Supplemental Materials

Yuan Jiang and Chunming Zhang

We present the proofs for Theorems 3–4. Throughout the supplemental materials, $\|\cdot\|$ is used only for the L_2 norm, and $\|\cdot\|_1$ and $\|\cdot\|_\infty$ denote the L_1 norm and the L_∞ norm respectively.

Proof of Theorem 3

Let $V_{ij}(\alpha) = \mathrm{q}_1(Y_{ni}; X_{ij}\alpha)X_{ij}$. $\widehat{\beta}^{\mathrm{CR}}_{n,j}$ are the solutions of the following equations

$$h_{n,j}(\alpha) = \frac{1}{n}\sum_{i=1}^{n} V_{ij}(\alpha) = 0, \quad j = 1, \dots, p_n.$$

With Condition 1(b), $h_{n,j}(\cdot)$ is an increasing function.

Part 1: $j \in \{1, \dots, s_n\}$.

To prove $\widehat{w}^{(\mathrm{I})}_{\max} = O_P(1/\gamma^{(\mathrm{I})}_n)$, it suffices to prove that there exists a small enough $\delta > 0$ such that

$$P\left(\min_{1\le j\le s_n} |\widehat{\beta}^{\mathrm{CR}}_{n,j}| \le \gamma^{(\mathrm{I})}_n \delta\right) = o(1). \tag{1}$$

Since $h_{n,j}(\cdot)$ is an increasing function and $h_{n,j}(\widehat{\beta}^{\mathrm{CR}}_{n,j}) = 0$,

$$P(|\widehat{\beta}^{\mathrm{CR}}_{n,j}| \le \gamma^{(\mathrm{I})}_n \delta) \le P\{h_{n,j}(-\gamma^{(\mathrm{I})}_n \delta) \le 0 \le h_{n,j}(\gamma^{(\mathrm{I})}_n \delta)\}. \tag{2}$$

By Taylor's expansion,

$$V_{ij}(\pm\gamma^{(\mathrm{I})}_n \delta) = \mathrm{q}_1(Y_{ni}; 0)X_{ij} + (\pm\gamma^{(\mathrm{I})}_n \delta)\mathrm{q}_2(Y_{ni}; \pm\gamma^{(\mathrm{I})}_n \delta^* X_{ij})X_{ij}^2,$$

with $\delta^* \in (0, \delta)$. Let $\mu_0 = F^{-1}(0)$ and $C_0 = q''(\mu_0)/F'(\mu_0) \neq 0$, thus

$$E\{V_{1j}(\pm\gamma^{(\mathrm{I})}_n \delta)\} = C_0 E(Y_n X_j) + (\pm\gamma^{(\mathrm{I})}_n \delta)E\{\mathrm{q}_2(Y_n; \pm\gamma^{(\mathrm{I})}_n \delta^* X_j)X_j^2\}.$$

Because $|E(Y_n X_j)| \ge c\gamma^{(\mathrm{I})}_n$, and $\max_{1\le j\le s_n} |E\{\mathrm{q}_2(Y_n; \pm\gamma^{(\mathrm{I})}_n \delta^* X_j)X_j^2\}|$ is bounded, we can choose δ small enough, such that for $1 \le j \le s_n$,

$$|E\{V_{1j}(\pm\gamma^{(\mathrm{I})}_n \delta)\}| \ge \frac{1}{2}|C_0||E(Y_n X_j)| \ge \frac{c}{2}|C_0|\gamma^{(\mathrm{I})}_n$$
$$\text{and}\quad \mathrm{sign}[E\{V_{1j}(\gamma^{(\mathrm{I})}_n \delta)\}] = \mathrm{sign}[E\{V_{1j}(-\gamma^{(\mathrm{I})}_n \delta)\}].$$

Assuming $E\{V_{1j}(\gamma_n^{(\mathrm{I})}\delta)\} < 0$ and $E\{V_{1j}(-\gamma_n^{(\mathrm{I})}\delta)\} < 0$ without loss of generality,

$$\begin{aligned}
&P\{h_{n,j}(-\gamma_n^{(\mathrm{I})}\delta) \le 0 \le h_{n,j}(\gamma_n^{(\mathrm{I})}\delta)\} \\
&\le P\left(\sum_{i=1}^{n}[V_{ij}(\gamma_n^{(\mathrm{I})}\delta) - E\{V_{ij}(\gamma_n^{(\mathrm{I})}\delta)\}] \ge -nE\{V_{1j}(\gamma_n^{(\mathrm{I})}\delta)\}\right) \\
&\le P\left(\sum_{i=1}^{n}[V_{ij}(\gamma_n^{(\mathrm{I})}\delta) - E\{V_{ij}(\gamma_n^{(\mathrm{I})}\delta)\}] \ge \frac{c}{2}|C_0|n\gamma_n^{(\mathrm{I})}\right) \\
&\le 2\exp\left(\frac{-c^2C_0^2n^2\gamma_n^{(\mathrm{I})2}/4}{C_1 n + C_2 c|C_0|n\gamma_n^{(\mathrm{I})}/2}\right), \qquad (3)
\end{aligned}$$

where the last inequality can be obtained similar to proving Lemma 1 (with possibly different C_1 and C_2). By (2), (3) and Bonferroni inequality, for a small enough $\delta > 0$,

$$P\Big(\min_{1\le j\le s_n}|\widehat{\beta}_{n,j}^{\mathrm{CR}}| \le \gamma_n^{(\mathrm{I})}\delta\Big) \le 2s_n\exp\left(\frac{-c^2C_0^2n^2\gamma_n^{(\mathrm{I})2}/4}{C_1 n + C_2 c|C_0|n\gamma_n^{(\mathrm{I})}/2}\right) = o(1). \qquad (4)$$

The equality in (4) follows from $\gamma_n^{(\mathrm{I})} = O(1)$, $\sqrt{n}\gamma_n^{(\mathrm{I})} \to \infty$ and $\log(s_n) = o(n\gamma_n^{(\mathrm{I})2})$.

Part 2: $j \in \{s_n + 1, \ldots, p_n\}$.

To prove $(\widehat{w}_{\min}^{(\mathrm{II})})^{-1} = o_P(\gamma_n^{(\mathrm{II})})$, it suffices to prove that for any $\epsilon > 0$,

$$P\left(\max_{s_n+1\le j\le p_n}|\widehat{\beta}_{n,j}^{\mathrm{CR}}| \ge \gamma_n^{(\mathrm{II})}\epsilon\right) = o(1). \qquad (5)$$

As $h_{n,j}(\widehat{\beta}_{n,j}^{\mathrm{CR}}) = 0$, it follows that

$$P\left(|\widehat{\beta}_{n,j}^{\mathrm{CR}}| \ge \gamma_n^{(\mathrm{II})}\epsilon\right) \le P\{h_{n,j}(\gamma_n^{(\mathrm{II})}\epsilon) \le 0\} + P\{h_{n,j}(-\gamma_n^{(\mathrm{II})}\epsilon) \ge 0\}. \qquad (6)$$

Similar to Part 1,

$$E\{V_{1j}(\gamma_n^{(\mathrm{II})}\epsilon)\} = C_0E(Y_nX_j) + \epsilon\gamma_n^{(\mathrm{II})}E\{\mathrm{q}_2(Y_n;\gamma_n^{(\mathrm{II})}\epsilon^*X_j)X_j^2\},$$

with $\epsilon^* \in (0,\epsilon)$. Since $E(Y_nX_j) = o(\gamma_n^{(\mathrm{II})})$ and $E\{\mathrm{q}_2(Y_n;\gamma_n^{(\mathrm{II})}\epsilon^*X_j)X_j^2\} \ge \eta$,

$$E\{V_{1j}(\gamma_n^{(\mathrm{II})}\epsilon)\} \ge \epsilon\eta\gamma_n^{(\mathrm{II})}/2, \text{ as } n \to \infty.$$

Again by an application of Bernstein's inequality as in (3), for large n,

$$\begin{aligned}
P\{h_{n,j}(\gamma_n^{(\mathrm{II})}\epsilon) \le 0\} &= P\Big(\frac{1}{n}\sum_{i=1}^{n}[V_{ij}(\gamma_n^{(\mathrm{II})}\epsilon) - E\{V_{ij}(\gamma_n^{(\mathrm{II})}\epsilon)\}] \le -E\{V_{1j}(\gamma_n^{(\mathrm{II})}\epsilon)\}\Big) \\
&\le P\Big(\sum_{i=1}^{n}[V_{ij}(\gamma_n^{(\mathrm{II})}\epsilon) - E\{V_{ij}(\gamma_n^{(\mathrm{II})}\epsilon)\}] \le -\epsilon\eta n\gamma_n^{(\mathrm{II})}/2\Big) \\
&\le 2\exp\Big(\frac{-\epsilon^2\eta^2n^2\gamma_n^{(\mathrm{II})2}/4}{C_1 n + C_2\epsilon\eta n\gamma_n^{(\mathrm{II})}/2}\Big). \qquad (7)
\end{aligned}$$

Similarly,

$$P\{h_{n,j}(-\gamma_n^{(\mathrm{II})}\epsilon) \geq 0\} \leq 2\exp\left(\frac{-\epsilon^2\eta^2 n^2 \gamma_n^{(\mathrm{II})2}/4}{C_1 n + C_2 \epsilon\eta n \gamma_n^{(\mathrm{II})}/2}\right). \tag{8}$$

Thus by (6), (7), (8) and Bonferroni inequality,

$$P\left(\max_{s_n+1\leq j\leq p_n} |\widehat{\beta}_{n,j}^{\mathrm{CR}}| \geq \gamma_n^{(\mathrm{II})}\epsilon\right) \leq 4(p_n - s_n)\exp\left(\frac{-\epsilon^2\eta^2 n^2 \gamma_n^{(\mathrm{II})2}/4}{C_1 n + C_2 \epsilon\eta n \gamma_n^{(\mathrm{II})}/2}\right) = o(1). \tag{9}$$

The equality in (9) follows from the conditions $\sqrt{n}\gamma_n^{(\mathrm{II})} \to \infty$, $\log(p_n - s_n) = o(n\gamma_n^{(\mathrm{II})})$ and $\log(p_n - s_n) = o(n\gamma_n^{(\mathrm{II})2})$.

Proof of Theorem 4

Minimizing (4.3) is equivalent to minimizing the following criterion functions,

$$\ell_{n,j}^{\mathrm{PCR}}(\alpha) = \frac{1}{n}\sum_{i=1}^{n} Q(Y_{ni}, F^{-1}(X_{ij}\alpha)) + \kappa_n|\alpha|, \quad j = 1, \ldots, p_n.$$

Part 1: $j \in \{1, \ldots, s_n\}$.

Similar to the proof of Theorem 3, we prove that for a small enough $\delta > 0$, there exist local minimizers $\widehat{\beta}_{n,j}^{\mathrm{PCR}}$ of $\ell_{n,j}^{\mathrm{PCR}}(\alpha)$ such that

$$P\Big(\min_{1\leq j\leq s_n} |\widehat{\beta}_{n,j}^{\mathrm{PCR}}| > \gamma_n^{(\mathrm{I})}\delta\Big) \to 1. \tag{10}$$

It suffices to prove that for a small enough $\delta > 0$ and some large enough $C_n > 0$, there exist some β_j with $|\beta_j| = 2\delta$ such that

$$P\Big(\min_{1\leq j\leq s_n}\Big\{\inf_{|\alpha|\leq\delta} \ell_{n,j}^{\mathrm{PCR}}(\gamma_n^{(\mathrm{I})}\alpha) - \ell_{n,j}^{\mathrm{PCR}}(\gamma_n^{(\mathrm{I})}\beta_j)\Big\} > 0\Big) \to 1, \tag{11}$$

and

$$P\Big(\min_{1\leq j\leq s_n}\Big\{\inf_{|\alpha|\geq C_n} \ell_{n,j}^{\mathrm{PCR}}(\gamma_n^{(\mathrm{I})}\alpha) - \ell_{n,j}^{\mathrm{PCR}}(\gamma_n^{(\mathrm{I})}\beta_j)\Big\} > 0\Big) \to 1. \tag{12}$$

(11) and (12) imply that with probability tending to one, there must exist local minimizers $\widehat{\beta}_{n,j}^{\mathrm{PCR}}$ of $\ell_{n,j}^{\mathrm{PCR}}(\alpha)$ such that $\gamma_n^{(\mathrm{I})}\delta < |\widehat{\beta}_{n,j}^{\mathrm{PCR}}| < \gamma_n^{(\mathrm{I})}C_n$ for $1 \leq j \leq s_n$, and this implies (10).

First, we prove (12). We notice that for every $n \geq 1$, when $|\alpha| \to \infty$,

$$\min_{1\leq j\leq s_n}\Big\{\ell_{n,j}^{\mathrm{PCR}}(\gamma_n^{(\mathrm{I})}\alpha) - \ell_{n,j}^{\mathrm{PCR}}(\gamma_n^{(\mathrm{I})}\beta_j)\Big\} \geq \kappa_n\gamma_n^{(\mathrm{I})}|\alpha| - \max_{1\leq j\leq s_n} \ell_{n,j}^{\mathrm{PCR}}(\gamma_n^{(\mathrm{I})}\beta_j) \to \infty.$$

Thus (12) holds.

Second, we prove (11). By Taylor's expansion, for $1 \leq j \leq s_n$,

$$\begin{aligned}\ell_{n,j}^{\mathrm{PCR}}(\gamma_n^{(\mathrm{I})}\alpha) = &\frac{1}{n}\sum_{i=1}^{n} Q(Y_{ni}, F^{-1}(0)) + \frac{\gamma_n^{(\mathrm{I})}}{n}\alpha\sum_{i=1}^{n} \mathrm{q}_1(Y_{ni}; 0)X_{ij} \\ &+ \frac{1}{2}\frac{\gamma_n^{(\mathrm{I})2}}{n}\alpha^2\sum_{i=1}^{n} \mathrm{q}_2(Y_{ni}; \gamma_n^{(\mathrm{I})}\alpha_j^* X_{ij})X_{ij}^2 + \gamma_n^{(\mathrm{I})}\kappa_n|\alpha|,\end{aligned}$$

with α_j^* between 0 and α. Thus, we have that

$$\begin{aligned}
&\min_{1\le j\le s_n}\left\{\inf_{|\alpha|\le\delta}\ell_{n,j}^{\mathrm{PCR}}(\gamma_n^{(\mathrm{I})}\alpha)-\ell_{n,j}^{\mathrm{PCR}}(\gamma_n^{(\mathrm{I})}\beta_j)\right\}\\
\ge&\min_{1\le j\le s_n}\inf_{|\alpha|\le\delta}\left\{\frac{\gamma_n^{(\mathrm{I})}}{n}(\alpha-\beta_j)\sum_{i=1}^n \mathrm{q}_1(Y_{ni};0)X_{ij}\right\}\\
&+\min_{1\le j\le s_n}\left[\frac{1}{2}\frac{\gamma_n^{(\mathrm{I})2}}{n}\inf_{|\alpha|\le\delta}\left\{\alpha^2\sum_{i=1}^n \mathrm{q}_2(Y_{ni};\gamma_n^{(\mathrm{I})}\alpha_j^*X_{ij})X_{ij}^2-\beta_j^2\sum_{i=1}^n \mathrm{q}_2(Y_{ni};\gamma_n^{(\mathrm{I})}\beta_j^*X_{ij})X_{ij}^2\right\}\right]\\
&+\min_{1\le j\le s_n}\inf_{|\alpha|\le\delta}\{\gamma_n^{(\mathrm{I})}\kappa_n(|\alpha|-|\beta_j|)\}\\
\equiv& I_1+I_2+I_3,
\end{aligned}$$

with α_j^* between 0 and α, and β_j^* between 0 and $\beta_{n,j}$.

Let $\mu_0=F^{-1}(0)$ and $C_0=q''(\mu_0)/F'(\mu_0)\ne 0$, for I_1,

$$\begin{aligned}
I_1\ge&\gamma_n^{(\mathrm{I})}\min_{1\le j\le s_n}\inf_{|\alpha|\le\delta}\{C_0(\alpha-\beta_j)E(Y_nX_j)\}\\
&+\gamma_n^{(\mathrm{I})}\min_{1\le j\le s_n}\inf_{|\alpha|\le\delta}\left[C_0(\alpha-\beta_j)\frac{1}{n}\sum_{i=1}^n\{Y_{ni}X_{ij}-E(Y_nX_j)\}\right]\\
&-\gamma_n^{(\mathrm{I})}\max_{1\le j\le s_n}\sup_{|\alpha|\le\delta}\left\{C_0\mu_0(\alpha-\beta_j)\frac{1}{n}\sum_{i=1}^n X_{ij}\right\}\\
\equiv& I_{1,1}+I_{1,2}+I_{1,3}.
\end{aligned}$$

We can see that

$$\begin{aligned}
|I_{1,3}|&\le\gamma_n^{(\mathrm{I})}|C_0\mu_0|\max_{1\le j\le s_n}\left\{\sup_{|\alpha|\le\delta}\left(3\delta\left|\frac{1}{n}\sum_{i=1}^n X_{ij}\right|\right)\right\}\\
&=O_P(\gamma_n^{(\mathrm{I})}\{\log(s_n)/n\}^{1/2})\delta,
\end{aligned}$$

since $\max_{1\le j\le s_n}|n^{-1}\sum_{i=1}^n X_{ij}|=O_P(\{\log(s_n)/n\}^{1/2})$ from Bernstein's inequality (Lemma 2.2.9 in van der Vaart and Wellner, 1996). Again $|I_{1,2}|=O_P(\gamma_n^{(\mathrm{I})}\{\log(s_n)/n\}^{1/2})\delta$ by a similar argument as in the proof of Theorem 3. Now we choose $\beta_j=-2\delta\mathrm{sign}\{C_0E(Y_nX_j)\}$ satisfying $|\beta_j|=2\delta$. Then

$$\begin{aligned}
I_{1,1}&=\gamma_n^{(\mathrm{I})}\min_{1\le j\le s_n}\inf_{|\alpha|\le\delta}\Big([\alpha+2\delta\mathrm{sign}\{C_0E(Y_nX_j)\}]C_0E(Y_nX_j)\Big)\\
&\ge\gamma_n^{(\mathrm{I})}\min_{1\le j\le s_n}\{\delta|C_0E(Y_nX_j)|\}\ge|C_0|c\gamma_n^{(\mathrm{I})}\mathcal{A}_n\delta.
\end{aligned}$$

For I_2 and I_3,

$$\begin{aligned}
|I_2|\le&\frac{1}{2}\frac{\gamma_n^{(\mathrm{I})2}}{n}\max_{1\le j\le s_n}\sup_{|\alpha|\le\delta}\left\{\alpha^2\left|\sum_{i=1}^n \mathrm{q}_2(Y_{ni};X_{ij}\gamma_n^{(\mathrm{I})}\alpha_j^*)X_{ij}^2\right|\right.\\
&\left.+\beta_j^2\left|\sum_{i=1}^n \mathrm{q}_2(Y_{ni};X_{ij}\gamma_n^{(\mathrm{I})}\beta_j^*)X_{ij}^2\right|\right\}\\
=&O_P(\gamma_n^{(\mathrm{I})2})\delta^2,
\end{aligned}$$

and $|I_3| = O(\gamma_n^{(\mathrm{I})}\kappa_n)\delta$.

Under the conditions $\mathcal{A}_n = \gamma_n^{(\mathrm{I})}$, $\kappa_n = o(\gamma_n^{(\mathrm{I})})$ and $\log(s_n) = o(n\gamma_n^{(\mathrm{I})2})$, we can choose a small enough $\delta > 0$ such that with probability tending to one, $I_{1,1}$ dominates $I_{1,2}$, $I_{1,3}$, I_2 and I_3. Thus (11) is proved.

Part 2: $j \in \{s_n + 1, \ldots, p_n\}$.

We prove that for any $\epsilon > 0$, there exist local minimizers $\widehat{\beta}_{n,j}^{\mathrm{PCR}}$ of $\ell_{n,j}^{\mathrm{PCR}}(\alpha)$ such that

$$P\Big(\max_{s_n+1 \le j \le p_n} |\widehat{\beta}_{n,j}^{\mathrm{PCR}}| \le \gamma_n^{(\mathrm{II})}\epsilon\Big) \to 1. \tag{13}$$

It suffices to prove that for any $\epsilon > 0$,

$$P\Big(\min_{s_n+1 \le j \le p_n} \Big\{\inf_{|\alpha|=\epsilon} \ell_{n,j}^{\mathrm{PCR}}(\gamma_n^{(\mathrm{II})}\alpha) - \ell_{n,j}^{\mathrm{PCR}}(0)\Big\} > 0\Big) \to 1. \tag{14}$$

By Taylor's expansion,

$$\begin{aligned}
&\min_{s_n+1 \le j \le p_n} \Big\{\inf_{|\alpha|=\epsilon} \ell_{n,j}^{\mathrm{PCR}}(\gamma_n^{(\mathrm{II})}\alpha) - \ell_{n,j}^{\mathrm{PCR}}(0)\Big\} \\
\ge\ & \min_{s_n+1 \le j \le p_n} \inf_{|\alpha|=\epsilon} \Big\{\frac{\gamma_n^{(\mathrm{II})}}{n}\alpha \sum_{i=1}^n \mathrm{q}_1(Y_{ni}; 0)X_{ij}\Big\} \\
& + \min_{s_n+1 \le j \le p_n} \inf_{|\alpha|=\epsilon} \Big\{\frac{\gamma_n^{(\mathrm{II})2}}{2n}\alpha^2 \sum_{i=1}^n \mathrm{q}_2(Y_{ni}; \gamma_n^{(\mathrm{II})}\alpha_j^* X_{ij})X_{ij}^2\Big\} + \inf_{|\alpha|=\epsilon} (\gamma_n^{(\mathrm{II})}\kappa_n|\alpha|) \\
\equiv\ & I_1 + I_2 + I_3,
\end{aligned}$$

with $\alpha_j^* \in (0, \alpha)$. For I_1, it is seen that

$$\begin{aligned}
|I_1| &\le \max_{s_n+1 \le j \le p_n} \sup_{|\alpha|=\epsilon} \Big\{\frac{\gamma_n^{(\mathrm{II})}}{n}\Big|\alpha \sum_{i=1}^n C_0(Y_{ni} - \mu_0)X_{ij}\Big|\Big\} \\
&\le \frac{\gamma_n^{(\mathrm{II})}}{n}|C_0|\epsilon \max_{s_n+1 \le j \le p_n} \Big|\sum_{i=1}^n Y_{ni}X_{ij}\Big| + \frac{\gamma_n^{(\mathrm{II})}}{n}|C_0\mu_0|\epsilon \max_{s_n+1 \le j \le p_n} \Big|\sum_{i=1}^n X_{ij}\Big|.
\end{aligned}$$

Since $|\sum_{i=1}^n Y_{ni}X_{ij}| \le |\sum_{i=1}^n \{Y_{ni}X_{ij} - E(Y_nX_j)\}| + |\sum_{i=1}^n E(Y_nX_j)|$ and similar to Part 1,

$$\max_{s_n+1 \le j \le p_n} \Big|\frac{1}{n}\sum_{i=1}^n X_{ij}\Big| = O_P(\{\log(p_n - s_n)/n\}^{1/2}),$$

$$\max_{s_n+1 \le j \le p_n} \Big|\frac{1}{n}\sum_{i=1}^n \{Y_{ni}X_{ij} - E(Y_nX_j)\}\Big| = O_P(\{\log(p_n - s_n)/n\}^{1/2}),$$

we have that $|I_1| \le O_P(\gamma_n^{(\mathrm{II})}\{\log(p_n - s_n)/n\}^{1/2})\epsilon + o_P(\gamma_n^{(\mathrm{II})}\mathcal{B}_n)\epsilon$.

The proof may be separated for case (1) and case (2) in Theorem 4 from here.

Case (1): For I_2, we have that

$$|I_2| \le \max_{s_n+1 \le j \le p_n} \sup_{|\alpha|=\epsilon} \Big\{\frac{\gamma_n^{(\mathrm{II})2}}{2n}\alpha^2\Big|\sum_{i=1}^n \mathrm{q}_2(Y_{ni}; \gamma_n^{(\mathrm{II})}\alpha_j^* X_{ij})X_{ij}^2\Big|\Big\} = O_P(\gamma_n^{(\mathrm{II})2})\epsilon^2.$$

Thus $I_3 = \gamma_n^{(\mathrm{II})}\kappa_n\epsilon$ dominates I_1 and I_2 with probability tending to one, since $\log(p_n - s_n) = o(n\kappa_n^2)$, $\mathcal{B}_n = O(\kappa_n)$ and $\gamma_n^{(\mathrm{II})} = o(\kappa_n)$. So (14) is proved.

Case (2): I_2 is always positive with Condition 1(b). Moreover,

$$
\begin{aligned}
I_2 \geq & \frac{1}{2}\gamma_n^{(\mathrm{II})2} \min_{s_n+1\leq j\leq p_n} \inf_{|\alpha|=\epsilon} \left[\alpha^2 E\{\mathrm{q}_2(Y_n;\gamma_n^{(\mathrm{II})}\alpha_j^* X_j)X_j^2\}\right] - \frac{1}{2}\gamma_n^{(\mathrm{II})2}\times \\
& \max_{s_n+1\leq j\leq p_n} \sup_{|\alpha|=\epsilon} \left(\alpha^2 \left|\frac{1}{n}\sum_{i=1}^{n}\left[\mathrm{q}_2(Y_{ni};\gamma_n^{(\mathrm{II})}\alpha_j^* X_{ij})X_{ij}^2 - E\{\mathrm{q}_2(Y_n;\gamma_n^{(\mathrm{II})}\alpha_j^* X_j)X_j^2\}\right]\right|\right) \\
\geq & \frac{1}{2}\gamma_n^{(\mathrm{II})2}\epsilon^2\eta - \frac{1}{2}\gamma_n^{(\mathrm{II})2}\epsilon^2 O_P(\{\log(p_n - s_n)/n\}^{1/2}).
\end{aligned}
$$

Thus, $I_3 = \gamma_n^{(\mathrm{II})}\kappa_n\epsilon$ and the term $\frac{1}{2}\gamma_n^{(\mathrm{II})2}\epsilon^2\eta$ in I_2 dominate all other terms with probability tending to one, since $\log(p_n - s_n) = o\{\max(n\kappa_n^2, n\gamma_n^{(\mathrm{II})2})\}$, $\mathcal{B}_n = O\{\max(\kappa_n, \gamma_n^{(\mathrm{II})})\}$ and $\log(p_n - s_n) = o(n)$. So (14) is also proved.