial t} \prod_{j\neq i}\big[1-p_{j|\theta}(t)+p_{j|\theta}(t)u\big] \\
&= \sum_{i=1}^{n} (u-1)\frac{\partial p_{i|\theta}(t)}{\partial t} \sum_{k=0}^{n-1} Q\big(N^{(-i)}(t)=k|\theta\big)u^k \\
&= \sum_{i=1}^{n} \frac{\partial p_{i|\theta}(t)}{\partial t}\Bigg\{-Q\big(N^{(-i)}(t)=0|\theta\big) - \sum_{k=1}^{n-1}\Big(Q\big(N^{(-i)}(t)=k|\theta\big) \\
&\quad - Q\big(N^{(-i)}(t)=k-1|\theta\big)\Big)u^k + Q\big(N^{(-i)}(t)=n-1|\theta\big)u^n\Bigg\}.
\end{aligned}$$

Hence for $k=0$, we have

$$-\frac{\partial Q(N(t)=0|\theta)}{\partial t} = \sum_{i=1}^{n}\frac{\partial p_{i|\theta}(t)}{\partial t} Q\big(N^{(-i)}(t)=0|\theta\big),$$

and for $1\le k\le n$,

$$\begin{aligned}
&-\frac{\partial Q(N(t)=k|\theta)}{\partial t} \\
&= \sum_{i=1}^{n}\frac{\partial p_{i|\theta}(t)}{\partial t}\Big(Q\big(N^{(-i)}(t)=k|\theta\big) - Q\big(N^{(-i)}(t)=k-1|\theta\big)\Big)
\end{aligned}$$

where $Q(N^{(-i)}(t)=n|\theta)=0$ for any $1\le i\le n$. Thus the conditional probability density function of $\tau^{(m)}$ given θ is

(13)
$$\begin{aligned}
f_{\tau^{(m)}|\theta}(t) &= -\frac{\partial Q(\tau^{(m)}>t|\theta)}{\partial t} = -\sum_{k=0}^{m-1}\frac{\partial Q(N(t)=k|\theta)}{\partial t} \\
&= \sum_{i=1}^{n}\frac{\partial p_{i|\theta}(t)}{\partial t} Q\big(N^{(-i)}(t)=m-1|\theta\big).
\end{aligned}$$

3.2 Pricing

In the following, $f_{\tau^{(m)}|\theta}$ is utilized to simplify the computation of expected premium payment legs and default payment legs in place of stochastic integrals. Assume that the underlying portfolio consists of n credit assets with unit total nominal. Let U_i be the nominal of underlying asset i, then $\sum_{i=1}^n U_i = 1$. If the underlying portfolio is the traded index *Dow Jones CDX NA IG* or *Dow Jones iTraxx Europe*, $n=125$ and the nominal is equally weighted, i.e. $U_i = \frac{1}{n}$, and the conditional marginal default distribution are identical, i.e. $p_{i|\theta}(t) = p_\theta(t)$ for all the i.

Let T be the maturity date. Premium payments are made periodically at $t_1 = I, t_2 = 2I, \dots, t_J = JI = T$. Let A_m and D_m be the attachment point and detachment point for tranche m, respectively. Let $L(t)$ be the accumulated loss up to time t, then the loss suffered by the holders of tranche m, is

(14)
$$\begin{aligned}
L_m(t) &= \min\{L(t), D_m\} - \min\{L(t), A_m\} \\
&= \min\{\big(L(t)-A_m\big)^+, D_m - A_m\}.
\end{aligned}$$

Because all the recovery rates δ_i are commonly assumed to be $\delta := 40\%$ for pricing *CDX NA IG* and *iTraxx* indexes in the market, the portfolio loss is

(15)
$$L(t) = \sum_{i=1}^{n} U_i(1-\delta_i)\mathbf{1}_{\{\tau_i\le t\}} = \frac{1-\delta}{n}N(t).$$

Therefore for $0\le t\le T$,

(16)
$$\begin{aligned}
L_m(t) &= \min\Bigg\{\Big(\frac{1-\delta}{n}N(t)-A_m\Big)^+, D_m - A_m\Bigg\} \\
&= \begin{cases} 0, & N(t)\in[0, \frac{nA_m}{1-\delta}]; \\ \frac{1-\delta}{n}N(t)-A_m, & N(t)\in(\frac{nA_m}{1-\delta}, \frac{nD_m}{1-\delta}]; \\ D_m - A_m, & N(t)\in(\frac{nD_m}{1-\delta}, n]. \end{cases}
\end{aligned}$$

The outstanding nominal for tranche m at time t is

(17)
$$O_m(t) = D_m - A_m - L_m(t).$$

Let $\{k_{m-1}+1,\dots,k_m\}$ be the integers in interval $(\frac{nA_m}{1-\delta}, \frac{nD_m}{1-\delta}]$. Let s_m be the annualized spread of tranche m. For calculating the premium payment leg, we assume that when the defaults happen between two premium payment dates, the remaining nominal is used to assess the premiums, then the expected premium payment leg is

(18)
$$\begin{aligned}
&\tilde{E}\Bigg[\sum_{j=1}^{J} I s_m B(0,t_j) O_m(t_j)\Bigg] \\
&= \tilde{E}\Bigg[\sum_{j=1}^{J} I s_m B(0,t_j)\big(D_m - A_m - L_m(t_j)\big)\Bigg].
\end{aligned}$$

Furthermore, $\tilde{E}[L_m(t_j)]$ can be written as

$$\tilde{E}\big[L_m(t_j)\big] = \sum_{l=1}^{k_m-k_{m-1}} \Big(\frac{k_{m-1}+l}{n}(1-\delta) - A_m\Big) Q\big(N(t_j)=k_{m-1}+l\big) + (D_m - A_m)Q\big(N(t_j) > k_m\big),$$

which yields

$$\tilde{E}\big[D_m - A_m - L_m(t_j)\big] = (D_m - A_m)Q\big(N(t_j) \le k_{m-1}\big) + \sum_{l=1}^{k_m-k_{m-1}} \Big(D_m - \frac{k_{m-1}+l}{n}(1-\delta)\Big) Q\big(N(t_j)=k_{m-1}+l\big).$$

On the other hand, the default payment leg for tranche m by (16) is

$$\tilde{E}\left[\int_0^T B(0,t)dL_m(t)\right] = \sum_{t\in[0,T]} B(0,t)\tilde{E}\big[\Delta L_m(t)\mathbf{1}_{N(t)\in(\frac{nA_m}{1-\delta},\frac{nD_m}{1-\delta}]}\big],$$

where $\Delta L_m(t) = L_m(t) - L_m(t-)$.

The jump of $L_m(t)$ happens only at times when there's a default, i.e., $\tau^{(l)}$ where $l \in \{k_{m-1}+1, \dots, k_m, k_m+1\}$. Therefore the discounted default payment till time t can be written as

$$\Big(\frac{k_{m-1}+1}{n}(1-\delta) - A_m\Big) B\big(0,\tau^{(k_{m-1}+1)}\big)\mathbf{1}_{\tau^{(k_{m-1}+1)}\le t} + \frac{1-\delta}{n}\sum_{l=2}^{k_m-k_{m-1}} B\big(0,\tau^{(k_{m-1}+l)}\big)\mathbf{1}_{\tau^{(k_{m-1}+l)}\le t} + \Big(D_m - \frac{k_m}{n}(1-\delta)\Big)B\big(0,\tau^{(k_m+1)}\big)\mathbf{1}_{\tau^{(k_m+1)}\le t}.$$

The expected default payment leg for tranche m is

(19)
$$\tilde{E}\left[\int_0^T B(0,t)dL_m(t)\right] = \Big(\frac{k_{m-1}+1}{n}(1-\delta) - A_m\Big)\tilde{E}\big[B\big(0,\tau^{(k_{m-1}+1)}\big)\mathbf{1}_{\tau^{(k_{m-1}+1)}\le T}\big] + \frac{1-\delta}{n}\sum_{l=2}^{k_m-k_{m-1}} \tilde{E}\big[B\big(0,\tau^{(k_{m-1}+l)}\big)\mathbf{1}_{\tau^{(k_{m-1}+l)}\le T}\big] + \Big(D_m - \frac{k_m}{n}(1-\delta)\Big)\tilde{E}\big[B\big(0,\tau^{(k_m+1)}\big)\mathbf{1}_{\tau^{(k_m+1)}\le T}\big],$$

where by (13)

$$\tilde{E}\big[B\big(0,\tau^{(l)}\big)\mathbf{1}_{\{\tau^{(l)}\le T\}}\big] = \tilde{E}_\theta\left[\int_0^T B(0,t)f_{\tau^{(l)}|\theta}(t)dt\right] = \tilde{E}_\theta\left[\sum_{i=1}^n \int_0^T B(0,t)Q\big(N^{(-i)}(t)=l-1|\theta\big)dp_{i|\theta}(t)\right]$$

Under the assumption of joint MGB2 distribution with identical marginal distributions, we have $p_{i|\theta}(t) = p_\theta(t) = G_p((t/b)^a/\theta)$ and $dp_{i|\theta}(t) = dp_\theta(t) = \frac{a}{b\theta^p\Gamma(p)}(t/b)^{ap-1} \times e^{-(t/b)^a/\theta}dt$. Furthermore, the expected discount factor of the l-th default is

$$\tilde{E}\big[B\big(0,\tau^{(l)}\big)\mathbf{1}_{\{\tau^{(l)}\in[0,T]\}}\big] = n\tilde{E}_\theta\left[\int_0^T B(0,t)Q\big(N^{(-i)}(t)=l-1|\theta\big)dp_\theta(t)\right] = n\binom{n-1}{l-1}\int_0^\infty\int_0^T B(0,t)p_\theta^{l-1}(t)\big(1-p_\theta(t)\big)^{n-l}f_\theta(\theta) \times dp_\theta(t)d\theta$$

The spread s_m of tranche m is then obtained by equating the expected premium payment leg (18) and default payment leg (19) under risk neutral probability.

In particular, for the equity tranche, i.e., $m=1$, we have $A_1 = 0, k_0 = 0$. Hence

(20)
$$Is_1\sum_{j=1}^J\sum_{l=0}^{k_1} B(0,t_j)\Big(D_1 - \frac{l}{n}(1-\delta)\Big)Q\big(N(t_j)=l\big) = \frac{1-\delta}{n}\sum_{l=1}^{k_1}\tilde{E}\big[B\big(0,\tau^{(l)}\big)\mathbf{1}_{\tau^{(l)}\le T}\big] + \Big(D_1 - \frac{k_1}{n}(1-\delta)\Big)\tilde{E}\big[B\big(0,\tau^{(k_1+1)}\big)\mathbf{1}_{\tau^{(k_1+1)}\le T}\big],$$

where

$$Q\big(N(t_j)=x\big) = \binom{n}{x}\int_0^\infty f(\theta)p_\theta^x(t_j)\big(1-p_\theta(t_j)\big)^{n-x}d\theta.$$

3.3 The monotonicity of equity tranche price on parameter a

To accommodate the marginal distribution of MGB2 distribution with constant intensity defined in exponential distribution, we match the first order moment, i.e., the expected time of default. Under the assumption of identical marginal distribution, we have

$$\tilde{E}(X_1) = \frac{b\Gamma(q-1/a)\Gamma(p+1/a)}{\Gamma(q)\Gamma(p)} = b\cdot\frac{B(p+1/a,q-1/a)}{B(p,q)}.$$

Obviously, the expected default time is increasing on scale parameter b. Due to the fact that beta function $B(x,c-x)$

is decreasing on $(0, c/2]$ and increasing on $[c/2, c)$ and that $q \geq 1/a$, we conclude that if $p < q$, the expected default time is decreasing on $a \in (1/q, 2/(q-p)]$ and increasing on $a \in (2/(q-p), \infty)$, and equal to b when $a = 1/(q-p)$; if $p \geq q$, the the expected default time is decreasing on $a \in (1/q, \infty)$.

For simplicity and uniqueness, we set $b = \frac{B(p,q)}{\lambda B(p+1/a, q-1/a)}$, where $1/\lambda$ is the expected default time in a exponential distribution. Thus for $i = 1, \ldots, n$,

$$p_\theta(t) = G_p\Bigg(\bigg(\frac{t\lambda B(p+1/a, q-1/a)}{B(p,q)}\bigg)^a/\theta\Bigg). \tag{21}$$

In this paper, we are interested in CDO prices produced by MGB2 distributions under various dependence structures. In consequence, we calibrate tranche prices with parameter a. To ensure the existence and uniqueness of the calibration procedure, we now proceed with three lemmas to show that the spread of equity tranche is a decreasing function of a under the above set up.

Lemma 3.1. *For fixed p and q, $p_\theta(t)$ is a decreasing function of a, if we choose $b = \frac{B(p,q)}{\lambda B(p+1/a,q-1/a)}$, where λ is the intensity of defaults satisfying $\lambda T < 1$.*

Proof. Consider digamma function $\phi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, the derivative of logarithm of gamma function $\ln(\Gamma(x))$. Using the derivative of beta function, we have

$$\frac{\partial B(p+\frac{1}{a}, q-\frac{1}{a})}{\partial a} = \frac{B(p+\frac{1}{a}, q-\frac{1}{a})}{a^2}\bigg[\phi\Big(q-\frac{1}{a}\Big) - \phi\Big(p+\frac{1}{a}\Big)\bigg]. \tag{22}$$

Note that the digamma function $\phi(x)$ is a nondecreasing function on $(0,\infty)$. Thus, if $q \leq p$, $\frac{\partial B(p+1/a,q-1/a)}{\partial a} < 0$; if $q > p$, $\frac{\partial B(p+1/a,q-1/a)}{\partial a} > 0$ on $a \in (2/(q-p), \infty)$ and $\frac{\partial B(p+1/a,q-1/a)}{\partial a} \leq 0$ on $a \in (1/q, 2/(q-p)]$.

Let $\kappa = \frac{\lambda t}{B(p,q)}$. It follows from (22) that

$$\begin{aligned}
\frac{\partial(\kappa B(p+1/a, q-1/a))^a}{\partial a} &= \big(\kappa B(p+1/a, q-1/a)\big)^a \\
&\quad \times \bigg[\frac{a}{B(p+1/a, q-1/a)}\frac{\partial B(p+1/a, q-1/a)}{\partial a} \\
&\quad + \ln\big(\kappa B(p+1/a, q-1/a)\big)\bigg] \\
&= \big(\kappa B(p+1/a, q-1/a)\big)^a\bigg\{\frac{1}{a}\bigg[\phi\Big(q-\frac{1}{a}\Big) - \phi\Big(p+\frac{1}{a}\Big)\bigg] \\
&\quad + \ln\bigg(\frac{\Gamma(p+1/a)\Gamma(q-1/a)}{\Gamma(p)\Gamma(q)}\bigg) + \ln(t\lambda)\bigg\}.
\end{aligned}$$

Considering the fact that $\ln(\Gamma(x))$ is strongly convex on $(0, \infty)$, we have $\ln(\Gamma(p)) > \ln(\Gamma(p+1/a)) - \phi(p+1/a)/a$ and $\ln(\Gamma(q)) > \ln(\Gamma(q-1/a)) + \phi(q-1/a)/a$, which then implies that

$$\frac{1}{a}\bigg[\phi\Big(q-\frac{1}{a}\Big) - \phi\Big(p+\frac{1}{a}\Big)\bigg] + \ln\bigg(\frac{\Gamma(p+1/a)\Gamma(q-1/a)}{\Gamma(p)\Gamma(q)}\bigg)$$

is negative. Together with the condition that $\lambda T < 1$,

$$\frac{\partial(\kappa B(p+1/a, q-1/a))^a}{\partial a} < 0.$$

Besides, taking derivative of (21) we get

$$\frac{\partial p_\theta(t)}{\partial a} = \frac{1}{\theta^p\Gamma(p)}\Big(\frac{t}{b}\Big)^{(p-1)a} e^{-(t/b)^a/\theta}\frac{\partial(\kappa B(p+1/a, q-1/a))^a}{\partial a}.$$

The sign of $\frac{\partial p_\theta(t)}{\partial a}$ only depends on that of $\frac{\partial(\kappa B(p+1/a,q-1/a))^a}{\partial a}$ and is negative as a consequence.

We note that in general, the maturity is within expected default time, i.e., $\lambda T < 1$ is a trivial assumption. ☐

Lemma 3.2. *The expected premium payment leg of equity tranche is an increasing function of a.*

Proof. Taking derivative of $Q(N(t_j) = x)$ with respect to a would end up with

$$\begin{aligned}
\frac{dQ(N(t_j) = x)}{da} &= \binom{n}{x}\int_0^\infty f(\theta)p_\theta^x(t_j)\big(1-p_\theta(t_j)\big)^{n-x} \\
&\quad \times \bigg[\frac{x}{p_\theta(t_j)} - \frac{n-x}{1-p_\theta(t_j)}\bigg]\frac{dp_\theta(t_j)}{da}d\theta.
\end{aligned} \tag{23}$$

Obviously, $Q(N(t_j) = 0)$ is an increasing function of a, and $Q(N(t_j) = n)$ is a decreasing function of a. For convenience, let

$$\begin{aligned}
A(t) &= \ln\bigg(\frac{\Gamma(p+1/a)\Gamma(q-1/a)}{\Gamma(p)\Gamma(q)}\bigg) \\
&\quad + \frac{1}{a}\bigg[\phi\Big(q-\frac{1}{a}\Big) - \phi\Big(p+\frac{1}{a}\Big)\bigg] + \ln(t\lambda).
\end{aligned}$$

We connect $\frac{dp_\theta(t)}{dt}$ with $\frac{dp_\theta(t)}{da}$ by observing that

$$\frac{dp_\theta(t)}{dt} = \frac{a}{b\theta^p\Gamma(p)}(t/b)^{ap-1}e^{-(t/b)^a/\theta},$$

and

$$\begin{aligned}
\frac{dp_\theta(t)}{da} &= \frac{1}{\theta^p\Gamma(p)}\Big(\frac{t}{b}\Big)^{ap-a}e^{\frac{-1}{\theta}(\frac{t}{b})^a}\bigg(\kappa B\Big(p+\frac{1}{a}, q-\frac{1}{a}\Big)\bigg)^a\frac{A(t)}{a} \\
&= \frac{1}{\theta^p\Gamma(p)}(t/b)^{ap}e^{-(t/b)^a/\theta}\frac{A(t)}{a} \\
&= \frac{dp_\theta(t)}{dt}\frac{tA(t)}{a}.
\end{aligned} \tag{24}$$

Replacing $\frac{dp_\theta(t)}{da}$ in (23) with (24) gives

$$\frac{dQ(N(t_j)=x)}{da} = \binom{n}{x}\int_0^\infty f(\theta)p_\theta^x(t_j)\big(1-p_\theta(t_j)\big)^{n-x}$$

$$\times\left[\frac{x}{p_\theta(t_j)} - \frac{n-x}{1-p_\theta(t_j)}\right]\frac{dp_\theta(t)}{dt}|_{t=t_j}d\theta\cdot\frac{t_jA(t_j)}{a}$$

$$=\int_0^\infty f(\theta)\big[f\big(\tau^{(x)}=t_j|\theta\big) - f\big(\tau^{(x+1)}=t_j|\theta\big)\big]d\theta\cdot\frac{t_jA(t_j)}{a}$$

$$=\big[f\big(\tau^{(x)}=t_j\big) - f\big(\tau^{(x+1)}=t_j\big)\big]\frac{t_jA(t_j)}{a}.$$

From the proof of Lemma 3.1, $A(t_j) < 0$. Let $\epsilon = D_1 - \frac{k_1}{n}(1-\delta)$, then the derivative of the first premium of first tranche with respect to parameter a is

$$\sum_{l=0}^{k_1}\sum_{j=1}^{J}B(0,t_j)\left(D_1-\frac{l}{n}(1-\delta)\right)\frac{dQ(N(t_j)=l)}{da}$$

$$=\sum_{j=1}^{J}\sum_{l=0}^{k_1}B(0,t_j)\left(\epsilon+\frac{k_1-l}{n}(1-\delta)\right)\frac{dQ(N(t_j)=l)}{da}$$

$$=\sum_{j=1}^{J}B(0,t_j)\frac{t_jA(t_j)}{a}\sum_{l=0}^{k_1}\left(\epsilon+\frac{l}{n}(1-\delta)\right)$$

$$\big[f\big(\tau^{(k_1-l)}=t_j\big) - f\big(\tau^{(k_1-l+1)}=t_j\big)\big]$$

$$=\sum_{j=1}^{J}B(0,t_j)\frac{t_jA(t_j)}{a}\Bigg[-\epsilon f\big(\tau^{(k_1+1)}=t_j\big)$$

$$+\frac{1-\delta}{n}\sum_{l=1}^{k_1}l\big[f\big(\tau^{(k_1-l)}=t_j\big) - f\big(\tau^{(k_1-l+1)}=t_j\big)\big]\Bigg]$$

$$=\sum_{j=1}^{J}B(0,t_j)\frac{t_jA(t_j)}{a}\Bigg[-\epsilon f\big(\tau^{(k_1+1)}=t_j\big)$$

$$-\frac{1-\delta}{n}\sum_{l=1}^{k_1+1}f\big(\tau^{(k_1-l+1)}=t_j\big)\Bigg] > 0.$$

Hence we complete the proof. □

Lemma 3.3. *The expected default payment leg of equity tranche is a decreasing function of a.*

Proof. Using integral by part, we have

$$\tilde{E}\big[B\big(0,\tau^{(l)}\big)\mathbf{1}_{\{\tau^{(l)}\in[0,T]\}}\big] = \int_0^T e^{-rt}dF_{\tau^{(l)}}(t)$$

$$= -e^{-rt}\big(1-F_{\tau^{(l)}}(t)\big)|_0^T - r\int_0^T e^{-rt}\big(1-F_{\tau^{(l)}}(t)\big)dt$$

$$= 1-e^{-rT}\big(1-F_{\tau^{(l)}}(T)\big) - r\int_0^T e^{-rt}\big(1-F_{\tau^{(l)}}(t)\big)dt$$

where $F_{\tau^{(l)}}(t)$ can be written as $1-F_{\tau^{(l)}}(t) = Q(\tau^{(l)}>t) = Q(N(t)<l) = \sum_{k=0}^{l-1}Q(N(t)=k)$. Substituting the above forms into the default payment leg of equity tranche gives

$$\frac{1-\delta}{n}\sum_{l=1}^{k_1}\tilde{E}\big[B\big(0,\tau^{(l)}\big)\mathbf{1}_{\tau^{(l)}\le T}\big]$$

$$+\left(D_1-\frac{k_1}{n}(1-\delta)\right)\tilde{E}\big[B\big(0,\tau^{(k_1+1)}\big)\mathbf{1}_{\tau^{(k_1+1)}\le T}\big]$$

$$= D_1 - e^{-rT}\sum_{k=0}^{k_1}\left(D_1-\frac{k}{n}(1-\delta)\right)Q\big(N(T)=k\big)$$

$$- r\int_0^T e^{-rt}\sum_{k=0}^{k_1}\left(D_1-\frac{k}{n}(1-\delta)\right)Q\big(N(t)=k\big)dt.$$

Using similar argument in Lemma 3.2, we conclude that the default payment leg is a decreasing function of a. □

Theorem 3.1. *The annualized spread of equity tranche is monotonically decreasing with respect to parameter a.*

Proof. From (20), Lemma 3.2 and Lemma 3.3, the theorem is easily derived. □

Theorem 3.1 allows us to predetermine any type of dependence structure by setting a pair of (p,q) and then using the monotonicity of equity tranche price with a for calibration.

4. COMPARISON OF MARKET AND MODEL CDO TRANCHE PREMIUMS

In this section, we will apply several known models to the Dow Jones iTraxx Europe index based on 125 names, using the data described in [5]. More specifically, the attachment and detachment points corresponding to the standard iTraxx CDO tranches are 3%, 6%, 9%, 12% and 22%. The 5 year credit spreads of the names lie between 9 bps and 120 bps with an average of 29 bps and a median of 26 bps. We assume constant credit spreads with respect to maturity for simplicity. The price of [0−3%] equity tranche is calibrated according to the market quote.

4.1 Prices

We calibrate a wide range of MGB2 models with p, q ranging from .005 to 10. Unlike the industry standard Gaussian copula where prices are predetermined once the correlation parameter is used for calibration, MGB2 distribution is more flexible with (p,q) available for creating various dependence structures. Due to the limited space, we only include three sets of CDO prices under MGB2 distribution. Prices under MGB2 distribution with $(p,q) = (5.35, 10)$ resemble those under Gaussian copula. As p decreases, the MGB2 distribution is able to fit the tranches [3−6%], [6−9%] better, although it tends to underestimate the prices of [9−12%], [12−22%]. Better fits are possible as we do not perform any optimization to match the market quotes.

Figure 1 illustrates the monotonicity of the tranche prices with respect to parameters (p,q) with either one fixed.

Table 1. Prices of iTraxx CDO tranches computed from market and model quotes

	Market	Gaussian	Clayton	t(5,5)	t(4,4)	t(3,3)	MGB2 (p,q)		
							(5.35, 10)	(4.7, 10)	(4.2, 10)
[0–3%]	916	916	916	916	916	916	916	916	916
[3–6%]	101	161	167	99	84	63	164	138	119
[6–9%]	33	47	49	36	33	27	46	33	23
[9–12%]	16	15	17	20	20	18	15	9	5
[12–22%]	9	1.5	3	9	10	11	2	1	.5
Spearman's rho		.21	.07	.26	.26	.25	.33	.30	.28

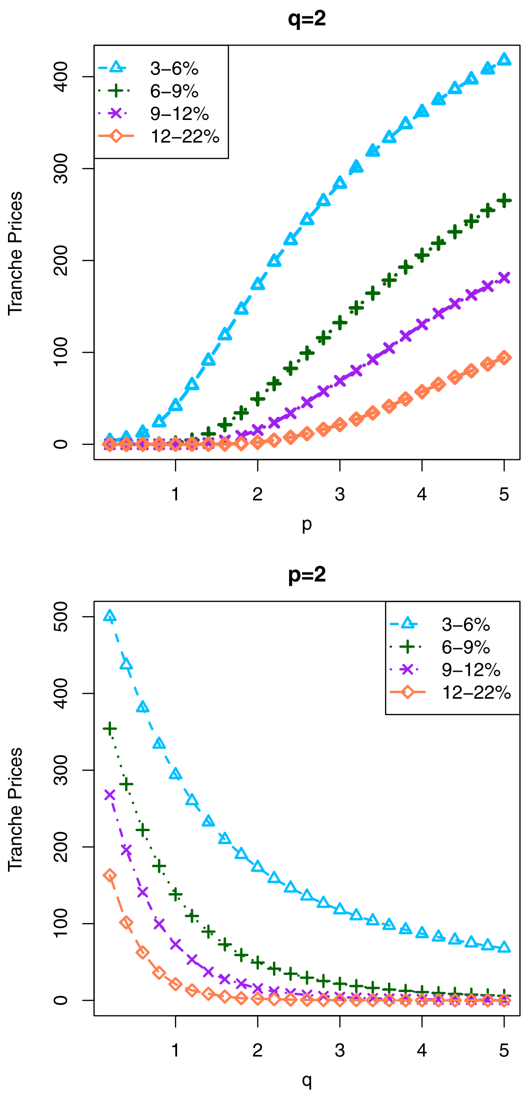

Figure 1. Tranche prices under the MGB2 model with q fixed (top) and p fixed (bottom).

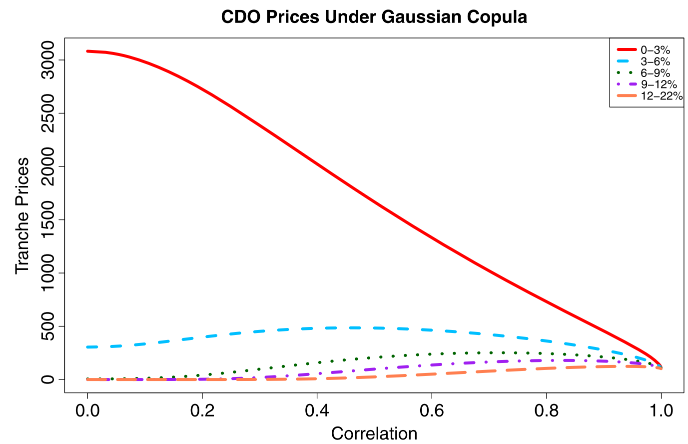

Figure 2. Tranche prices under the industry standard one factor Gaussian copula.

The prices of non-equity tranches are increasing functions of p when q is fixed, and decreasing functions of q when p is fixed, and the rate of changes depend on tranches and (p, q).

We consider the non-parametric measure of dependence spearman's rho of MGB2 distribution, which can be obtained by

$$\rho(C_{p,q}) = 12 \int_0^1 \int_0^1 B_{p+q,p}\left(\frac{1 - B_{p,q}^{-1}(u)}{1 - B_{p,q}^{-1}(u) B_{p,q}^{-1}(v)}\right) v du dv - 3.$$

The Spearman's rho for p ranging from .2 to 5 and $q = 2$ in the top panel of Figure 1 lies between [.045, .710] and increases as p increases; The Spearman's rho for q ranging from .2 to 5 and $p = 2$ in the bottom panel of Figure 1 lies between [.260, .946] and decreases with increasing q. Table 1 reports the Spearman's rho of models in consideration. MGB2 copulas have the strongest dependence, followed by double t copula, and Clayton copula has the weakest dependence according to Spearman's rho.

4.2 Implied correlation and base correlation

The implied correlation is a paradigm for implying credit default dependencies but cannot be implied for some market CDO tranches [15]. [2] investigated situations where implied correlation smile can arise as a result of model misspecification in the industry standard one-factor Gaussian copula and empirical features like fat tails in return distributions, heterogeneous pair-wise correlation, heterogeneous spreads and so on.

Figure 2 simulates the tranche prices under the industry standard one-factor Gaussian copula with correlation vary-

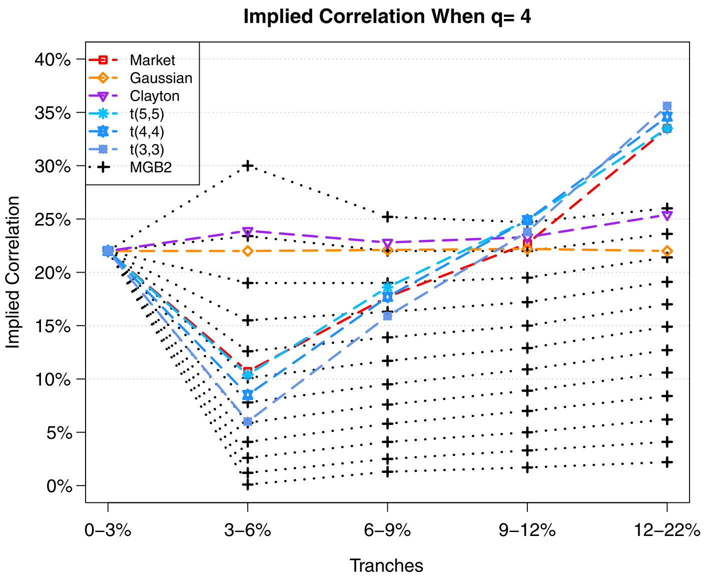

Figure 3. The dashed lines represent implied correlation for increasing p's for MGB2 model from the bottom up.

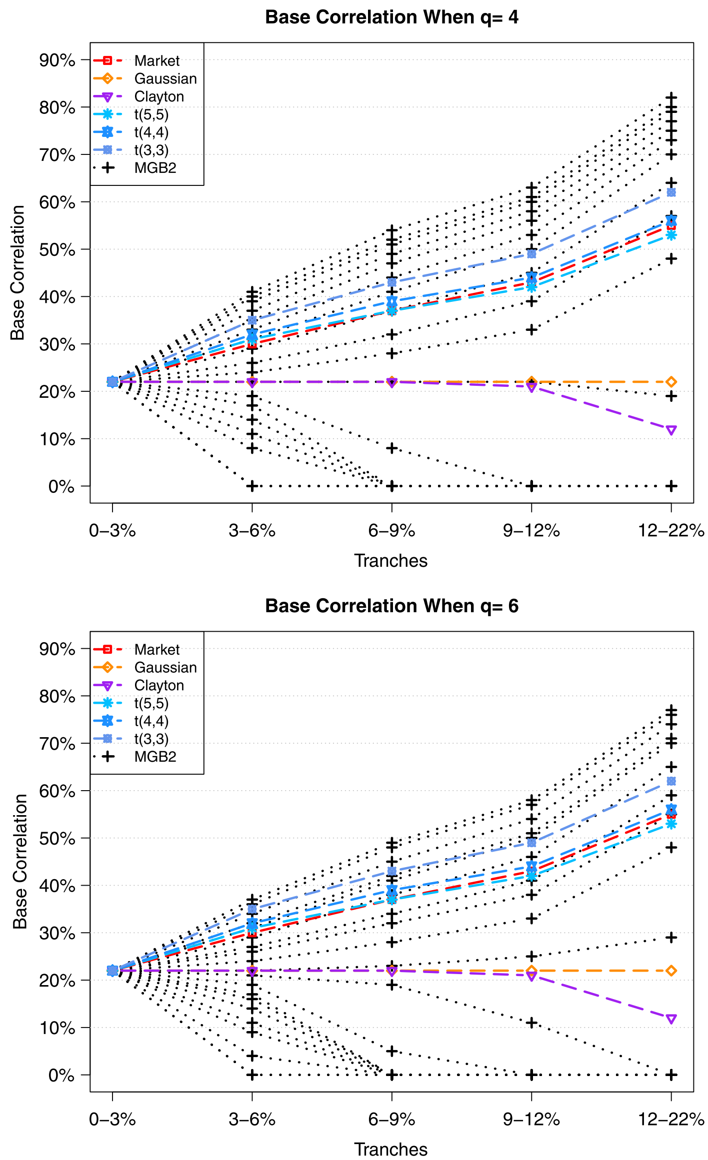

Figure 4. The dashed lines represent implied correlation for increasing p's from the top down.

ing between 0 and 1. As stated in [4], the CDO tranche premium of equity or senior type are monotonic with respect to the correlation parameter ρ in the one-factor Gaussian copula. However, it is not the case for the tranches with neither an attachment point equal to zero or a detachment point equal to 100%. As illustrated in Figure 2, prices corresponding to other tranches show vaulted patterns with respect to the correlation parameter. Among the consequences are prices without a proper implied correlation or prices with two possible values of correlation parameter, both of which confronting the wide range of prices MGB2 distribution is able to produce.

Figure 3 demonstrates some typical patterns of implied correlation based on CDO prices when q is set to be 4. The MGB2 copula approaches Gaussian copula with $\rho = c/(1+c)$ when $q \to \infty$ and $p/q \to c$; it has tail dependence index only related to (p, q); see [16]. We are able to observe tranche prices resembling Gaussian copula for different pairs of (p, q), which agrees with the consensus that tail dependence is of little help in explaining model quotes.

The drawbacks of implied correlation encourage another widely used criteria for implying credit default dependencies, the base correlation, introduced by [13]. Base correlation is defined as the inputs required for a series of equity tranches that give the tranche values consistent with quoted spreads, using the standardized large pool model. Figure 4 demonstrates some typical patterns of base correlation based on CDO prices coming from the MGB2 models. The MGB2 distribution is capable of producing a wide variety of patterns in base correlation, among which some match the base correlation of the market quote well. For instance, for $(p, q) = (2.32, 4)$ and $(3.11, 6)$, MGB2 model fits very well. The implied base correlation for the model quotes in Figure 4 are summarized in Table 2. Table 2 suggests MGB2 model has the same capacity as double t model to match the market data in terms of base correlation, and it is much better than the Gaussian and Clayton model.

The tranche spreads corresponding to the two MGB2 models in Table 2 are $(916, 114, 21, 4, .3)$ and $(916, 124, 25, 6, 1)$. As we have commented earlier, the MGB2 models tend to underestimate the prices of $[9-12\%], [12-22\%]$. However, the implied base correlations still show sufficient resemblance to those derived from double t model. In fact, our simulation shows that the

Table 2. Implied base correlation for iTraxx CDO tranches computed from market and model quotes

	Market	Gausssian	Clayton	t(5,5)	t(4,4)	t(3,3)	MGB2 (p,q)	
							$(2.32, 4)$	$(3.11, 6)$
[0−3%]	0.22	0.22	0.22	0.22	0.22	0.22	0.22	0.22
[3−6%]	0.30	0.22	0.22	0.31	0.32	0.35	0.29	0.27
[6−9%]	0.37	0.22	0.22	0.37	0.39	0.43	0.37	0.34
[9−12%]	0.43	0.22	0.21	0.42	0.44	0.49	0.45	0.41
[12−22%]	0.55	0.22	0.12	0.53	0.56	0.62	0.64	0.59

implied base correlation is especially sensitive to the price of the [3−6%] tranche, but not so for less risky tranches. This suggests that implied base correlation might not be a sufficient measure for the rich credit dynamics.

5. CONCLUSION

We proposed MGB2 distribution to evaluate the prices of synthetic CDO's tranches. The essence of our model is MGB2 copula, which marks Gaussian copula as a limit case and has great flexibility in dependence structure. We found that tail dependence index is not helpful to explain the implied correlation smile, which is consistent with one of the results in [5]. From the perspective of base correlation, MGB2 model is so flexible that it can generate many patterns of base correlation with varying parameters. Furthermore, MGB2 model can match the market implied base correlation as well as the double t model does, which the Gaussian model and Clayton model cannot make.

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REFERENCES

[1] Andersen, L. and Sidenius, J. (2004). Extensions to the Gaussian copula: Random recovery and random factor loadings. *Journal of Credit Risk* **1** 29–70.

[2] Ağca, Ş., Agrawal, D. and Islam, S. (2007). Implied correlations: Smiles or smirks? *Journal of Derivatives* **16** 7–35.

[3] Brigo, D., Pallavicini, A. and Torresetti, R. (2010). *Credit Models and the Crisis: A Journey into CDOs, Copulas, Correlations and Dynamic Models*. Wiley, Chichester.

[4] Burtschell, X., Gregory, J. and Laurent, J. P. (2007). Beyond the Gaussian copula: Stochastic and local correlation. *Journal of Credit Risk* **3** 31–62.

[5] Burtschell, X., Gregory, J. and Laurent, J. P. (2009). A comparative analysis of CDO pricing models. *Journal of Derivatives* **16** 9–37.

[6] Hofert, M. and Scherer, M. (2011). CDO pricing with nested Archimedean copulas. *Quantitative Finance* **11** 775–787. MR2800641

[7] Hull, J. C. and White, A. D. (2004). Valuation of a CDO and an n-th to default CDS without Monte Carlo simulation. *Journal of Derivatives* **12** 8–23.

[8] Kalemanova, A., Schmid, B. and Werner, R. (2007). The normal inverse Gaussian distribution for synthetic CDO pricing. *Journal of Derivatives* **14** 80–94.

[9] Laurent, J. P. and Cousin, A. (2009). An overview of factor modeling for CDO pricing. In: *Frontiers in Quantitative Finance—Volatility and Credit Risk Modeling*, Cont R., ed., Wiley, Chichester, pp. 185–216.

[10] Laurent, J. P. and Gregory, J. (2005). Basket default swaps, CDOs and factor copulas. *Journal of Risk* **7** 103–122.

[11] Li, D. X. (2000). On default correlation: A copula function approach. *Journal of Fixed Income* **9** 43–54.

[12] Madan, D. B., Konikov, M. and Marinescu, M. (2006). Credit and basket default swaps. *Journal of Credit Risk* **2** 67–87.

[13] McGinty, L., Beinstein, E., Ahluwalia, R. and Watts, M. (2004). Introducing base correlations. *Credit Derivatives Strategy*, JP Morgan.

[14] Moosbrucker, T. (2006). Pricing CDOs with correlated variance gamma distributions. *Journal of Fixed Income* **12** 1–30.

[15] Torresetti, R., Brigo, D. and Pallavicini, A. (2006). Implied correlation in CDO tranches: A paradigm to be handled with care. *Available at SSRN 946755*.

[16] Yang, X., Frees, E. W. and Zhang, Z. (2011). A generalized beta copula with applications in modeling multivariate long-tailed data. *Insurance: Mathematics and Economics* **49** 265–284. MR2811994

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